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# A note on a class of Fourier transforms

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Abstract. We consider functions $f \in L^2(\mathbf{R}^n)$ for which

$$\int_{\mathbf{R}^n} |\hat{f}(t)|^2 (1 + \log^+ |t|)^{2\beta} dt < \infty, \ \beta > 0,$$

where $\hat{f}$ is the Fourier transform of $f$, and we identify a kernel $\mathcal{K}_\beta$ such that $f$ satisfies this integral condition if, and only if,

$$f(x) = (\mathcal{K}_\beta * F)(x) = \int_{R^n} \mathcal{K}_\beta(x - t)\, F(t)\, dt$$

for some function $F \in L^2(\mathbf{R}^n)$. We also address the question of 'Fourier inversion' for this class by showing that certain Bochner-Riesz means of the transforms of $f = \mathcal{K}_\beta * F$ converge to $f$ outside small exceptional sets of points in $\mathbf{R}^n$ of capacity zero.

## 1. Introduction

It was conjectured by Lusin in 1915 that the Fourier series of a periodic function $f \in L^2(-\pi, \pi)$ converges almost everywhere, that is, if $c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-ikx} f(x) dx, k \in \mathbf{Z}$, denote the Fourier coefficients of $f$, then the partial sums

$$s_n(f)(x) = \sum_{k=-n}^{k=n} c_k e^{ikx}$$

converge almost everywhere to $f(x)$ as $n \to \infty$. The conjecture remained unproven for several decades and, as doubts began to arise regarding its veracity, some research was directed towards constructing a counterexample. It came as a major surprise therefore when, in a famous and very difficult paper [2], Lennart Carleson proved Lusin's conjecture in 1966. This result was widely celebrated within mathematics and particularly, perhaps, by those analysts who (like this author) had been nurtured mathematically on Zygmund's *Trigonometric Series*! Carleson's result was extended to $L^p$ functions, $p > 1$, by Hunt [6].

We are concerned with Fourier transforms, and the question that arises in this context is whether Carleson's result has an analogue in $\mathbf{R}^n$, specifically whether for a function $f \in L^2(\mathbf{R}^n)$, the spherical partial integral

$$S_R f(x) = \int_{|t| \le R} \hat{f}(t) \exp(2\pi i x \cdot t) dt, \ R > 0, \ x \in \mathbf{R}^n, \ n \ge 2, \tag{1}$$

converges almost everywhere to $f(x)$ in $\mathbf{R}^n$ as $R \to \infty$, where $\hat{f}$ is the Fourier transform of $f$. This question remains open, but by analogy with partial results established for Fourier series prior to Carleson's paper (see [14, 1.13, p. 163]), it is natural to begin

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seeking answers by investigating functions $f$ which satisfy conditions such as

$$\int_{\mathbf{R}^n} |\hat{f}(t)|^2 (1+\log^+|t|)^{2\beta} dt < \infty, \ \beta > 0, \tag{2}$$

a stronger requirement than $f \in L^2(\mathbf{R}^n)$. We note that it has been shown by Carberry and Soria [4] (see also [5]) that if $f$ satisfies (2) with $\beta = 1$ then $S_R(f) \to f$, as $R \to \infty$, almost everywhere. We focus in this article on providing a characterisation of functions $f$ for which (2) holds and, to this end, we define a kernel $\mathcal{K}_\beta$ which, as we shall prove in section 2, has the property that $f$ satisfies (2) if, and only if, $f = \mathcal{K}_\beta * F$ for some $F \in L^2(\mathbf{R}^n)$. A *kernel* $K$ is a non-negative, unbounded, and integrable function on $\mathbf{R}^n$ which is radially symmetric and decreasing, i.e. $K(x) = K(t)$ if $|x| = |t|$ and $K(x) \le K(t)$ if $|x| \ge |t|$. We write $L^2_K(\mathbf{R}^n)$ to denote the class of potentials

