

## PROBLEM PAGE

Both the new problems this time are inequalities.

1. If  $1 < p \leq 2$  and  $\alpha = \frac{\pi}{2p}$  show that

$$\left(\frac{\cos \theta}{\cos \alpha}\right)^p \geq 1 + (\tan \alpha) \cos(p\theta), \quad \text{for } 0 \leq \theta \leq \alpha.$$

This was shown to me by Matts Essen of Uppsala. It can be done by elementary calculus, but function theorists may like to speculate on how the inequality arises 'naturally'.

The other problem was submitted by Bob Grove of Auburn University, Alabama.

2. Suppose that  $0 = \phi_1 \leq \dots \leq \phi_n < \pi$ , that  $A = [\sin(|\phi_i - \phi_j|)]$ , and that  $\|A\| = \max(\|Ax\| : \|x\| = 1)$ . Show that

$$\|A\| \leq \cot\left(\frac{\pi}{2n}\right),$$

and characterize the case of equality.

Now for the solutions of some previous problems.

1. Consider the sequence of digits

198423768.....

obtained using the rule:

"after 1984 every digit which appears is the final digit of the sum of the previous four digits."

Does 1984 appear later in the sequence and, if so, when? What about 1985?

This problem was suggested by Pat Fitzpatrick who says that it originated in a Hungarian mathematical magazine.

First note that if we reduce all digits mod 2 then the sequence is

1100011000 .....

which is periodic with period 5. Hence 1985, which reduces to 1101 mod 2, can never appear.

To see that 1984 must reappear note that there are only  $10^4$  four digit numbers. Hence, some block of four digits must repeat, say abcd. Since the sequence of digits can be generated backwards in a unique manner from any given block of four digits, we can arrive at a second occurrence of 1984 by working backwards from the second occurrence of abcd.

Working with a computer one finds that 1984 reappears after 1560 steps. However, Pat points out that this fact can be ascertained even in the event of a power failure, using a little algebra to reduce the effort. Here is the idea.

The problem can be written in the form

$$a_{n+4} = a_{n+3} + a_{n+2} + a_{n+1} + a_n \pmod{10}$$

where  $a_0 = 1, a_1 = 9, a_2 = 8, a_3 = 4$ . We know that this sequence has period 5 when reduced mod 2 and so it is enough (since 2 and 5 are coprime) to find the period  $n$  when the sequence is reduced mod 5. The original sequence will then have period  $5n$ .

Recasting the problem in matrix form we have

$$\underline{u}_{n+1} = A\underline{u}_n \pmod{10}, \quad \text{for } n = 0, 1, 2, \dots,$$

where

$$\underline{u}_n = \begin{pmatrix} a_n \\ a_{n+1} \\ a_{n+2} \\ a_{n+3} \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

We are then seeking the smallest  $n$  such that

$$A^n \underline{u}_0 \equiv \underline{u}_0 \pmod{5}.$$

Now the vectors  $\underline{u}_0, \underline{u}_1, \underline{u}_2, \underline{u}_3$  are linearly independent over  $\mathbb{Z}_5$  since

$$\begin{vmatrix} 1 & 9 & 8 & 4 \\ 9 & 8 & 4 & 2 \\ 8 & 4 & 2 & 3 \\ 4 & 2 & 3 & 7 \end{vmatrix} = -149 \equiv -1 \pmod{5},$$

As

$$A^n \underline{u}_i \equiv \underline{u}_i \pmod{5}, \text{ for } i = 0, 1, 2, 3,$$

this shows that

$$A^n \equiv I \pmod{5}.$$

The matrix  $A$  satisfies its own characteristic equation, that is,

$$x^4 - x^3 - x^2 - x - 1 = 0 \quad (*)$$

and, since this polynomial is irreducible over  $\mathbb{Z}_5$ , the smallest field containing  $\mathbb{Z}_5$  and  $A$  is  $GF(5^4)$ . (I am grateful to Bob Margolis at the Open University for a short refresher course on Galois theory!) This means that the multiplicative order of  $A$  in  $GF(5^4)$  is a divisor of  $5^4 - 1 = 624 = 2^4 \cdot 3 \cdot 13$ . It is now a tedious but elementary exercise to check (with the aid of  $(*)$ ) that the multiplicative order of  $A$  in  $GF(5^4)$  is 312.

So the answer to the original problem is indeed  $1560 = 5 \times 312$ .

**Remarks 1.** It is easy to check that  $A^5 \equiv I \pmod{2}$  and so the above discussion shows that the period is 1560 whenever  $\det(\underline{u}_0 \underline{u}_1 \underline{u}_2 \underline{u}_3)$  is relatively prime to 10.

2. Tim Lister at the Open University noticed that 9126 appears in 1984 ... after exactly 780 steps (half a period). In fact

$$A^{156} \equiv -I \pmod{5},$$

since the non-zero elements of  $GF(5^4)$  form a cyclic multiplicative group, and so

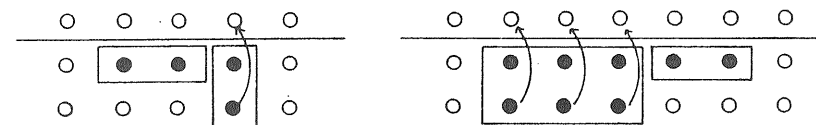
$$A^{780} \equiv -I \pmod{10}.$$

So this is no coincidence!

