$$(1) \quad \lambda_k(n) \leq k^{2n}/\binom{nk}{n}$$

$$(2) \quad \theta_k \leq (k-1)^{k-1}/k^{k-2}$$

and their conjecture states that (2) is an equality for all k. (The positive solution of the van der Waerden conjecture yields $\theta_k \geq k/e$ here.)

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Mathematics Department,
University College,
Dublin.

SOME APPLICATIONS OF THE CLASSIFICATION OF FINITE SIMPLE GROUPS TO PERMUTATION GROUP THEORY [1]

Martin W. Liebeck

The classification of finite simple groups has made it possible to prove many new and striking results in the theory of finite permutation groups. We survey some of these results and describe some of the methods used in proving them. We also present a theorem on maximal subgroups of finite classical groups which is of use in extending the techniques.

(A) The Classification Theorem This states that any finite simple group is isomorphic to one of the following groups:

cyclic	Z_p	
alternating	A_n $(n \geq 5)$	
groups of Lie type	classical:	$PSL(n,q)$, $PSp(2m,q)$, $PSU(n,q)$, $P\Omega^{\pm}(n,q)$
groups of Lie type	Chevalley:	$G_2(q)$, $F_4(q)$, $E_6(q)$, $E_7(q)$, $E_8(q)$
	twisted:	${}^2B_2(q)$, ${}^2G_2(q)$, ${}^2F_4(q)$, ${}^3D_4(q)$, ${}^2E_6(q)$
26 sporadic groups		

See [5] for descriptions of the groups of Lie type.

(B) Some Recent Applications to Permutation Groups. As explained, for example, in Sections 2 and 3 of [2], at the heart of

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the theory of permutation groups lies the study of primitive permutation groups. Many old problems in this field have been solved using the classification theorem. Here are some examples.

THEOREM 1 ([2], Section 5). All finite 2-transitive groups are known. Any 6-transitive permutation group of finite degree n must be A_n or S_n.

THEOREM 2 ([3]). For almost all positive integers n the only primitive groups of degree n are A_n and S_n. More precisely, if $e(x) = |\{n \leq x|$ there is a primitive group G of degree n with $A_n \not\leq G\}|$ then $e(x) \sim 2x/\log x$.

THEOREM 3 (Sims' Conjecture:[4]). There is a function $f:\mathbb{N} \to \mathbb{N}$ such that if G is primitive on a finite set Ω and for $\alpha \in \Omega$, G_α has an orbit $(\neq\{\alpha\})$ of size d, then $|G_\alpha| < f(d)$.

THEOREM 4 ([9]). All primitive groups of degree kp, with p prime and k < p, are known.

THEOREM 5 ([8], Example 5). All primitive groups G of degree p^r with p prime, and G having no elementary abelian regular normal subgroup, can be classified.

THEOREM 6 ([2], Theorem 6.1). Let G be primitive of degree n. Then one of the following holds:

(i) G has an elementary abelian regular normal subgroup;

(ii) G is a known group;

(iii) $|G| < n^{10\log\log n}$.

(C) Methods of Reduction. The basic tool for reducing problems about general primitive groups to problems about primitive *simple* groups is the following theorem of O'Nan and Scott (see Theorem 4.1 of [2]; note that there is an error in the statement

IN [2] - possibility (ii) below is omitted).

O'Nan-Scott THEOREM 7. Let G be primitive of finite degree n on Ω and let N = soc G, the product of the minimal normal subgroups of G. Then $N = T_1 \times \ldots \times T_r$ with all $T_i \simeq T$, a fixed simple group, and one of the following holds:

(i) $T \simeq Z_p$, $N \simeq (Z_p)^r$ and $G \leq AGL(r,p)$;

(ii) T is nonabelian, $N_\alpha = 1$ $(\alpha \in \Omega)$ and $n = |T|^r$;

(iii) T is nonabelian and either

(a) *wreath action:* $T = \mathrm{soc}\, G_0$ for some primitive group G_0 of degree n_0 and $G \leq G_0 \mathrm{\ wr\ } S_r$, with $n = n_0^r$, or

(b) *diagonal action:* $N_\alpha = D_1 \times \ldots \times D_m$ where $r = km$ for some $k > 1$, D_i is a diagonal subgroup of $T_{(i-1)k+1} \times \ldots \times T_{ik}$ and $n = |T|^{r-m}$.

As an example of the use of the O'Nan-Scott Theorem we consider Theorem 4 above, and prove

PROPOSITION 8. If G is primitive of degree kp with p prime and k < p then T = soc G is simple (so that $T \lhd G \leq \mathrm{Aut}\, T$).

Proof. Let $N = \mathrm{soc}\, G = T_1 \times \ldots \times T_r$ as above. If case (i) of the O'Nan-Scott Theorem occurs then k = 1 and $G \leq AGL(1,p)$. If (ii) holds then $kp = |T|^r$ with T a nonabelian simple group, which is impossible. Similarly case (iii)(b) cannot hold. Finally, in case (iii)(a) we must have r = 1 and $n_0 = kp$, so that T = soc G is simple.

