

Two Extensions of Cauchy’s Double Alternant

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ABSTRACT. Two extensions of Cauchy’s double alternant are evaluated in closed form that may serve also as parametric generalizations of the remarkable determinant identity of a skew-symmetric matrix discovered by Schur (1911) and its multiplicative counterpart due to Laksov–Lascoux–Thorup (1989).

1. INTRODUCTION AND OUTLINE

There exist numerous determinant identities in the literature (cf. [9, 13]). For example, the determinants of Vandermonde and Cauchy’s “double alternant”

$$\Lambda_m = \det_{1 \leq i, j \leq m} \left[x_i^{m-j} \right] = \prod_{1 \leq i < j \leq m} (x_i - x_j),$$

$$\det_{1 \leq i, j \leq m} \left[\frac{1}{x_i + y_j} \right] = \frac{\prod_{1 \leq i < j \leq m} (x_i - x_j)(y_i - y_j)}{\prod_{1 \leq i, j \leq m} (x_i + y_j)};$$

play an important role in symmetric functions and group characters (cf. [5, 8, 10]). In general, a matrix $T(x_1, x_2, \dots, x_m)$ of order $m \times n$ in m variables is called an alternant (cf. [9, §321]) when the elements of the first row of T are all functions of variable x_1 , the elements of the second row the like functions of x_2 , and so on. For example

$$\begin{bmatrix} e^x, \sin x, \cos x \\ e^y, \sin y, \cos y \\ e^z, \sin z, \cos z \end{bmatrix}.$$

Likewise, a matrix $\mathbb{T}(x_1, x_2, \dots, x_m; y_1, y_2, \dots, y_n)$ is a double alternant if \mathbb{T} is an alternant respect to both rows in variables $\{x_1, x_2, \dots, x_m\}$ and columns in variables $\{y_1, y_2, \dots, y_n\}$. Suppose that $f(x, y)$ is a bivariate function, we have the following general double alternant

$$\left[f(x_i, y_j) \right]_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} = \begin{bmatrix} f(x_1, y_1), f(x_1, y_2), \dots, f(x_1, y_n) \\ f(x_2, y_1), f(x_2, y_2), \dots, f(x_2, y_n) \\ \vdots \quad \quad \quad \vdots \quad \quad \quad \dots \quad \quad \quad \vdots \\ f(x_m, y_1), f(x_m, y_2), \dots, f(x_m, y_n) \end{bmatrix}.$$

There exist several generalizations (cf. [1, 3, 6]) of the determinants for Cauchy’s double alternant. By employing the calculus of divided differences, the second author [2] evaluated determinants for a large class of variants of Cauchy’s double alternant. As a complements to the results appearing in [2], we shall examine, in this little article, the

2020 *Mathematics Subject Classification.* 15A15, 15B57.
Key words and phrases. Determinant, Cauchy’s double alternant, Divided difference.
 Received on 18-12-2023; revised 9-8-2024.
 DOI: 10.33232/BIMS.0094.55.60.

determinants of two particular matrices. Let T and $\{x_k, y_k\}_{1 \leq k \leq m}$ be indeterminates. Define two matrices by

$$U_m = [u_{i,j}]_{1 \leq i, j \leq m} : u_{i,j} = \frac{x_i + T y_j}{x_i + y_j}, \quad (1)$$

$$V_m = [v_{i,j}]_{1 \leq i, j \leq m} : v_{i,j} = \frac{x_i + T y_j}{1 + x_i y_j}. \quad (2)$$

We shall prove the following two surprisingly elegant determinant identities.

Theorem 1 ($m \in \mathbb{N}$).

$$\det U_m = \frac{\prod_{1 \leq i < j \leq m} (x_i - x_j)(y_i - y_j)}{(1 - T)^{1-m} \prod_{1 \leq i, j \leq m} (x_i + y_j)} \times \left\{ \prod_{i=1}^m x_i - T \prod_{i=1}^m (-y_i) \right\}.$$

Theorem 2 ($m \in \mathbb{N}$).

$$\begin{aligned} \det V_m &= \frac{\prod_{1 \leq i < j \leq m} (x_j - x_i)(y_i - y_j)}{\prod_{1 \leq i, j \leq m} (1 + x_i y_j)} \\ &\times \frac{1}{2} \left\{ \prod_{i=1}^m (x_i + \sqrt{T})(1 + y_i \sqrt{T}) + \prod_{i=1}^m (x_i - \sqrt{T})(1 - y_i \sqrt{T}) \right\}. \end{aligned}$$

When $T = 0$, the corresponding identities in both theorems are equivalent to the Cauchy double alternant. For $T = -1$ and even $m = 2n$, these identities reduce, in the case $x_k = y_k$ for all k , to the following remarkable Pfaffian formulae discovered by Schur [11] and Laksov–Lascoux–Thorup [7] (see also [12]), respectively:

$$\begin{aligned} \det_{1 \leq i, j \leq 2n} \begin{bmatrix} x_i - x_j \\ x_i + x_j \end{bmatrix} &= \prod_{1 \leq i < j \leq 2n} \left(\frac{x_i - x_j}{x_i + x_j} \right)^2, \\ \det_{1 \leq i, j \leq 2n} \begin{bmatrix} x_i - x_j \\ 1 + x_i x_j \end{bmatrix} &= \prod_{1 \leq i < j \leq 2n} \left(\frac{x_i - x_j}{1 + x_i x_j} \right)^2. \end{aligned}$$

