ASYMPTOTIC BEHAVIOUR OF GRAVITATIONAL INSTANTONS ON $\mathbb{R}^4$

Niall Ó Murchadha

Instantons are objects which play an important rôle in the transition from a classical to a quantum model for many physical theories. An instanton is a solution of the 'euclidean' classical field equations, satisfying appropriate boundary conditions. "Euclidean" means that the signature of the spacetime is changed from (-, +, +, +) to (+, +, +, +) but the form of the field equations is left unchanged. In particular this means that the D'Alembertian

$$\Box = -\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

gets changed into a four-dimensional Laplacian

$$\nabla^2 = \frac{\partial^2}{\partial w^2} + \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

Thus there is an interest in gravitational instantons, asymptotically flat solutions to the Einstein equations, on Riemannian rather than pseudo-riemannian manifolds. Edward Witten [1], in a recent paper, has shown that no gravitational instantons exist on $\mathbb{R}^4$. His proof is very simple but assumes that the (non-existent) solution falls off like r^{-4} (where $r^2 = w^2 + x^2 + y^2 + z^2$). Witten argues that this is a reasonable assumption on the basis that the monopole in four dimensions falls off like r^{-2}, the dipole like r^{-3} and the quadrupole like r^{-4}. Since gravity is a quadrupole theory, then the solutions should fall off like r^{-4}. This argument cannot be trusted, because ordinary general relativity, in addition to the quadrupole term also contains a monopole term, the Newtonian gravitational potential.

Therefore, it would be more reasonable to assume that the gravitational instantons fall off like r^{-2}, rather than r^{-4}. In this article, I will show that if I assume that the instanton

vanishes at infinity, I can prove that it must fall off like r^{-4} and thus complete the Witten proof. This result will depend only on very simple properties of the four-dimensional Laplacian.

Near infinity, the gravitational field is weak and I can legitimately ignore the higher order non-linear terms by comparison with the linear terms. Thus the leading part of the gravitational field must satisfy the (euclidean) linear theory of gravity equations.

The structure of linearized gravity bears a strong resemblance to electromagnetism. In Maxwell's equations the field variable is the vector potential A_μ. The field equations are not hyperbolic, due to the gauge freedom (many different A_μs give the same physical effects). When, however, we reduce the gauge freedom by imposing the Lorentz gauge condition

$$\eta^{\mu\nu}A_{\mu,\nu} = 0 \qquad (1)$$

[η is the Minkowski metric diagram (-1, +1, +1, +1)]

then Maxwell's equations take the nice hyperbolic form

$$\Box A_\mu = 0 \qquad (2)$$

The Lorentz condition does not completely eliminate the gauge freedom. If we have a scalar ϕ satisfying

$$\Box\phi = 0 \qquad (3)$$

and a vector A_μ satisfying (1) and (2), then

$$A'_\mu = A_\mu + \phi_{,\mu} \qquad (4)$$

also satisfies (1) and (2).

The linear theory of gravity looks just like Maxwell's equations except that the field variable is a symmetric tensor $h_{\mu\nu}$ rather than a vector. Again, the field equations are not hyperbolic until we impose the gauge condition

$$\eta^{\alpha\beta}h_{\mu\alpha,\beta} = 0 \qquad (5)$$

The field equations now become

$$\Box h_{\mu\nu} = 0 \qquad (6)$$

Gauge condition (5) does not entirely eliminate the gauge freedom. If we have an $h_{\mu\nu}$ which satisfies (5) and (6), and a vector λ_μ which satisfies

$$\Box \lambda_\mu = 0 \qquad (7)$$

then

$$h'_{\mu\nu} = h_{\mu\nu} + \lambda_{\mu,\nu} + \lambda_{\nu,\mu} \qquad (8)$$

also satisfies (5) and (6).