The O'Nan-Scott Theorem thus focuses attention of primitive permutation representations of finite simple groups. Such representations are determined by the conjugacy classes of maximal subgroups.

(D) Maximal Subgroups. Let T be a finite simple group and G a group with $T \triangleleft G \leq \mathrm{Aut}\, T$ (i.e. $T = \mathrm{soc}\, G$). We describe some recent progress in determining the maximal subgroups of such groups G.

(1) *T sporadic* It seems certain that the maximal subgroups of G can be determined explicitly in this case; in fact this has already been accomplished for 14 of the 26 groups.

(2) *T exceptional of Lie type* Here the determination of the maximal subgroups of G requires special techniques for each type of group. It seems possible that a complete determination will eventually be achieved; this has been done for several of the cases of low rank.

(3) *T alternating* In this case much can be said. Let $T = A_c$ and suppose that $c > 6$, so that G is A_c or S_c. Let G act naturally on $C = \{1,\dots,c\}$ and let H be a maximal subgroup of G. Then one of the following occurs:

- (i) H is intransitive on C: then $H = (S_k \times S_{n-k}) \cap G$ for some k with $1 \leq k \leq n-1$;
- (ii) H is transitive and imprimitive on C: then H permutes b blocks of size a, where $ab = c$ and $a > 1$, $b > 1$, so $H = (S_a \text{ wr } S_b) \cap G$;
- (iii) H is primitive on C: in this case our results on primitive groups (such as O'Nan-Scott or Theorem 6, for example) apply to H.

Finally we discuss the case

(4) *T classical* I have recently obtained (in [10]) the following result on the orders of maximal subgroups of G in this case:

THEOREM 9 ([10]. Let G_0 be a simple classical group with natural projective module V of dimension n over GF(q) (for example $G_0 = PSL(n,q)$, etc.), and let G be a group such that

$G_0 \triangleleft G \leq \mathrm{Aut}\, G_0$. If H is a maximal subgroup of G then either

- (I) H is a known group, and $H \cap G_0$ has well-described (projective) action on V, or
- (II) $|H| < q^{3n}$.

Note. $|G|$ is roughly q^{n^2} (if $G_0 = PSL(n,q)$) or $q^{\frac{1}{2}n^2}$(otherwise), so for large n the maximal subgroups H under (II) are of very small order. Theorem 9 improves substantially the results of [6] and [7] (note, however, that these were obtained without the use of the classification theorem).

The known groups under (I) comprise the following:

(a) stabilisers of
- (i) certain subspaces of V,
- (ii) certain decompositions of V as a direct sum or tensor product of subspaces,
- (iii) fields $F_1 \subset GF(q)$ or $F_0 \supset GF(q)$, of prime index.

(b) classical groups of dimension n over GF(q) contained in G;

(c) A_c or S_c in a representation of smallest degree over GF(q) ($n = c-1$ or $c-2$).

The proof of Theorem 9 uses the following fundamental structure theorem of Aschbacher:

THEOREM 10 ([11]). Let G_0, G and H be as in the statement of Theorem 9. Then either

- (A) H is known, or
- (B) there is a nonabelian finite simple group S with $S \triangleleft H \leq \mathrm{Aut}\, S$, and the representation of the covering group $\overline{S}$ of S on V is absolutely irreducible.

The groups under (A) are as in (a) and (b) above, plus a few extra subgroups. So to prove Theorem 9 we must essentially show that if H satisfies (B) of Theorem 10 then either

$|H| < q^{3n}$ or H is A_c or S_c as in (c) above. This is achieved by obtaining lower bounds for degrees of absolutely irreducible modular representations of groups H satisfying (B) of Theorem 10.

(E) Some Deductions. As a corollary to Theorem 9 we obviously have:

COROLLARY 11. If G is a classical group of dimension n over $GF(q)$ and G acts primitively on a set Ω then either

(I) G^{Ω} is 'known' (i.e. it is the action of G on the cosets of a known subgroup), or

(II) $|\Omega| > |G|/q^{3n}$.

In order to demonstrate an application of this we return to our consideration of Theorem 4. Let G be primitive on of degree kp with p prime and $k < p$. By Proposition 8, $T = \operatorname{soc} G$ is simple and, excluding the case $G \leq AGL(1,p)$ as well known, T is also nonabelian. Clearly $|G:G_\alpha| < r^2$ where r is the largest prime divisor of $|T|$ $(\alpha \in \Omega)$. When T is alternating, exceptional or sporadic it is possible to determine the possibilities for G_α and hence for G^{Ω}. And if T is classical then Corollary 11 gives the possibilities for G^{Ω}; for example, if $T = PSL(n,q)$ then $|\Omega| = |G:G_\alpha| < (q^n-1)^2$ and so, provided $n \geq 6$, G^{Ω} must fall under (I) of Corollary 11.

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Downing College,
Cambridge,
England.