$$(K * F)(x) = \int_{R^n} K(x-t)\, F(t)\, dt\,,$$

where $K$ is a kernel on $\mathbf{R}^n$ and $F \in L^2(\mathbf{R}^n)$, with $n \ge 2$. (From here on we shall write $L^2_K$ for $L^2_K(\mathbf{R}^n)$, and $L^2$ for $L^2(\mathbf{R}^n)$.) We note [11, p.3] that if $f \in L^2_K$ then $f \in L^2$ and hence has a Fourier transform $\hat{f} \in L^2$ by the Plancherel theorem. It follows that $\hat{f}$ is integrable in $\{x : |x| \le R\}$ for every fixed $R > 0$, and the integral for the mean $S_R f$ in (1), and the mean $T^\lambda_R f$ in (3) below, are thus well-defined for $f \in L^2_K$.

An important alternative summability method to the one based on the mean $S_R f$ is *Bochner-Riesz summability* ([11, pp.170-172], [8], [13]) with

$$T^\lambda_R f(x) = \int_{|t|\le R} (1 - \frac{|t|^2}{R^2})^\lambda \, \hat{f}(t) \exp(2\pi i x{\cdot}t) dt, \ \lambda > 0, \tag{3}$$

a more amenable mean than the spherical partial integral. In section 3, using the characterisation $f = \mathcal{K}_\beta * F$, we derive a result on the convergence of $T^\lambda_R f$ means, outside sets of capacity zero, for functions satisfying (2).

## 2. The main theorem

We begin with the definition of the kernel $\mathcal{K}_\beta$. We set

$$\mathcal{K}_\beta(x) = \int_0^1 \frac{P_s(x)}{s\left(\log \frac{2}{s}\right)^{\beta+1}}\, ds, \quad x \in \mathbf{R}^n, \ \beta > 0,$$

where, for $n \ge 1$ and $s > 0$,

$$P_s(x) = \frac{\lambda_n\, s}{(s^2 + |x|^2)^{(n+1)/2}}\,, \quad \lambda_n = \Gamma(\frac{n+1}{2})/\pi^{(n+1)/2},$$

is the Poisson kernel for $\mathbf{R}^{n+1}_+ = \{(x,s) : x \in \mathbf{R}^n, s > 0\}$. Since $\int_{\mathbf{R}^n} P_s(x) dx = 1$ for each $s > 0$ [11, p. 9], and $P_s$ is radially symmetric and decreasing, it follows that $\mathcal{K}_\beta$ is a kernel.

To prepare for our theorem we present three lemmas, the first two of which provide estimates for $\mathcal{K}_\beta$ and $\hat{\mathcal{K}}_\beta$, and the third establishes an equivalence relation for the classes $L^2_K$ which is central to the proof of the theorem. We will not use Lemma 1 in the proof but, as it answers obvious questions, we include the lemma for the sake of completeness.

**Lemma 2.1.** *We have*

$$\frac{c_\beta}{|x|^n \left(\log \frac{2}{|x|}\right)^{\beta+1}} \le \mathcal{K}_\beta(x) \le \frac{c'_\beta}{|x|^n \left(\log \frac{2}{|x|}\right)^{\beta+1}}\,, \quad 0 < |x| \le 1. \tag{1}$$

*We also have* $\mathcal{K}_\beta(x) \le c_\beta |x|^{-(n+1)}$ *for* $|x| > 1$.

In Lemmas 2.1 and 2.2, $c_\beta$ and $c'_\beta$ denote positive quantities which depend on $\beta$ or $n$ or both, but are not necessarily the same at each occurrence.

