

"THE CONVOLUTION PRODUCT AND SOME APPLICATIONS"

By W. Kecs

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The problem of multiplying distributions has occupied mathematicians for decades. In many ways, this is an excellent problem, leading to some intriguing mathematics which often has genuine use in other areas. Probably, the most successful multiplication is the convolution product, and properties of this product and its uses in engineering are the subject of Kecs' book.

Given two  $L^1$  (i.e. integrable) functions on  $R^n$ ,  $f$  and  $g$ , the convolution  $f * g$  is defined by  $f * g(x) = \int f(y)g(x-y)dy$ . An application of Fubini's theorem shows that  $f * g = g * f$  is again in  $L^1$ . Now, every  $L^1$  function  $h$  defines a continuous linear form  $T_h$  or distribution, on the space of infinitely differentiable functions on  $R^n$  with compact support, via the mapping  $T_h(\phi) = \int \phi(x)h(x)dx$ . (Of course, a definition of a topology is needed to entitle us to say that  $T_h$  is continuous, but we will omit this.) Applying this to  $h = f * g$ , we can see that  $T_{f * g}(\phi) = T_f(T_g(\phi))$  where  $\check{g}(x) = g(-x)$ . In more suggestive notation, we can write  $T_{f * g}(\phi) = T_{f_x}(S_{g_y}(\phi(x+y)))$ . This, hopefully, motivates the following definition.

For distributions  $S, T$ , let  $S * T$  be the distribution defined by  $S * T(\phi) = S_x(T_y(\phi(x+y)))$ . (In fact,  $S * T$  cannot be defined for all pairs  $(S, T)$ , just as  $f * g$  cannot be defined for all pairs of functions  $(f, g)$ , and it is of much interest to find when this convolution does make sense. However, it is defined for "very many" pairs  $(S, T)$ , and we will only consider these.) The most useful, and straightforward, properties of convolutions are that  $S * T = T * S$  and that  $\delta * T = T$  for all  $T$ , where  $\delta$  is the Dirac distribution, which takes a function  $\phi$  to  $\phi(0)$ . Also, if  $P(D)$  is a differential polynomial, we can define  $P(D)T$  to be the distribution given by  $P(D)T(\phi) = T(P(-D)\phi)$ . Then,

we can verify that  $P(D)S * T = (P(D)S) * T$ . A distribution  $E$  is called a fundamental solution for the differential polynomial  $P(D)$  if  $P(D)E = \delta$ . Fundamental solutions are major building blocks in the theory of differential equations, because if  $T$  is a distribution, then  $P(D)(E * T) = (P(D)E) * T = \delta * T = T$ , and so  $E * T$  is a solution to the problem  $P(D)X = T$  (always assuming that all convolution products make sense). This observation is, in a general way, the motivating force behind the interest in convolution products

The book under review is the second in the Eastern Europe Reidel series, Mathematics and its Applications. As the editor of the series states, it is "hoped to contribute something towards better communications among the practitioners in diversified fields", by making available to western audiences monographs emanating from the Soviet Union, Eastern Europe and Japan. It is unfortunate that this worthwhile objective has been thwarted by Reidel which has priced this volume (\$69.50, for a 330 page book, printed in Romania) well beyond the reach of much of its intended audience. This is a pity, since Kecs' book deserves a larger audience than it will receive.

Put briefly, the book introduces distributions and operations on distributions in the first three chapters, with the aim (Chapters 4 and 5) of describing applications of convolution equations in engineering. Chapter 1 is an introduction to distributions, together with basic underlying definitions from functional analysis. Chapters 2 and 3 deal with convolution products and Fourier and Laplace transforms. Much of the material here is completely standard, with the usual presentation of the basic properties of convolutions and transforms, and the relations between them. Several interesting features of these chapters do stand out, such as an exposition of the author's work on the partial convolution product and a discussion of Mikusinski's operational calculus with several good examples. The idea for this operational calculus is as follows. We consider  $C(R_+)$ , the space of continuous complex valued functions on  $R_+$ , defining a product in the following manner:

for  $f, g \in C(\mathbb{R}^+)$ ,  $f \otimes g$  is the function in  $C(\mathbb{R}^+)$  given by

$$f \otimes g(x) = \int_0^x f(t)g(x-t)dt.$$

By a theorem of Titchmarsh,  $C(\mathbb{R}^+)$  is an integral domain with this product, and Mikusinski was led to the quotient algebra  $Q(\mathbb{R}^+)$ . It turns out that  $Q(\mathbb{R}^+)$  contains the usual differential and integral operators, as well as many distributions. In particular, the Dirac  $\delta$  and the distributions are in  $Q(\mathbb{R}^+)$ , where  $s(f) = f' - f'(0)$  for  $C^1$  functions  $f$ . As a consequence, one can apply Laplace transform techniques, using  $s$ , to solve differential equations with constant coefficients, integral equations, etc.

The main body of the book is the last two chapters. Chapter 4 deals with convolution equations in spaces of distributions. It is here that the problem of finding fundamental solutions for various operators is addressed, with Kec examining the role of special spaces of distributions. The Cauchy initial value problem is considered, in a number of settings, and applications are made to the wave equation, heat equation, etc. Finally, in Chapter 5, the author applies the methods of the previous chapter to solve differential equations arising in electrical and mechanical engineering and in viscoelasticity.

The text is readable, although the English is not always idiomatic. It is evident that the translator has little mathematical experience. Thus, for example, we find ourselves considering the "body of real or complex numbers" and the open unit "bubble" of a normed space. A more substantive criticism can be made of the author's approach, from the mathematician's point of view. Routine results, such as properties of the convolution, are usually proved in full detail. On the other hand, the discussion is often incomplete in terms of (mathematically) more interesting results. For example, no attempt is made to discuss topological properties of the space of distributions, beyond some mention of the weak-\* topology. No mention is made of such beautiful results as the Titchmarsh-Lions

theorem on the support of convolutions, the Paley-Wiener theorems, etc. Indeed, the relation of analytic function theory to this subject seems to have been largely ignored. Unlike Schwartz's "Mathematics for the Physical Sciences", this book (which might be considered as "convolution equations for the engineering sciences") has no exercises.

These doubts having been raised, it must in fairness be mentioned that it seems remarkable that, as Kec shows, one can get many, apparently non-trivial, results in engineering mathematics using only the material developed in this volume. Thus, it may well be that the book serves the very useful purpose of introducing engineers to this fruitful area of mathematics.

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#### "MATHEMATICAL SNAPSHOTS"

By *H. Steinhaus*

Published by *Oxford University Press*, (311 pp.). Stg £5.95.

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By *Morris Kline, Professor Emeritus of Mathematics at the Courant Institute of Mathematical Sciences, New York University.*

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The book must be distinguished from numerous books on riddles, puzzles and paradoxes. Such books may be amusing but in almost all cases the mathematical content is minor