3. In his book "Geometry", Coxeter credits Lagrange as the first to notice that the final digits of the Fibonacci numbers form a periodic sequence with period 60. However, he gives no algebraic discussion of this fact.

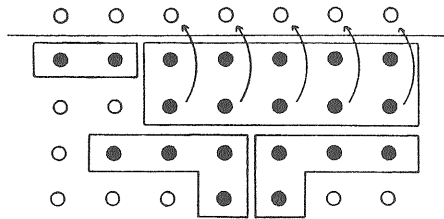
1. Playing solitaire on an unlimited board, on which is drawn a horizontal line, you are required to lay out pegs below the line in such a way that a single peg can be manoeuvred as high as possible above the line.

The diagrams below illustrate positions which enable a peg to reach the second, third and fourth rows, respectively. The blocking of the pegs indicates, informally, the order of play.



Position 1

Position 2

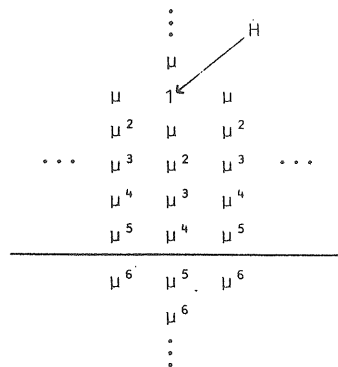


Position 3

Notice that, while playing position 2, we obtain position 1 moved up one row. Similarly, while playing position 3, we obtain position 2 moved up one row.

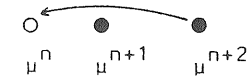
However, there is no arrangement of pegs below the line which enables a single peg to reach the fifth row above the line. Here is the beautiful proof of this fact given in volume 2 of "Winning Ways for your Mathematical Plays" by Berlekamp, Conway and Guy (warning: these books are addictive!).

Choose a particular hole H on the fifth row. It is enough to show that H cannot be reached. Assign to H the value 1 and then assign to any other hole the value  $\mu^n$ , where n is the length of the shortest path (parallel to the axes) to H. Here  $\mu$  is a number between 0 and 1 to be chosen in a moment.



Any position can now be assigned a value by adding the values of the pegs.

We choose  $\mu$  in such a way that a move of the following type



leaves the value of the position unchanged. Evidently we require that

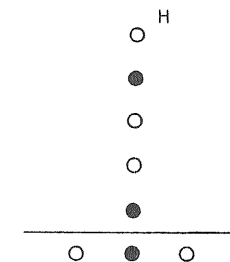
$$\mu^n = \mu^{n+1} + \mu^{n+2}, \text{ that is, } \mu^2 + \mu - 1 = 0,$$

and so  $\mu = \frac{1}{2}(\sqrt{5} - 1) \approx 0.618$ , a not-unfamiliar number!

It is now easy to check that no move can increase the value of a position, and so to reach H it is necessary to start with a position whose value is at least 1. If such a position exists, with all pegs below the line, we may assume that it contains only finitely many pegs.

However, a straightforward calculation shows that the total value of all the holes lying below the line is 1. Hence no such position exists.

**Remarks 1.** This valuation of positions makes the attainment of row 4 look rather a modest achievement, since the hole below H has value only  $\mu$ . It is much harder, for example, to reach the following position



which has value

$$\mu + \mu^4 + \mu^5 = 1 - \mu^4 \approx 0.854.$$

It seems unlikely that one can reach every position (below H) which has value less than 1, but I don't know of a counter-example.

2. A similar calculation reveals that for the analogous problem in three dimensions it is impossible for a peg to reach the eighth row. The seventh row can be reached, however.

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## CONFERENCE REPORT

### GROUPS IN GALWAY 10-11 MAY 1985

A Conference on Group Theory, sponsored by University College, Galway, the Royal Irish Academy, and the Irish Mathematical Society was again held at University College, Galway, on Friday and Saturday 10th and 11th May 1985. The invited speakers were T.O. Hawkes (Warwick), G.D. James (Cambridge), T.J. Laffey (UCD), P.D. MacHale (UCC), T.G. Murphy (TCD), and S.J. Tobin (UCG). Lectures were also given by D.W. Lewis (UCD) and M. Ó'Searcóid (UCC).

Seán Tobin opened the Conference with his talk entitled 'Razmyslov Algebras' (see *I.M.S. Newsletter* 13 (March 1985) pp. 57-65). He explained Razmyslov's construction of a non-soluble group of exponent  $p^2$ , for odd primes  $p$ .

Tim Murphy, in his lecture on 'Tensor Groups', discussed the duality between linear groups and the sets of tensors which they fix. He also illustrated the use of Penrose's notation for tensors, and his hand-out demonstrated the qualities of the TEX computer typesetting system.

Gordon James described 'A  $q$ -Analogue of the Symmetric Group Algebra'. Given a field  $F$  of characteristic  $p$ , and a nonzero element  $q$  of  $F$ , he defined an  $F$ -algebra  $H$ , whose properties (for example the number of its simple right ideals, and the dimension of its centre) can be obtained from those of the group algebra over  $F$  of the symmetric group by substituting  $q$  for  $p$ .

Tom Laffey's title was 'Some Maximal Subgroups of the General Linear Group'. He began by presenting a short proof (due to Radjavi) of the fact that every element of  $SL_n(F)$  is a commutator in  $GL_n(F)$ , except when  $n = |F| = 2$ . He then proved that the invertible monomial matrices form a maximal subgroup