*Proof of Lemma 2.1.* For notational simplicity we write $\gamma$ for $\beta+1$ throughout this proof. Since $a^2+b^2 \le (a+b)^2 \le 2(a^2+b^2)$ for $a, b \ge 0$, we note that it is enough to show that the inequalities in (1) are satisfied by

$$\begin{aligned}\int_0^1 \frac{ds}{\left(\log \frac{2}{s}\right)^\gamma (|x|+s)^{n+1}} &= |x|^{-n} \int_0^{1/|x|} \frac{dr}{\left(\log \frac{2}{r|x|}\right)^\gamma (1+r)^{n+1}} \\ &= |x|^{-n} I(x),\end{aligned}$$

say. Next, if $0 < |x| \le 1$ and we write $\varphi_x(r)$ for the integrand in $I(x)$, we have, since $r \ge |x|/2$ implies $2/r|x| \le 4/|x|^2$,

$$I(x) \ge \int_{|x|/2}^{1/|x|} \varphi_x(r)dr \ge 2^{-\gamma} \left(\log \frac{2}{|x|}\right)^{-\gamma} \int_{1/2}^1 \frac{dr}{(1+r)^{n+1}} \ge 2^{-(\gamma+n+2)} \left(\log \frac{2}{|x|}\right)^{-\gamma}.$$

This gives the lefthand inequality in (1). To obtain the second inequality we note that $1/|x|^{1/2} \le 1/|x|$ when $0 < |x| \le 1$, and write $I(x) = I_1(x) + I_2(x)$, where in $I_1$ we integrate over $(0, 1/|x|^{1/2})$ and in $I_2$ over $(1/|x|^{1/2}, 1/|x|)$. Since $r \le 1/|x|^{1/2}$ implies $2/r|x| \ge (2/|x|)^{1/2}$,

$$I_1(x) = \int_0^{1/|x|^{1/2}} \varphi_x(r)dr \le 2^\gamma \left(\log \frac{2}{|x|}\right)^{-\gamma} \int_0^\infty \frac{dr}{(1+r)^{n+1}} \le 2^\gamma \left(\log \frac{2}{|x|}\right)^{-\gamma}.$$

We have $r|x| \le 1$ in $I_2(x)$, so

$$I_2(x) \le (\log 2)^{-\gamma} \int_{1/|x|^{1/2}}^{1/|x|} \frac{dr}{r^2} \le (\log 2)^{-\gamma} |x|^{1/2} \le c_\beta \left(\log \frac{2}{|x|}\right)^{-\gamma}$$

since $|x| \le 1$ implies $\left(\log \frac{2}{|x|}\right)^\gamma \le c'_\beta (2/|x|)^{1/2}$, for a big enough constant $c'_\beta$. Inequality (1) follows. The estimate for $|x| > 1$ is easily obtained and the Lemma is proved.

**Lemma 2.2.** *If $\mathcal{K}_\beta$ is the kernel defined as above for $\beta > 0$, then the Fourier transform $\hat{\mathcal{K}}_\beta(x) = \int_{\mathbf{R}^n} \mathcal{K}_\beta(t) \exp(-2\pi i x \cdot t) dt$ satisfies*

$$\frac{c_\beta}{(1+\log^+|x|)^\beta} \le \hat{\mathcal{K}}_\beta(x) \le \frac{c'_\beta}{(1+\log^+|x|)^\beta}, \quad x \in \mathbf{R}^n. \tag{2}$$

*Proof of Lemma 2.2.* We note to begin with, since $\hat{P}_s(x) = \exp(-2\pi s|x|)$ [11, p. 5], that, by an interchange of integrals,

$$\begin{aligned}\hat{\mathcal{K}}_\beta(x) &= \int_{\mathbf{R}^n} \left( \int_0^1 \frac{P_s(t)}{s\left(\log \frac{2}{s}\right)^{\beta+1}} ds \right) \exp(-2\pi i x \cdot t) dt \\ &= \int_0^1 \frac{\exp(-2\pi s|x|)\, ds}{s\left(\log \frac{2}{s}\right)^{\beta+1}} = \int_0^{1/|x|} \frac{\exp(-2\pi r)\, dr}{r\left(\log \frac{2|x|}{r}\right)^{\beta+1}}.\end{aligned} \tag{3}$$

