PERSONAL ITEMS

Dr Ray Ryan of the Mathematics Department, UCG, is visiting the Department of Mathematics at Kent State University for the academic year 1985-'86.

Professor Robin Harte of the Mathematics Department, UCC, will be at the University of Iowa on sabbatical leave during the academic year 1985-'86.

Dr Gabrielle Kelly of the Statistics Department, UCC, will be at Columbia University on sabbatical leave during the academic year 1985-'86.

Dr Niall Ó Murchadha of the Department of Experimental Physics, UCC, will spend the period September 1985-March 1986 on leave at the University of British Columbia.

Mr Micheál Ó Searcóid has been appointed to a temporary lectureship at the Mathematics Department, UCC, for the academic year 1985-'86. His research interests are in Operator Theory.

Dr Alastair Wood has been appointed to the Westinghouse Chair of Applied Mathematical Sciences at NIHE, Dublin.

A MATRIX JOKE

Robin Harte

1. If $x = (x_{ij}) \in A^{n,n}$ is an nxn matrix with entries x_{ij} in a ring A with identity 1, under what conditions does it have a two-sided inverse $x^{-1} \in A^{n,n}$? If the ring A is commutative, then the answer is very nearly the same as for the real or the complex numbers:

$$x \text{ invertible in } A^{n,n} \iff |x| \text{ invertible in } A, \qquad (1.1)$$

where $|x|$ denotes the *determinant* of x, defined [5, Chapter 5] in any one of the usual ways. If the ring A is not commutative then the formulae for the determinant become ambiguous, unless we restrict to matrices $x = (x_{ij})$ which are *commutative*, in the sense that their entries form a commutative set $\{x_{ij}\}$. With this restriction implication (1.1) was demonstrated for 2x2 matrices of Hilbert space operators by Halmos [1, Problem 55], extended to nxn matrices of Banach algebra elements using the spectral mapping theorem [3, Example 2.4], and is now given in full generality by Halmos again [2, Problem 70]. In this note we will demonstrate that (1.1) holds separately for left and right inverses, at least for 2x2 matrices: the argument seems to depend on a joke.

2. Suppose that $x = (x_{ij})$ is a commutative nxn matrix over the ring A, with determinant $|x| \in A$, and *cofactor* $x^{\sim} \in A^{n,n}$, in the sense of the usual 'adjugate' or 'classical adjoint' matrix of x: then we recall *Cramer's rule*,

$$x^{\sim}x = xx^{\sim} = |x|\underline{1}, \qquad (2.1)$$

and

$$\underline{1}^{\sim} = \underline{1},$$

where $\underline{1} = (\delta_{ij})$ is the identity matrix. If also $y = (y_{ij})$ is another commutative matrix, and if in addition *the entries of* *y commute with the entries of* x, then we have the product formula

$$(xy)^\sim \;=\; y^\sim x^\sim , \qquad (2.3)$$

and hence also

$$|xy| \;=\; |x|\,|y| \;=\; |y|\,|x| . \qquad (2.4)$$

Backward implication in (1.1) is clear from (2.1); conversely if a commutative matrix x has a two-sided inverse x^{-1} in $A^{n,n}$, and if $x^{-1} = y$ is commutative and has its entries commuting with those of x, then (2.4) will guarantee that |x| is invertible in A. The second Halmos argument [2, Problem 70] demonstrates this by noting that if $z \in A^{n,n}$ and $t \in A$ are arbitrary then there is implication

$$xz \;=\; zx \;\Rightarrow\; zx^{-1} \;=\; x^{-1}z \qquad (2.5)$$

and

$$x(t\underline{1}) \;=\; (t\underline{1}) \;\Leftrightarrow\; \mathrm{AND}_{ij}(x_{ij}t \;=\; t x_{ij}) . \qquad (2.6)$$

3. The analogue of (1.1) holds separately for left and right inverses: if $x \in A^{n,n}$ is commutative then

$$x \textit{ left invertible in } A^{n,n} \;\Leftrightarrow\; |x| \textit{ left invertible in } A \qquad (3.1)$$

and

$$x \textit{ right invertible in } A^{n,n} \;\Leftrightarrow\; |x| \textit{ right invertible in } A . \qquad (3.2)$$

