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# A simpler proof of Lima's dilogarithm identity

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Abstract. From a closed-form expression for a hyperbolic integral, I derived in 2012 a non-trivial two-term dilogarithm identity for $\mathrm{Li}_2\left(\sqrt{2}-1\right)+\mathrm{Li}_2\left(1-1/\sqrt{2}\,\right)$. In recent works published in this Bulletin, Campbell (2021) has applied a series transform obtained via Fourier-Legendre theory to find a new proof for that identity, whereas Stewart (2022), working independently, used known functional relations for the dilogarithm function to develop three other proofs, which has renewed interest in this subject. In this short note, I show how Hill's five-term relation can be applied to suitable algebraic points in order to get a simpler proof of that identity.

## 1. Introduction

The dilogarithm function is a classical function introduced by Leibnitz in 1696, defined as $\mathrm{Li}_2(z) := \sum_{n=1}^{\infty} z^n/n^2$, which converges for all complex $z$ with $|z| \le 1$. This function can be extended to all $z \in \mathbb{C}\backslash(1,\infty)$ through the integral representation

$$\mathrm{Li}_2(z) := -\int_0^z \frac{\ln(1-t)}{t}\,dt\,. \tag{1}$$

Although this integral cannot be expressed as a finite combination of elementary functions, as follows from a theorem by Liouville (1837) [10], closed-forms are currently known for only a few special values, namely $\mathrm{Li}_2(0)=0$, $\mathrm{Li}_2(1/2)=\pi^2/12-\ln^2 2/2$, $\mathrm{Li}_2(-1)=-\pi^2/12$, $\mathrm{Li}_2(1)=\pi^2/6$, $\mathrm{Li}_2(\pm i)=-\pi^2/48\pm i\,G$, $\mathrm{Li}_2(1\pm i)=\pi^2/16\pm i\,(G+\pi\,\ln 2/4)$, $\mathrm{Li}_2(1/2\pm i/2)=5\,\pi^2/96-\ln^2 2/8\pm i\,(G-\pi\,\ln 2/8)$, $\mathrm{Li}_2(-\phi)=-\pi^2/10-\ln^2\phi$, $\mathrm{Li}_2(-1/\phi)=-\pi^2/15+\frac{1}{2}\ln^2\phi$, $\mathrm{Li}_2(1/\phi)=\pi^2/10-\ln^2\phi$, $\mathrm{Li}_2(1/\phi^2)=\pi^2/15-\ln^2\phi$, where $G:=\sum_{n=0}^{\infty}(-1)^n/(2n+1)^2$ is Catalan's constant and $\phi:=\left(1+\sqrt{5}\,\right)/2$ is the golden ratio. In fact, closed-form expressions remain scarce even for two-term linear combinations with rational coefficients of this function at algebraic points (some examples are given in Refs. [4] and [8, Chaps. 1 and 2], and references therein). Interestingly, in 2012, on investigating a hyperbolic version of the trigonometric change of variables introduced by Beukers, Calabi and Kolk to show that $\sum_{n=1}^{\infty} 1/n^2=\pi^2/6$ (the so-called *Basel problem*) [1], I found that (see Theorem 3 of Ref. [9])

$$\int_{\alpha/2}^{\infty} \ln(\tanh z)\,dz = \frac{\alpha^2}{4}-\frac{\pi^2}{16}\,,$$

where $\alpha := \ln(\sqrt{2}+1)$. This allowed me to derive the following two-term dilogarithm identity (see Theorem 4 of Ref. [9]):

$$\mathrm{Li}_2\left(\sqrt{2}-1\right)+\mathrm{Li}_2\left(1-\frac{1}{\sqrt{2}}\right)=\frac{\pi^2}{8}-\frac{\alpha^2}{2}-\frac{\ln^2 2}{8}\,. \tag{2}$$

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Since then, this result has attracted the interest of some mathematicians, among them Campbell, who in 2021 used a series transform obtained via Fourier-Legendre theory (see Refs. [3] and [6, Sec. 7.14]) to present an independent proof in this Bulletin [2], a work which he complemented one year later with a careful investigation of previous equivalent results, which has led to a result of 1915 by Ramanujan, as seen in Eqs. (3) and (4) of Ref. [4]. Also in 2022, Stewart has found three other distinct proofs for Eq. (2) by exploring some known functional relations for the dilogarithm function [11].

However, all these approaches involve complex mathematical steps or are somewhat lengthy. In this short note, I apply Hill's five-term relation to get a simpler proof of Eq. (2).

## 2. Main Result

According to a conjecture of 1995 by Kirillov [7], it should be possible to derive all two-term dilogarithm identities from Hill's five-term relation (1830) [5][1]

$$L(x\,y) = L(x) + L(y) - L\left(\frac{x\,(1-y)}{1-x\,y}\right) - L\left(\frac{y\,(1-x)}{1-x\,y}\right) \,, \tag{3}$$

where $x$ and $y$ are two complex numbers such that $|x| < 1$ and $0 < y < 1$, or $|y| < 1$ and $0 < x < 1$, or $x < 1$ and $0 < y < 1$, or $y < 1$ and $0 < x < 1$. Note that, for simplicity, it is stated in terms of the normalized Rogers' dilogarithm $L(z) := \frac{6}{\pi^2}\left[\mathrm{Li}_2(z) + \frac{1}{2}\ln z\,\ln(1-z)\right]$, as usual. On taking Kirillov's conjecture as a motivation, after many attempts I have succeeded in finding a suitable pair of algebraic arguments $x$ and $y$ for which Hill's five-term relation reduces to the identity in Eq. (2).

*Proof of Eq.* (2). On taking $x = 2-\sqrt{2}$ and $y = 1/\sqrt{2}$, for which $x\,y = \sqrt{2}-1$, in Hill's five-term relation, our Eq. (3), one finds

$$L\left(\left(2-\sqrt{2}\right)\,\frac{1}{\sqrt{2}}\right) = L\left(2-\sqrt{2}\right) + L\left(\frac{1}{\sqrt{2}}\right) - L\left(\frac{3-2\sqrt{2}}{2-\sqrt{2}}\right) - L\left(\frac{1/\sqrt{2}-\sqrt{2}+1}{2-\sqrt{2}}\right), \tag{4}$$

which promptly simplifies to

$$L\left(\sqrt{2}-1\right) = L\left(2-\sqrt{2}\right) + L\left(\frac{1}{\sqrt{2}}\right) - L\left(1-\frac{1}{\sqrt{2}}\right) - L\left(\frac{1}{2}\right) \,. \tag{5}$$

Now, one applies Euler's reflection formula (1768) $L(z) = 1 - L(1-z)$ to both $L\left(2-\sqrt{2}\,\right)$ and $L\left(1/\sqrt{2}\,\right)$. Since Euler's reflection yields $L(1/2) = 1/2$, one finds

$$L\left(\sqrt{2}-1\right) = 1 - L\left(\sqrt{2}-1\right) + 1 - L\left(1-\frac{1}{\sqrt{2}}\right) - L\left(1-\frac{1}{\sqrt{2}}\right) - \frac{1}{2} \,, \tag{6}$$

which promptly reduces to

$$L\left(\sqrt{2}-1\right) + L\left(1-\frac{1}{\sqrt{2}}\right) = \frac{3}{4} \,, \tag{7}$$

which is just the Rogers equivalent of Eq. (2). □

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