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The Sieve of Eratosthenes and a Partition of the Natural Numbers

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Abstract. The sieve of Eratosthenes is a method for finding all the prime numbers less than some maximum value M by repeatedly removing multiples of the smallest remaining prime until no composite numbers less than or equal to M remain. The sieve provides a means of partitioning the natural numbers. We examine this partition and derive an expression for the densities of the constituent "Eratosthenes sets". The densities must sum to unity, yielding an interesting result, equation (14), that may be new.

1. The Sieve of Eratosthenes

The *primorial*, P_K — often denoted $K\#$ — is defined to be the product of the first K primes:

$$P_K = \prod_{k=1}^{K} p_k\,.$$

The sequence of primorials is $\{2, 6, 30, 210, 2310, \dots\}$ and the terms of the sequence grow as K^K. It is convenient to set $M = P_K$. The algorithm of Eratosthenes goes as follows: starting from the set $I_M = \{1, 2, 3, \dots, M\}$,

- eliminate all multiples of 2 greater than 2;
- eliminate all remaining multiples of 3 greater than 3;
- eliminate all remaining multiples of 5 greater than 5;
- ... ...
- eliminate all remaining multiples of p_K greater than p_K.

All that remains is the set of the first m prime numbers, $\{2, 3, 5, \dots, p_m\}$, where p_m is the largest prime not exceeding $M = P_K$.

For all $k \in \mathbb{N}$, let D_k be the set of numbers in $\mathbb{N}$ that are divisible by p_k. For $k = 1, 2, \dots, K$, we define the set $D_{k,M} = D_k \cap I_M$ to be the set of numbers in I_M that are divisible by p_k. Thus, $D_{1,M}$ is the set of even numbers up to M, $D_{2,M}$ the multiples of 3 up to M, and so on.

The k-th "Eratosthenes set", E_k, is the set containing p_k together with all the numbers removed at stage k. Thus, E_1 is the set of all multiples of 2, that is, all the even numbers; E_2 is the set of odd multiples of 3; E_3 is the set of multiples of 5 not divisible by 2 or 3; E_4 is the set of multiples of 7 not divisible by 2, 3 or 5; and so on.

The Eratosthenes set E_k may be defined symbolically:

$$E_k = \{n \in \mathbb{N} : (p_k \mid n) \wedge (p_\ell \nmid n \text{ for } \ell < k)\}\,.$$

Some initial values of E_k are shown in Table 1.

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TABLE 1. Arrangement of the natural numbers as multiples of the prime numbers in sequence. The k-th row contains the "Eratosthenes set" E_k.

2	4	6	8	10	12	14	16	18	20	22	24	26	28	30	...
3	9	15	21	27	33	39	45	51	57	63	69	81	87	93	...
5	25	35	55	65	85	95	115	125	145	155	175	185	205	215	...
7	49	77	91	119	133	161	203	217	259	287	301	329	343	371	...
11	121	143	187	209	253	319	341	407	451	473	517	583	649	671	...
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋱

We denote by $E_{k,M}$ the set $E_k \cap I_M$. It is the set containing all multiples of p_k up to M that are *not multiples of any smaller prime*. We see immediately that $E_{1,M} = D_{1,M}$, that $E_{2,M} = D_{2,M} \setminus D_{1,M} = D_{2,M} \cap D^{\complement}_{1,M}$ and, more generally, that

$$E_{k,M} = D_{k,M} \setminus (D_{1,M} \cup D_{2,M} \cup \cdots \cup D_{k-1,M}) = D_{k,M} \cap (D_{1,M} \cup D_{2,M} \cup \cdots \cup D_{k-1,M})^{\complement} .$$

Using De Morgan's law, we may write

$$E_{k,M} = D_{k,M} \cap (D^{\complement}_{1,M} \cap D^{\complement}_{2,M} \cap \cdots \cap D^{\complement}_{k-1,M}) . \tag{1}$$

Since all primes p_k for $k \leq K$ divide M, the sizes of the D-sets are known: $|D_{k,M}| = M/p_k$ and so $|D^{\complement}_{j,M}| = M - M/p_j = M(1 - 1/p_j)$.