The euclidean linear gravity equations look just like (5), (6), (7) and (8) except that the Minkowski metric is replaced by the euclidean metric $\delta_{\alpha\beta}$. The field equations are

$$\nabla^2 h_{\mu\nu} = 0 \qquad (9)$$

The gauge condition is

$$\delta^{\alpha\beta}h_{\mu\alpha,\beta} = 0 \qquad (10)$$

and the residual gauge freedom is represented by a vector satisfying

$$\nabla^2 \lambda_\mu = 0 \qquad (11)$$

If $h_{\mu\nu}$ falls off at infinity, one can always impose gauge condition (10), just as we can always use the Lorentz gauge in electromagnetism. Therefore we have from (9) that each component of $h_{\mu\nu}$ is a harmonic function of the four dimensional Laplacian. The leading (monopole) term of the Laplacian is $1/r^2$. There are four dipole terms, w/r^4, x/r^4, y/r^4, z/r^4, nine $1/r^4$ harmonic functions, sixteen of order $1/r^5$ and so on. These can be found by taking repeated derivatives of $1/r^2$, because of course any derivative of a harmonic function is a harmonic function.

If we have a solution to (9) which vanishes at infinity it must fall off at least as fast as r^{-2}. If it has an r^{-2} term it must be of the form

$$h_{\mu\nu} = \begin{Bmatrix} A/r^2, & B/r^2, & C/r^2, & D/r^2 \\ B/r^2, & E/r^2, & F/r^2, & G/r^2 \\ C/r^2, & F/r^2, & H/r^2, & K/r^2 \\ D/r^2, & G/r^2, & K/r^2, & L/r^2 \end{Bmatrix}$$

where (A,B,...,L) are ten arbitrary constants. But $h_{\mu\nu}$ must also satisfy the four divergence conditions of (10) as well. This means that each row of $h_{\mu\nu}$ must be divergence-free. Looking just at the first row we get

$$-2A\frac{w}{r^4} - 2B\frac{x}{r^4} - 2C\frac{y}{r^4} - 2D\frac{z}{r^4} = 0 \qquad (12)$$

The only solution to this is A = B = C = D = 0. The other three divergence equations require that all the others of the ten "arbitrary" constants must be zero. Thus there is no r^{-2} solution to (9) and (10).

A counting argument may be illuminating at this point. There are ten components of $h_{\mu\nu}$ and one harmonic function which gives ten arbitrary constant coefficients. The gauge condition (10) involves first derivatives of $h_{\mu\nu}$, so we get the four independent $1/r^3$ harmonic functions (see (12)). We have four divergence conditions and therefore sixteen conditions on the ten coefficients. The only solution is that they all vanish.

If $h_{\mu\nu}$ does not fall off like $1/r^2$, the next possible fall-off is $1/r^3$. In this case we can have that each of the ten components of $h_{\mu\nu}$ is a sum of the four $1/r^3$ harmonic functions. This gives us forty arbitrary constant coefficients. The first derivatives of the $1/r^3$ harmonic functions will give us the nine (linearly independent) $1/r^4$ harmonic functions. Thus the four divergence conditions will give us thirty-six conditions on the forty coefficients. This means that we have four linearly independent $1/r^3$ solutions to (9) and (10).

Of course, this needs that the thirty-six conditions be linearly independent. They are.

This is not the end of the road, however. We still have to account for the residual gauge freedom (11). Since $h_{\mu\nu}$ falls off like $1/r^3$, we seek solutions to (11) which fall off like $1/r^2$ (see (8)). The general $1/r^2$ solution to (11) is of the form

$$\lambda_\mu = (\alpha/r^2, \beta/r^2, \gamma/r^2, \delta/r^2)$$

with four arbitrary constants $(\alpha,\beta,\gamma,\delta)$. Thus we have four linearly independent pure gauge $1/r^3$ solutions to (9) and (10). These are the only solutions that are left after imposing the thirty-six conditions on the forty coefficients. They do not correspond to real solutions because they can be totally eliminated by a gauge transformation.

The next term to consider is $1/r^4$. Now each component of $h_{\mu\nu}$ can be a sum of the nine $1/r^4$ harmonic functions of the Laplacian giving ninety constant coefficients. Since there are sixteen $1/r^5$ harmonic functions, each of the four divergence conditions will give sixteen conditions, sixty-four in all. Therefore we have twenty-six independent solutions falling off like $1/r^4$ of equations (9) and (10). The residual gauge freedom is represented by a vector that falls off like $1/r^3$. Each component of the vector is a sum of the four $1/r^3$ harmonic functions. Thus sixteen of the twenty-six solutions of (9) and (10) are pure gauge, leaving ten independent solutions which cannot be eliminated by a gauge choice.

We see then that if the instanton falls off at infinity, it must fall off like r^{-4}, and so the Witten assumption is not only reasonable but correct, and his proof of the absence of any gravitational instanton goes through.