Assume that $|x| > 1$. Then

$$\begin{aligned}\hat{\mathcal{K}}_\beta(x) &\ge \int_{1/4|x|^2}^{1/2|x|} \frac{\exp(-2\pi r)\, dr}{r\left(\log \frac{2|x|}{r}\right)^{\beta+1}} \\ &\ge e^{-2\pi} 3^{-\beta-1} (\log 2|x|)^{-\beta-1} \int_{1/4|x|^2}^{1/2|x|} \frac{1}{r}\, dr \\ &= e^{-2\pi} 3^{-\beta-1} (\log 2|x|)^{-\beta} \ge c_\beta (1+\log|x|)^{-\beta} = c_\beta (1+\log^+|x|)^{-\beta}.\end{aligned}$$

This gives the lower bound in (2). To obtain the upper bound for $|x| > 1$, we note from the second equality in (3) that

$$\beta\hat{K}_\beta(x) = \int_0^1 \mathrm{e}^{-2\pi s|x|} d\left(\log\frac{2}{s}\right)^{-\beta} = \mathrm{e}^{-2\pi|x|}(\log 2)^{-\beta} + 2\pi|x|\, J(x), \tag{4}$$

where

$$J(x) = \int_0^1 \mathrm{e}^{-2\pi s|x|}(\log\frac{2}{s})^{-\beta} ds.$$

We write $J$ as $J_1 + J_2$ where $J_1 = \int_0^{1/|x|^{1/2}}$, $J_2 = \int_{1/|x|^{1/2}}^1$. Then

$$\begin{aligned} J_1(x) &\leq (\log 2|x|^{1/2})^{-\beta} \int_0^{1/|x|^{1/2}} \mathrm{e}^{-2\pi s|x|} ds \\ &= \frac{2^{\beta-1}(\log 4|x|)^{-\beta}}{2\pi|x|} \int_0^{2\pi|x|^{1/2}} \mathrm{e}^{-u} du \leq c_\beta\, |x|^{-1}(1+\log^+|x|)^{-\beta}, \end{aligned}$$

since $|x| > 1$. Next, by a similar argument,

$$J_2(x) = \int_{1/|x|^{1/2}}^1 \leq (\log 2)^{-\beta}\frac{1}{2\pi|x|}\int_{2\pi|x|^{1/2}}^{2\pi|x|} \mathrm{e}^{-u} du \leq c_\beta\,(1+\log^+|x|)^{-\beta}/|x|,$$

choosing $c_\beta$ large enough. Since $J = J_1 + J_2$, and $\mathrm{e}^{-2\pi|x|} \leq c_\beta(1+\log^+|x|)^{1-\beta}$, $|x| \geq 1$, the required upper bound follows from (4) when $|x| > 1$. The proof of the inequalities (2) for the case $|x| \leq 1$, i.e. that $c_\beta \leq \hat{\mathcal{K}}_\beta(x) \leq c'_\beta$, is easy and is omitted. The proof of Lemma 2.2 is complete.

*Remark* If we apply the result $(\exp(-2\pi s|x|))^\wedge = P_s(x)$ [11, p. 6] to the middle integral in (3) we see that $(\hat{\mathcal{K}}_\beta)^\wedge = \mathcal{K}_\beta$.

**Lemma 2.3.** *Let $K$ be a kernel with $\hat{K} > 0$ and suppose that $f \in L^2$. Then*

$$\int_{\mathbf{R}^n} \hat{K}(t)^{-2}|\hat{f}(t)|^2 dt < \infty \Longleftrightarrow f \in L^2_K.$$

*Proof.* Note first, by $L^2$ transform theory, that if $g \in L^2$ then the transform $\hat{g} \in L^2$ and $(\hat{g})^\wedge(-t) = g(t)$.