The Inclusion-Exclusion Principle. The inclusion-exclusion principle provides a valuable means of calculating the sizes of unions of sets [1] We denote the cardinality of a finite set A by $|A|$. The size of the union of two finite sets is

$$|A \cup B| = |A| + |B| - |A \cap B| , \tag{2}$$

where the intersection term prevents double counting. For the union of three sets,

$$|A \cup B \cup C| = (|A| + |B| + |C|) - (|A \cap B| + |A \cap C| + |B \cap C|) + |A \cap B \cap C| . \tag{3}$$

This idea can be generalised using the inclusion-exclusion principle to give the magnitude of the union of n finite sets:

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{i=1}^{n} |A_i| - \sum_{1 \leqslant i < j \leqslant n} |A_i \cap A_j| + \sum_{1 \leqslant i < j < k \leqslant n} |A_i \cap A_j \cap A_k| - \cdots + (-1)^{n+1} |A_1 \cap \cdots \cap A_n| . \tag{4}$$

Thus, the size of the union of sets is expressed as a combination of sizes of intersections.

Density. We define the density of a set $A \subseteq I_M$ (relative to M) to be $\rho(A) = |A|/M$. Then $\rho(D_{k,M}) = 1/p_k$ and $\rho(D^{\complement}_{j,M}) = (1 - 1/p_j)$. We note that division of equations (2)–(4) by M converts the cardinalities to densities. Thus, for example, (2) becomes

$$\rho(A \cup B) = \rho(A) + \rho(B) - \rho(A \cap B) .$$

Clearly, density is additive for disjoint sets. Thus,

$$\rho(D_{k,M}) = \rho(D_{k,M} \cap (D_{\ell,M} \uplus D^{\complement}_{\ell,M})) = \rho(D_{k,M} \cap D_{\ell,M}) + \rho(D_{k,M} \cap D^{\complement}_{\ell,M})$$

and, as p_k and p_ℓ are coprime, $\rho(D_{k,M} \cap D_{\ell,M}) = 1/p_k p_\ell$ and $\rho(D_{k,M} \cap D^{\complement}_{\ell,M}) = (p_\ell - 1)/p_k p_\ell$, so that

$$\rho(D_{k,M} \cap D_{\ell,M}) = \rho(D_{k,M})\rho(D_{\ell,M}) \quad \text{and} \quad \rho(D_{k,M} \cap D^{\complement}_{\ell,M}) = \rho(D_{k,M})\rho(D^{\complement}_{\ell,M}) \tag{5}$$

Moreover,

$$\begin{aligned}\rho(D^{\complement}_{k,M}\cap D^{\complement}_{\ell,M}) &= \rho((D_{k,M}\cup D_{\ell,M})^{\complement}) = 1-\rho(D_{k,M}\cup D_{\ell,M})\\ &= 1-[\rho(D_{k,M})+\rho(D_{\ell,M})-\rho(D_{k,M}\cap D_{\ell,M})]\\ &= 1-\left(\frac{1}{p_k}+\frac{1}{p_\ell}\right)+\frac{1}{p_k p_\ell} = \frac{p_k-1}{p_k}\frac{p_\ell-1}{p_\ell}\\ &= \rho(D^{\complement}_{k,M})\rho(D^{\complement}_{\ell,M})\,. \end{aligned} \tag{6}$$

By means of the inclusion-exclusion principle, we easily extend the product relationships (5) and (6) to show that the density of the set $E_{k,M}$ in (1) is the product of the densities of the component sets on the right side:

$$\rho(E_{k,M}) = \rho(D_{k,M})\rho(D^{\complement}_{1,M})\rho(D^{\complement}_{2,M})\dots\rho(D^{\complement}_{k-1,M})\,. \tag{7}$$

Using explicit expressions for the terms on the right, the density of the set $E_{k,N}$ is

$$\rho(E_{k,M}) = \frac{1}{p_k}\frac{(p_1-1)}{p_1}\frac{(p_2-1)}{p_2}\;\dots\;\frac{(p_{k-1}-1)}{p_{k-1}} = \frac{1}{P_k}\prod_{j=1}^{k-1}(p_j-1)\,, \tag{8}$$

where $P_k = p_1 p_2 \dots p_k$. We observe that the numbers $\rho_{k,M} := \rho(E_{k,M})$ are generated by a recurrence relation

$$\rho_{k+1,M} = \left(\frac{p_k-1}{p_{k+1}}\right)\rho_{k,M}\,, \tag{9}$$

with initial value $\rho_{1,M} = \frac{1}{2}$. This enables us to compute the sequence $\{\rho_{k,M}\}$. The first eight density values are given in Table 2.