It is important to notice that the proof in this article is entirely local (although "local at infinity"). I do not

assume that the Einstein equations are satisfied everywhere, I only need that they are satisfied near infinity. Thus, the result here covers cases where one looks for solutions to the euclidean Einstein equations with sources; such solutions will exist, but they must fall off at infinity like $1/r^4$. The ten independent $1/r^4$ solutions that I find must have some interpretation as moments (quadrupole ?) of the sources.

I would like to stress that this proof of the non-existence of gravitational instantons holds only for instantons on manifolds with the topology of $\mathbb{R}^4$. There do exist instantons with $\mathbb{R}^3 \times S^1$ topology [2]. The absence of the $\mathbb{R}^4$ instanton is interpreted as meaning that cold flat space is stable, whereas the existence of $\mathbb{R}^3 \times S^1$ instantons shows that hot flat space is unstable (for example, via the spontaneous formation of a small black hole).

<u>Notes for the Initiated</u>:

(i) Function spaces: To make this whole scheme work I have to assume that $h_{\mu\nu}$ belongs to some weighted space, i.e. that if $h_{\mu\nu}$ falls off like $r^{-\alpha}$ then $h_{\mu\nu;}$ falls off like $r^{-(\alpha+1)}$ and so on. Weighted Sobolev spaces or weighted Holder spaces will do. This is used in several places, particularly when I ignore the non-linear terms in favour of the linear terms, and again when I claim that I can make the gauge choice (10). This involves solving an elliptic equation.

(ii) Gauge freedom: This arises from the fact that the exact theory is geometrical; thus I can make coordinate transformations at will. The linear theory inherits this, and the gauge transformation (8) is nothing more than the Lie derivative of the metric along a coordinate transformation. This is why all solutions to (9) and (10) which can be written in the form $\lambda_{\alpha,\beta} + \lambda_{\beta,\alpha}$ can be eliminated.

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References

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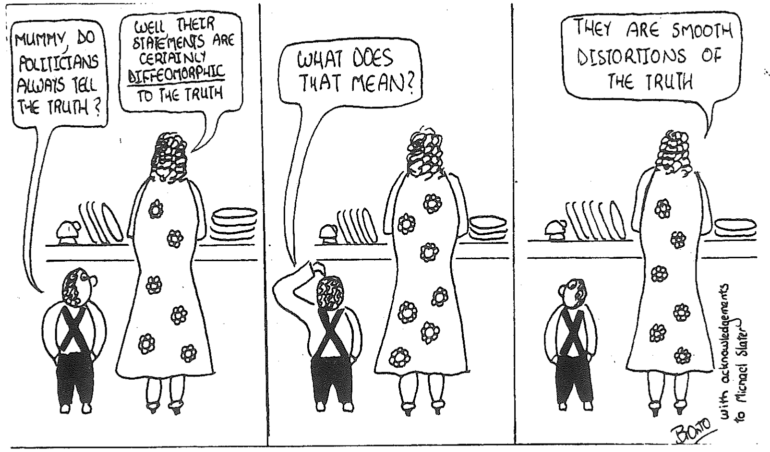
Experimental Physics Department,
University College,
Cork,
Ireland.



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KNOWING 'ABOUT' MATHEMATICS: A FOCUS ON TEACHING

John O'Donoghue

> *"Mathematical Education" may be seen then as an operational activity based on a number of areas of study with the analysis of the communication of mathematics as its objective.*
>
> (G.T. Wain)

INTRODUCTION

Since all of us have a good intuitive idea of what is meant by mathematical education it is acceptable to start by presenting a definition. The above definition may not suit everyone's tastes but then definitions rarely find universal acceptance. It is not my intention to argue a case for mathematical education as a discipline but rather to focus attention on some important aspects of mathematical education as an activity. This particular definition serves to focus attention on the *communication* of mathematics. All of us at some time or another have been concerned with this aspect of mathematics teaching as students, teachers, lecturers or professors. Many of us have resolved to improve matters given the opportunity. My particular concern has been to improve teacher preparation so that better mathematics teaching results in secondary schools.

The purpose of this paper is to draw attention to a neglected aspect of mathematics teaching at third level which is vitally important for future teachers of mathematics. A case is made for better treatment of this aspect, and finally an outline of an experimental course is given.