**Irish Math. Soc. Bulletin**
Number 93, Summer 2024, 52–54
ISSN 0791-5578

# Teo Banica: Introduction to Quantum Groups, Springer, 2023. ISBN:978-3-031-23816-1, EUR 50.28, 425 pp.

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For each non-zero $q \in [-1, 1]$, Woronowicz [9] defined a universal unital C*-algebra $C(\mathrm{SU}_q(2))$ generated by elements $\alpha$, $\gamma$ subject to a number of relations, including $\alpha\gamma = q\gamma\alpha$. The algebra admits a *comultiplication* into the minimal tensor product

$$\Delta : C(\mathrm{SU}_q(2)) \to C(\mathrm{SU}_q(2)) \otimes C(\mathrm{SU}_q(2)),$$

and the crucial point is that at $q = 1$, the algebra is commutative, $C(\mathrm{SU}_1(2)) \cong C(\mathrm{SU}(2))$, the algebra of continuous functions on the matrix group SU(2), and, via $C(\mathrm{SU}(2)) \otimes C(\mathrm{SU}(2)) \cong C(\mathrm{SU}(2) \times \mathrm{SU}(2))$, the comultiplication is a transpose

$$\Delta f(u, v) = f(uv) \qquad (u, v \in \mathrm{SU}(2),\ f \in C(\mathrm{SU}(2))).$$

Thus $C(\mathrm{SU}_q(2))$ is said to be a $q$-deformation of $C(\mathrm{SU}(2))$, a quantum SU(2). This exhibits the very basic philosophy of quantum groups: they are given by an algebra $A$ that satisfies some specific axioms such that whenever an algebra satisfying those same axioms is commutative, it is an algebra of functions on a group, and the comultiplication is the transpose of the group law. When such an algebra is non-commutative, it can be viewed as an algebra of functions on a quantum group, $\mathbb{G}$, and the algebra written $A = C(\mathbb{G})$, but this quantum group $\mathbb{G}$ is a so-called *virtual object*, i.e. it is not a set.

The quantum groups of Drinfeld and Jimbo [3, 4] are also $q$-deformations, and while $\mathrm{SU}_q(2)$ fits naturally into their framework, a 1987 paper of Woronowicz [10] split the field into two schools Woronowicz vs Drinfeld–Jimbo, and the book author is firmly of the Woronowicz school (often called the C*-algebraic approach). Woronowicz proved foundational theorems for compact quantum groups, but it was the 1990s examples of Wang [7, 8] of quantum $\mathrm{U}_N$, quantum $\mathrm{O}_N$, and quantum $\mathrm{S}_N$ that gave a real impetus to the field.

In this book, Banica concentrates largely on quantum subgroups of the quantum unitary group $\mathrm{U}_N^+$ and the quantum orthogonal group $\mathrm{O}_N^+$. These compact quantum groups are *free* or *liberated* versions of the classical orthogonal and unitary groups, not deformations in the sense of $\mathrm{SU}_q(2)$. Consider an invertible matrix $u \in M_N(A)$ with entries in a unital C*-algebra $A$. If its entries are self-adjoint $u_{ij}^* = u_{ij}$, and the inverse $u^{-1}$ is given by the transpose $u^T$, it is said to be a quantum orthogonal matrix. The algebra of continuous functions on $\mathrm{O}_N$ may be realised as a universal commutative C*-algebra

$$C(\mathrm{O}_N) = \mathrm{C}^*_{\mathrm{comm}}(u_{ij} \,|\, u \text{ an } N \times N \text{ quantum orthogonal matrix}).$$

The generators $u_{ij} \in C(\mathrm{O}_N)$ are coordinate functions $u_{ij}(g) = g_{ij}$. If the algebra is *liberated* from commutativity, the relations still define a unital C*-algebra

$$A_o(N) = \mathrm{C}^*(u_{ij} \,|\, u \text{ an } N \times N \text{ quantum orthogonal matrix}),$$

Received on 07-12-2023; revised 15-12-2023.
DOI:10.33232/BIMS.0093.52.54.

and this algebra admits a $*$-homomorphism $\Delta : A_o(N) \to A_o(N) \otimes A_o(N)$

$$\Delta(u_{ij}) = \sum_{k=1}^{N} u_{ik} \otimes u_{kj},$$

giving $(A_o(N), u)$ the structure of a compact quantum group $A_o(N) = C(\mathrm{O}_N^+)$.

In the 25 years since Wang's examples, Banica and co-authors have studied these quantum groups over the course of 99 papers, the highlight of which might be the seminal work of Banica and Speicher [1]. This 25-year study has culminated in Banica beginning to write a series of tomes, of which the one under review is the first to be published. It is structured in four parts as follows.

