Some Aspects of Brendan Goldsmith's Contributions to Abelian Groups

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*Dedicated to Brendan Goldsmith on the occasion of his **75**th birthday)*

ABSTRACT. We give a brief overview of the contribution to Abelian group theory by Brendan Goldsmith on the occasion of his 75th birthday.

1. SOME SHORT PRELIMINARY HISTORICAL DETAILS

The well-known mathematician Brendan Goldsmith was born in Belfast, Ireland, on 22 January 1949. He attended Queen's University and later Oxford University from where he graduated with a D. Phil. in Mathematics in 1978 under the supervision of Professor Anthony L. S. Corner from Oxford writing a dissertation entitled "*An Investigation in Abelian Group Theory*". His thesis was concerned with realising endomorphism rings of certain classes of Abelian groups and p-adic modules modulo the ideal of so-called *inessential endomorphisms*, an approach which became important in subsequent work of R. Göbel and others on Abelian groups.

He was appointed in 1974 to a lectureship at the College of Technology, Kevin Street, and worked for ten years in that rôle. During this period he was a founding member of the Irish Mathematical Society and subsequently served as President of the Society. In 1983 he was appointed Head of School of Mathematics and introduced new programmes in mathematics and computer science. Ten year later, in late 1993, he was appointed President of DIT, becoming the first president of the new institute. In that same year he worked with Tom Laffey and the late David Simms on the organisation for the Royal Irish Academy of a conference to celebrate the sesqui-centennial of Hamilton's discovery of quaternions; the special issue of the Academy's Proceedings dedicated to the conference was edited by the three of them.

He often recalls the great challenges he experienced as the new president of DIT in those early days and, in particular, the difficulty in acquiring the new campus at Grangegorman; the campus is now home to a central part of the recently established Technological University, Dublin and Goldsmith presently continues to work from there as an emeritus professor. According to the existing current on-line databases, Professor Goldsmith has 6 students and 9 descendants. They are as follows: one from Trinity College (Dublin), three from Dublin Institute of Technology (DIT) and two from University of Duisburg-Essen, Germany, respectively; the latter two students had Goldsmith as external examiner for their doctorates.

Goldsmith's scientific interests are mainly in the areas of Abelian groups and modules over (possibly commutative) rings. He has published widely in journals including Proceedings and Transactions of AMS, Journal and Bulletin of LMS, Quarterly

J. Mathematics, J. Algebra, J. Pure and Applied Algebra, Commun. Algebra, Forum Mathematicum, Mediterranean J. Math., J. Physics A, J. Commut. Algebra, Illinois J. Math., Archiv Math., J. Group Theory, Rendiconti Sem. Mat. Univ. Padova and, of course, IMS Bulletin. He has also been an editor of several books in these areas: Models, Modules and Abelian Groups (de Gruyter 2008) - a volume in memory of A.L.S. Corner; Groups, Modules and Model Theory; Surveys and Recent Developments (Springer 2017) - a volume in memory of R. Göbel; a guest editor of a special volume honoured to the 95th birthday of László Fuchs in the Rend. Sem. Mat. Univ. Padova **144** (2020); and he is also currently involved as a guest editor in the production of two issues of the Journal of Commutative Algebra to celebrate the forthcoming 100th birthday of László Fuchs in June 2024.

An interesting feature of Goldsmith's research work has been his extensive collaborations: to date he has some 30 collaborators, including multiple collaborations with myself, R. Göbel, A.L.S. Corner, L. Salce, L. Strüngmann, P. Vámos and P. Zanardo.

2. Recent Papers on Abelian Groups and Modules

Here I will concentrate initially on the joint cooperation between Goldsmith and myself. Our collaboration began in 2009. To date we have succeeded to write and publish jointly ten very interesting papers in prestigious international journals and our joint work is still active (see, e.g., [11] and [12]) by involving two other well-established specialists in Abelian groups like Andrey R. Chekhlov from Tomsk State University of Russia and Patrick W. Keef from Whitman College in Walla Walla, WA, United States.

(1) In a series of papers [1], [2] and [3], we introduced the notions of *socle-regularity* and, respectively, *strong socle-regularity* for Abelian p-groups. The notions arise because of the difficulty in describing completely the fully invariant and characteristic subgroups of an arbitrary p-group; we restricted attention to consideration of the socle of groups (i.e., the subgroup of elements of order p). An interesting outcome was the surprising connection between the two notions: a group G is socle-regular if, and only if, its square $G \oplus G$ is strongly socle-regular; the result has a clear resemblance to an earlier result of Files and Goldsmith published in PAMS solving an old problem of Kaplansky on the connection between the concepts of transitivity and full transitivity.

(2) In paper [4] we used a similar approach replacing fully invariant subgroups with the well-known projection-invariant subgroups. Paper [5] concerns strengthening the classical concept of fully transitive group to groups which we named *projectively fully transitive* Abelian p-groups. The final paper in this initial period of collaboration works in the same vein but focuses on subgroups C which are *commutator invariant* in the sense that $f(C) \leq C$ for all endomorphisms f of the form $f = [\phi, \psi] = \phi\psi - \psi\phi$ for endomorphisms ϕ, ψ.

After a reloaded break of seven years, we renewed our scientific collaboration with a series of successful papers in another two areas of Abelian group theory; these were written together with the Russian algebraist A. R. Chekhlov.

(4) Common generalizations to the standard notions of fully invariant and characteristic subgroups are, respectively, fully inert and characteristically inert subgroups and the latter ones allow us to define in [7] and [8] *fully inert socle-regular* and *characteristically inert socle-regular* Abelian p-groups. They are, certainly, closely related to the aforementioned articles [1] and [2], being their natural expansions.

(5) In our three further papers [9], [10] and [11], we study arbitrary (possibly mixed) Abelian groups which are devoted to the classification of *Bassian* and *generalized Bassian* groups as well as of some their derivations. Unfortunately, we were able to

classify only Bassian groups and some large types of generalized Bassian groups conjecturing that the second kind is the direct sum of a Bassian group and an elementary group. This problem seems to be quite difficult and is **not** answered yet.

During this period, Goldsmith also had a number of other important collaborations. In particular, he and Luigi Salce have published a series of interesting papers relating to Abelian p-groups revisiting and progressing ideas going back to Pierce's fundamental work in this area. He has also collaborated with his former student, Ketao Gong, on ideas relating to quotient and cyclic subgroup transitivity. He has also worked with his student, Noel White, on ideas deriving from an old unpublished work of Corner; for instance, one of these has appeared recently in this Bulletin.

Goldsmith's most recent publication in Forum Mathematicum has just appeared in February 2024 and is a joint work with Fuchs, Salce and Strüngmann on *Cellular Covers of Modules over Valuation Domains*. As an aside, the joint age of the authors is over 300, perhaps giving the lie to the notion that mathematical research is only for the young!

In conclusion, I personally would like to emphasize that I really have had the personal privilege to enjoy working with this well-known algebraist as the expert work with him considerably influenced on my own scientific interests definitely!

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