Assume that $\int_{\mathbf{R}^n} \hat{K}(t)^{-2}|\hat{f}(t)|^2 dt < \infty$ and set $F(t) = \hat{K}(t)^{-1}\hat{f}(t)$, so that $F \in L^2$. Then

$$\hat{f}(t) = \hat{K}(t)F(t) = \hat{K}(t)(\hat{F})^\wedge(-t) = \hat{K}(t)\hat{H}(t) = (K * H)^\wedge(t)$$

where $H(t) = \hat{F}(-t)$, so $H \in L^2$, and we have applied the multiplication formula from [7, theorem 5.8], with $p = 1$ and $q = 2$, for the last equality. Hence

$$(\hat{f})^\wedge = ((K * H)^\wedge)^\wedge, \quad i.e.\ f(-t) = (K * H)(-t),$$

or $f(t) = (K * H)(t)$, $t \in \mathbf{R}^n$. This proves the first part of the Lemma.
For the second part assume that $f \in L^2_K$ so that $f = K * Q$ where $Q \in L^2$. Then $\hat{f} = \hat{K}\hat{Q}$ and $\hat{Q}(t) = \hat{K}(t)^{-1}\hat{f}(t)$. Since $\hat{Q} \in L^2$, the converse implication follows and the proof is complete.

Our main theorem now follows immediately from Lemmas 2.2 and 2.3 (with $K = \mathcal{K}_\beta$).

**Theorem 2.4.** *If $f \in L^2$ then*

$$\int_{\mathbf{R}^n} |\hat{f}(t)|^2(1+\log^+|t|)^{2\beta} dt < \infty,\ \beta > 0, \tag{5}$$

*if, and only if, $f \in L^2_{\mathcal{K}_\beta}$.*

## 3. Bochner-Riesz summability

We obtain our final theorem on the convergence of certain Bochner-Riesz means in $L^2_K$ by simply combining two known results. We begin by noting from [11, Corollary 4.16 (b), p.172] that

*if $f \in L^2$ has a Lebesgue point at $x \in \mathbf{R}^n$, that is, if*

$$\lim_{r\to 0} \frac{1}{m(B(x,r))} \int_{B(x,r)} |f(t) - f(x)|\, dt = 0,$$

*where $m(B(x,r))$ denotes the Lebesgue measure of the ball $B(x,r)$, then the Bochner-Riesz mean $T^\lambda_R f(x)$ in (1.3) converges to $f(x)$ for all $\lambda > (n-1)/2$ as $R \to \infty$.*

Since $L^2_K \subset L^2$, as noted above, and $L^2$ functions have Lebesgue points almost everywhere, it follows that for $f \in L^2_K$ the set of $x \in \mathbf{R}^n$ for which (1.3) fails to converge for $\lambda > (n-1)/2$ has Lebesgue measure zero. We strengthen this by combining the Stein-Weiss result with the following consequence of [12, Theorem 1]:

*if $f \in L^2_K$ then $f$ has a Lebesgue point at all points in $\mathbf{R}^n$ except possibly for a set of points of $C_{K,2}$-capacity zero.*

The capacity referred to here is the particular case $p = 2$ of the $L^p$-capacities of Meyers ([9], [1, Chapter 2]). For a brief summary of the basic properties of these capacities see [10, pp. 341-2]. If $C_{K,2}(E) = 0$, then $E$ has measure zero, and Meyers' capacities provide a way of differentiating between sets of measure zero.

Taking $K = \mathcal{K}_\beta$ we immediately deduce the following convergence result for functions in $L^2_{\mathcal{K}_\beta}$.

**Theorem 3.1.** *If $f \in L^2_{\mathcal{K}_\beta}$, or equivalently if (2.5) holds, and $\lambda > (n-1)/2$, then $\lim_{R\to\infty} T^\lambda_R f(x) = f(x)$ for all $x \in \mathbf{R}^n$ outside an exceptional set of $C_{\mathcal{K}_\beta,2}$-capacity zero.*

It has been shown in [3, Theorem A] that if $f \in L^2$ then $\lim_{R\to\infty} T^\lambda_R f(x) = f(x)$ *almost everywhere* in $\mathbf{R}^n$ for *all* $\lambda > 0$, and an obvious question here therefore is whether the range of $\lambda$ in Theorem 3.1 can be extended to all positive values. There is also the question of whether the characterisation $f = \mathcal{K}_\beta * F$ can be used to obtain convergence results in $L^2_{\mathcal{K}_\beta}$ for the spherical partial integral $S_R f$. These questions are open.

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