We shall confine ourselves to the proof of (3.1) when n = 2:

THEOREM *If a, b, c, d are mutually commuting elements of A then*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \textit{ left invertible in } A^{2,2} \;\Leftrightarrow\; ad-bc \textit{ left invertible in } A . \qquad (3.3)$$

Proof. From (2.1) we have

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \;=\; \begin{pmatrix} ad-bc & 0 \\ 0 & ad-bc \end{pmatrix} \;=\; \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} , \qquad (3.4)$$

which gives backward implication in (3.1). Conversely if

$$\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} \;=\; \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} , \qquad (3.5)$$

with no commutativity assumptions on a', b', c', d' in A, then (3.4) gives

$$\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}\begin{pmatrix} ad-bc & 0 \\ 0 & ad-bc \end{pmatrix} \;=\; \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} . \qquad (3.6)$$

We now come to what we think is the joke: if you take apart (3.6) and then reassemble its four constituent equations, you get

$$\begin{pmatrix} d' & -b' \\ -c' & a' \end{pmatrix}\begin{pmatrix} ad-bc & 0 \\ 0 & ad-bc \end{pmatrix} \;=\; \begin{pmatrix} a & b \\ c & d \end{pmatrix} . \qquad (3.7)$$

The joke is now over: another application of (3.5) gives two (possibly equal) left inverses for ad-bc in A:

$$(a'd'-b'c')(ad-bc) \;=\; (d'a'-c'b')(ad-bc) \;=\; 1 \qquad (3.8)$$

The analogue of (3.3) for right inverses, or indeed for left and for right zero-divisors, may be left to the reader. It is also possible to extend the argument of (3.3) to 3x3 matrices, although the joke is not nearly so funny. We shall give elsewhere [4] an inductive proof of (3.1) and (3.2) based on a proof of (1.1) due to Tom Laffey.

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FREE TOPOLOGICAL GROUPS

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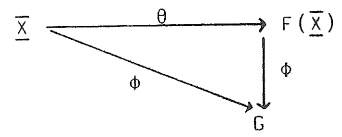
The purpose of this paper is to provide a brief expository sketch of [1].

If $\underline{X}$ is any set, the free group $F(\underline{X})$ is defined abstractly as follows: $F(\underline{X})$ is a group such that if G is any group and if $\phi : \underline{X} \to G$ is any map of $\underline{X}$ into G then there is a homomorphism $\Phi : F(\underline{X}) \to G$ so that the diagram below commutes:



(*)

The embedding θ is fixed and is independent of ϕ and of G.

The existence of $F(\underline{X})$ is assured by the construction described next.

A *word* is a finite sequence $x_1^{\varepsilon_1} x_2^{\varepsilon_2} \ldots x_n^{\varepsilon_n}$ in which x_i is an element of $\underline{X}$ and each $\varepsilon_i = \pm 1$. The product of two words $x_1^{\varepsilon_1} \ldots x_n^{\varepsilon_n}$ and $y_1^{\delta_1} \ldots y_m^{\delta_m}$ is the word $x_1^{\varepsilon_1} \ldots x_n^{\varepsilon_n} y_1^{\delta_1} \ldots y_m^{\delta_m}$. The collection W of all words is thus an associative semigroup. The subsemigroup S generated by all words of the form $x_1^{\varepsilon_1} \ldots x_n^{\varepsilon_n}$ in which $x_1 = x_2 = \ldots = x_n$ and

$$\sum_{i=1}^{n} \varepsilon_i = 0$$

leads to the quotient structure W/S, a group $F(\underline{X})$ for which $x_n^{-\varepsilon_n} \ldots x_1^{-\varepsilon_1}$ is a representative of the inverse of the element represented by $x_1^{\varepsilon_1} \ldots x_n^{\varepsilon_n}$.

If $\underline{X}$ is a topological space, the natural object corresponding to $F(\underline{X})$ is a topological group for which the same diagram (*) obtains and where θ is a fixed topological embedding, G is