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Some Easy Inequalities for a Triangle

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Abstract. We present some results about inequalities in a triangle which we have not been able to find elsewhere.

Even if a mathematical result is easy to prove, it still deserves to be recorded, if it has not appeared previously in the literature. In this note we present some pretty results about inequalities in a triangle which we have not been able to find elsewhere. Our notation is standard — ABC is a triangle with side-lengths a, b and c, with $2s = a + b + c$; R is its circumradius, r is its inradius and Δ is its area. We note that

$$16\Delta^2 = (a + b + c)(a + b - c)(b + c - a)(c + a - b),$$

an adaptation of Heron's formula. We need the following well-known preliminary results (see [1], for example).

Lemma 1.

$$4R = \frac{abc}{\Delta} \text{ and } r = \frac{\Delta}{s}.$$

Lemma 2 (Euler 1767). *$R \geq 2r$, with equality if and only if the triangle is equilateral.*

Theorem 1.

$$R \geq \sqrt{\frac{abc}{a+b+c}} \geq 2r$$

Proof. By Lemma 1, $4Rrs = (abc/\Delta)(\Delta/s)s = abc$, so $2Rr = (abc)/(a + b + c)$.

By Lemma 2, this becomes $(abc)/(a + b + c) \geq 4r^2$, so $\sqrt{(abc)/(a + b + c)} \geq 2r$, as claimed.

Similarly, $2Rr \leq R^2$ and so $R^2 \geq (abc)/(a + b + c)$ and $R \geq \sqrt{(abc)/(a + b + c)}$.

Thus $R \geq \sqrt{(abc)/(a + b + c)} \geq 2r$, with equality if and only if the triangle is equilateral. □

Theorem 2. *$(abc)(a + b + c) \geq 16\Delta^2$.*

Proof. $R \geq 2r$ becomes

$$\frac{abc}{4\Delta} \geq \frac{2\Delta}{s} = \frac{4\Delta}{(a+b+c)}.$$

Thus $(abc)(a + b + c) \geq 16\Delta^2$. Again, by Lemma 2, we have equality if and only if the triangle is equilateral. □

Theorem 3. *$abc \geq (a + b - c)(b + c - a)(c + a - b)$.*

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Proof. By Theorem 2, $(abc)(a+b+c) \geq 16\Delta^2 = (a+b+c)(a+b-c)(b+c-a)(c+a-b)$. Since $a+b+c$ is non-zero, we may cancel it to get $(abc) \geq (a+b-c)(b+c-a)(c+a-b)$. □

Theorem 4. $(a+b+c)^3 \geq 27(a+b-c)(b+c-a)(c+a-b)$.

Proof. By the arithmetic mean/geometric mean inequality, we have $(a+b+c)^3 \geq 27abc$ which, by Theorem 3, is at least $27(a+b-c)(b+c-a)(c+a-b)$, and the result follows. Clearly, we have equality if and only if $a=b=c$. □

Topic For Investigation: What is the range of values of $(abc)/(a+b+c)$ if a, b and c are *positive integers* satisfying all three of the triangle inequalities?

References

[1] H.S. Hall and S.R. Knight. Elementary Trigonometry. Macmillan, London, 1955.

Des MacHale is Emeritus Professor of Mathematics at University College Cork where he taught for forty years. His main interests are in finite groups and rings, but he also dabbles in Number Theory, Euclidean Geometry, Trigonometric Inequalities, Combinatorial Geometry and Problem Posing and Solving. He has written several biographical books on George Boole but some would say his magnum opus is *Comic Sections Plus, the Book of Mathematical Jokes, Humour, Wit and Wisdom*, cf. `https://www.logicpress.ie/authors/machale`.

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