*Part I: Quantum Groups* The preliminaries begin with the definition of a Hilbert space, and then neatly summarises the basics of C*-algebras, including the important examples of the group C*-algebras $\mathrm{C}^*(G)$ and their reduced versions $\mathrm{C_r}(G)$. In my opinion, the existing literature on compact quantum groups does not go far enough in discussing axioms. In contrast, Banica spends a good ten pages carefully teasing out why compact quantum groups are defined as they are. One of the reasons that compact quantum groups have been so well-studied is that they have a very rich representation theory, which extends the theories of Peter–Weyl and Tannaka–Krien. This book gives a comprehensive treatment of the representation theory, and presents the relatively recent, more concrete version of Tannaka duality, that is due to Malacarne [6].

*Part II: Quantum Rotations* Where $u \in M_N(C(\mathbb{G}))$ is the so-called fundamental representation, the intertwiner spaces $\mathrm{hom}(u^{\otimes k}, u^{\otimes l})$, which form a tensor category, are crucial in the representation theory of compact quantum groups. In the classical setting, for $\mathrm{U}_N$ and $\mathrm{O}_N$, Brauer's work shows that the elements of $\mathrm{hom}(u^{\otimes k}, u^{\otimes l})$ are linear combinations of intertwiners $T_\pi$ parameterised by coloured partitions in a partition category, $\pi \in D(k, l)$. In fact, any such partition category $D$ yields a tensor category $C_D$, which we learned in Part I yields a compact quantum group, a so-called *easy quantum group* (Theorem 2.7). If the partition category is crossing (non-planar partitions), then $fg = gf$ for all $f, g \in C(\mathbb{G})$, the algebra of functions is commutative, and the associated compact quantum group is classical. On the other hand, if the partition category is non-crossing (planar partitions), the algebra of functions is non-commutative, and the associated compact quantum group is genuinely quantum. That intertwiners $T_\pi$, $T_\sigma$ parameterised with partitions $\pi$, $\sigma \in D$ respect the tensor category structure of $C_D$ is given in terms of a diagrammatic calculus (e.g. horizontal and vertical concatenation,"upside-down turning", "the semi-circle", etc.), but unfortunately there are no pictures accompanying these descriptions. Of particular interest here is the study in relation to free probability. Where $u \in M_N(C(\mathrm{O}_N^+))$ is the fundamental representation of the quantum orthogonal group, and $\int_{\mathrm{O}_N^+} : C(\mathrm{O}_N^+) \to \mathbb{C}$ the Haar state, the moments

$$\int_{\mathrm{O}_N^+} \mathrm{tr}(u)^k = C_k,$$

the Catalan numbers, and thus the law of the 'main character' $\mathrm{tr}(u)$ with respect to the Haar state is the famous semi-circle law of Wigner.

*Part III: Quantum Permutations* Banica defines the quantum permutation groups $\mathrm{S}_N^+$ as discovered by Wang in the 1990s [8], which, in the case of finite sets, answered a famous question of Alain Connes: "*What is the quantum automorphism group of a space?*". Of particular importance in the study of quantum permutation groups was the author's development and study of the quantum automorphism group $G^+(X)$ of a finite graph $X$, which has since met applications in quantum information [5]. This is a very active field, with a long-standing problem being settled as recently as November 2023.

However, while this book introduces quantum automorphism groups, it concentrates mostly on the relationship between $S_N^+$ and other easy compact quantum groups (there is a tome focussed on quantum permutation groups in production).

*Part IV: Advanced Topics* In Part I, the author chose 'lighter' axioms

> *"...with the aim of focussing on what is beautiful and essential..."*

Firstly, the author assumes that the antipode $S : C(\mathbb{G}) \to C(\mathbb{G})$ is a *-antihomomorphism. This assumption, stronger than Woronowicz's, forces the algebras of functions to be of Kac type, satisfying $S^2 = \text{id}$ (so that $\text{SU}_q(2)$ falls outside the author's formalism). Secondly, algebras of continuous functions $C(\mathbb{G})$ have a dense Hopf*-algebra $\mathcal{O}(\mathbb{G})$, but in general this algebra admits more than one completion to a C*-algebra. Banica chooses to work in one particular completion (the universal one): the counit $\varepsilon : \mathcal{O}(\mathbb{G}) \to \mathbb{C}$ extends to this completion, but on the other hand the Haar state is not necessarily faithful. For certain applications, for example quantum automorphism groups of finite graphs, what completion (if any) is used is often irrelevant. Part IV looks at this (and other assorted topics) in more detail (with our own G.J. Murphy involved in the treatment in the full formalism [2]).

This book has fantastic character, is out to inspire, well-written, and is full of what the author calls "philosophy". The author claims the presentation is elementary: "*a standard first year, graduate level textbook*". It is an excellent reference for many facets of the theory, but is not encyclopaedic, with sometimes only brief proofs, and with the intentional avoidance of getting stuck in technical mud. The locally compact case is not treated (despite the fact that many of the latest results in the compact case are actually proven in this more technically demanding setting). The book has very interesting exercises: a mix of calculations (for self-study), and some much more open-ended problems that would be ideal for group discussion and development (and perhaps undergraduate projects).

This book is for anyone with an interest or use for compact quantum groups, but the breadth of the exposition probably means that there is something interesting in there for everyone.

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