TABLE 2. Density of the Eratosthenes sets E_k for $k \le 8$.

k	1	2	3	4	5	6	7	8	$\dots$
p_k	2	3	5	7	11	13	17	19	$\dots$
P_k	2	6	30	210	2310	30,030	510,510	9,699,690	$\dots$
ρ_k	$\frac{1}{2}$	$\frac{1}{6}$	$\frac{1}{15}$	$\frac{4}{105}$	$\frac{8}{385}$	$\frac{16}{1001}$	$\frac{192}{17{,}017}$	$\frac{3072}{323{,}323}$	$\dots$

2. Passage from I_N to $\mathbb{N}$

For arbitrary $N \in \mathbb{N}$, let $I_N = \{1, 2, \dots, N\}$ and let $D_{k,N}$ denote $D_k \cap I_N$, the set of all multiples of p_k not exceeding N. Then $|D_{k,N}| = [N/p_k]$ and $\rho(D_{k,N}) = [N/p_k]/N$. Since, for any real x, we have $x-1 < [x] \le x$, it follows that $(N/p_k)-1 < [N/p_k] \le N/p_k$, and thus $(1/p_k)-(1/N) < \rho(D_{k,N}) \le 1/p_k$. Therefore, the limit of $\rho(D_{k,N})$ exists, so that

$$\rho(D_k) := \lim_{N\to\infty}\rho(D_{k,N}) = \frac{1}{p_k} \qquad \text{and also} \qquad \rho(D^c_k) = 1-\rho(D_k) = 1-\frac{1}{p_k}\,.$$

In this way, we can pass from I_N to $\mathbb{N}$, obtaining the densities of all the Eratosthenes sets in $\mathbb{N}$. In particular, the values of ρ_k in Table 2 are also the densities of the first eight (infinite) Eratosthenes sets relative to the natural numbers. Equations (7) and (8) remain valid in the limit $M \to \infty$, as does the recurrence relation for $\rho_k := \lim_{M\to\infty}\rho_{k,M}$. Thus,

$$\rho_{k+1} = \left(\frac{p_k-1}{p_{k+1}}\right)\rho_k\,. \tag{10}$$

Convergence. We now show that the series $\sum \rho_n$ converges. The simple ratio test is inadequate, as $\lim \rho_{n+1}/\rho_n = 1$, telling us nothing. A more subtle and discriminating test is required.

In his classical text, *Introduction to the Theory of Infinite Series*, Bromwich [2, §12.1] describes an extension of the ratio test, originating with Ernst Kummer and refined by Ulisse Dini. To test a series $\sum a_n$ for convergence, we select a sequence $\{d_n\}$ such that the series $\sum d_n^{-1}$ is divergent. The criterion is as follows.

$$\text{Let} \quad t_n = d_n \left[\frac{a_n}{a_{n+1}} \right] - d_{n+1} \,. \quad \text{Then} \begin{cases} \text{if } \lim t_n > 0, & \sum a_n \text{ converges;} \\ \text{if } \lim t_n < 0, & \sum a_n \text{ diverges.} \end{cases} \tag{11}$$

If $\lim t_n = 0$, there is no conclusion and another choice of $\{d_n\}$ is required. The selection of the sequence $\{d_n\}$ depends on the series being tested.

This test can be used to show that the series $\sum \rho_n$ converges. From (9), the ratio of successive terms is $\rho_n/\rho_{n+1} = p_{n+1}/(p_n - 1)$. In his paper on infinite series, Euler [3] showed that the series $\sum 1/p_n$ diverges. Choosing $d_n = p_n$, we have

$$t_n = p_n \left[\frac{p_{n+1}}{p_n - 1} \right] - p_{n+1} = \left[\frac{p_{n+1}}{p_n - 1} \right] > 1 \,,$$

which fulfils the convergence criterion $\lim t_n > 0$, so the series converges. We will show below that the sum to infinity is 1, but the convergence rate is quite slow. Writing $\sigma_N = \sum_{k=1}^{N} \rho_k$ we have $\sigma_{10} = 0.842$, $\sigma_{1,000} = 0.938$, and $\sigma_{100,000} = 0.960$.

Partitioning the Natural Numbers. Defining $E_0 = \{1\}$, we obtain a partition of the natural numbers $\mathbb{N}$:

$$\mathbb{N} = \biguplus_{n=0}^{\infty} E_n \,, \tag{12}$$

where the sets E_n may be listed explicitly:

$$\begin{array}{lcl} E_0 & = & \langle 1 \rangle \\ E_1 & = & \langle 2, 4, 6, 8, 10, 12, \dots \rangle \\ E_2 & = & \langle 3, 9, 15, 21, 27, \dots \rangle \\ E_3 & = & \langle 5, 25, 35, 55, 65, 85, \dots \rangle \\ & & \dots\dots\dots\dots\dots\dots\dots \\ E_K & = & \langle p_K, p_K^2, p_K p_{K+1}, \dots \rangle \\ & & \dots\dots\dots\dots\dots\dots\dots \end{array}$$

The disjoint union in (12) contains all the positive integers, each occurring just once, providing a partition of $\mathbb{N}$.

Totient Function Expression for ρ_k. Euler's totient function $\varphi(n)$ counts the natural numbers up to n that are coprime to n. In other words, $\varphi(n)$ is the number of integers k in the range $1 \le k \le n$ for which the greatest common divisor $\gcd(k, n)$ is equal to 1. Clearly, for prime numbers, $\varphi(p) = p - 1$. Gauss first proved that

$$\sum_{d|N} \varphi(d) = N$$

[4, Th. 63]. This states that the sum of the numbers $\varphi(d)$, extended over all the divisors d of any number N, is equal to N itself.

The number of values x coprime to $\prod_{j=1}^{k} m_j$ is, by definition, given by $\varphi(m_1 m_2 \ldots m_k)$. But Euler's function is multiplicative for products of coprime numbers $\{m_1, m_2, \ldots, m_k\}$:

$$\varphi(m_1 m_2 \ldots m_k) = \prod_{i=1}^{k} \varphi(m_j)\,.$$

Thus, for $M = P_K$, we have

$$\varphi(P_K) = \varphi\left(\prod_{j=1}^{K} p_j\right) = \prod_{j=1}^{K} \varphi(p_j) = \prod_{j=1}^{K} (p_j - 1) = P_K \prod_{j=1}^{K} \left(1 - \frac{1}{p_j}\right)\,.$$

Now, using (8), we can write the density in terms of the totient function:

$$\rho_k = \frac{1}{P_k} \prod_{j=1}^{k-1} (p_j - 1) = \frac{1}{P_k} \prod_{j=1}^{k-1} \varphi(p_j) = \frac{\varphi(P_{k-1})}{P_k}\,. \tag{13}$$

For example, for $K = 4$ we have $p_K = 7$, $P_K = 210$ and $P_{K-1} = 30$ and, counting explicitly, $\varphi(30) = |\{1, 7, 11, 13, 17, 19, 23, 29\}| = 8$. Thus,

$$\rho_4 = \frac{\varphi(P_3)}{P_4} = \frac{8}{210} = \frac{4}{105}\,,$$

as already shown in Table 2.

An Interesting Result. Defining the cumulative density $\sigma_k = \sum_{j=1}^{k} \rho_k$ and noting that, as the sets E_k are mutually disjoint, σ_k must approach 1, we obtain the relationship

$$\sum_{k=1}^{\infty} \left[\frac{\varphi(P_{k-1})}{P_k}\right] = 1\,, \tag{14}$$

where we define $P_0 = 1$.

This result must be well known, although it has not been found in a cursory search of the literature. Its originality and significance will be the subject of a future study.

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