2. PROOF OF THEOREM 1

For the matrix U_m , by subtracting the last row from the other rows, we can check that the resulting matrix becomes

$$U'_m = [u'_{i,j}]_{1 \leq i, j \leq m} : u'_{i,j} = \begin{cases} \frac{(1-T)(x_i - x_m)y_j}{(x_i + y_j)(x_m + y_j)}, & 1 \leq i < m; \\ \frac{x_m + T y_j}{x_m + y_j}, & i = m. \end{cases}$$

By extracting the common row factor $(1-T)(x_i - x_m)$ for $1 \leq i < m$ and the common column factor $\frac{y_j}{x_m + y_j}$ for $1 \leq j \leq m$, we find the following determinant equality

$$\det U_m = \det U'_m = \det U''_m \times (1-T)^{m-1} \prod_{i=1}^{m-1} (x_i - x_m) \prod_{j=1}^m \frac{y_j}{x_m + y_j}, \quad (3)$$

where the matrix U''_m is given by

$$U''_m = [u''_{i,j}]_{1 \leq i, j \leq m} : u''_{i,j} = \begin{cases} \frac{1}{x_i + y_j}, & 1 \leq i < m; \\ \frac{x_m + T y_j}{y_j}, & i = m. \end{cases}$$

Expanding the determinant along the last row leads us to the equality

$$\det U''_m = \sum_{k=1}^m (-1)^{m+k} \frac{x_m + Ty_k}{y_k} \det U''_m[m, k], \quad (4)$$

where $U''_m[m, k]$ is the sub-matrix of U''_m with the m th row and the k th column being removed. By applying the Cauchy double alternant, we can evaluate

$$\begin{aligned} \det U''_m[m, k] &= \frac{\prod_{1 \leq i < j < m} (x_i - x_j) \prod_{1 \leq i < j \leq m, (i, j \neq k)} (y_i - y_j)}{\prod_{1 \leq i < m} \prod_{1 \leq j \leq m, (j \neq k)} (x_i + y_j)} \\ &= (-1)^{m-k} \frac{\prod_{i=1}^m (x_i + y_k)}{\prod_{i \neq m} (x_m - x_i)} \prod_{j \neq k} \frac{x_m + y_j}{y_k - y_j} \frac{\prod_{1 \leq i < j \leq m} (x_i - x_j) (y_i - y_j)}{\prod_{1 \leq i, j \leq m} (x_i + y_j)}. \end{aligned}$$

Here and henceforth for simplicity, $\prod_{\ell \neq k}$ stands for the product with the index ℓ running from 1 to m except for $\ell = k$. Substituting this into (4) gives rise to the following expression

$$\det U''_m = \frac{\prod_{1 \leq i < j \leq m} (x_i - x_j) (y_i - y_j)}{\prod_{1 \leq i, j \leq m} (x_i + y_j)} \frac{\prod_{j=1}^m (x_m + y_j)}{\prod_{i \neq m} (x_m - x_i)} \sum_{k=1}^m \frac{x_m + Ty_k}{y_k} \frac{\prod_{i \neq m} (x_i + y_k)}{\prod_{j \neq k} (y_k - y_j)}.$$

Denote by $\Delta[y_1, y_2, \dots, y_m]f(y)$ the divided difference (cf. [2]) of the function $f(y)$ at the points $\{y_k\}_{k=1}^m$, which can be expressed by Newton's symmetric sum

$$\Delta[y_1, y_2, \dots, y_m]f(y) = \sum_{k=1}^m \frac{f(y_k)}{\prod_{j \neq k} (y_k - y_j)}.$$

Then we can evaluate the last sum (cf. [4]) as

$$\begin{aligned} \sum_{k=1}^m \frac{x_m + Ty_k}{y_k} \frac{\prod_{i \neq m} (x_i + y_k)}{\prod_{j \neq k} (y_k - y_j)} &= \Delta[y_1, y_2, \dots, y_m] \left\{ \frac{x_m + Ty}{y} \prod_{i \neq m} (x_i + y) \right\} \\ &= \Delta[y_1, y_2, \dots, y_m] \left\{ Ty^{m-1} + \frac{\prod_{i=1}^m x_i}{y} \right\} \\ &= T - (-1)^m \prod_{i=1}^m \frac{x_i}{y_i}, \end{aligned}$$

which leads us to the following simpler formula

$$\det U''_m = \frac{\prod_{1 \leq i < j \leq m} (x_i - x_j) (y_i - y_j)}{\prod_{1 \leq i, j \leq m} (x_i + y_j)} \frac{\prod_{j=1}^m (x_m + y_j)}{\prod_{i \neq m} (x_m - x_i)} \left\{ T - (-1)^m \prod_{i=1}^m \frac{x_i}{y_i} \right\}.$$

Substituting this into (3) and then simplifying the resulting expression, we confirm the determinant identity stated in Theorem 1. \square

3. PROOF OF THEOREM 2

By following exactly the same procedure as done in the last section, we can explicitly evaluate the determinant for the matrix V_m . Subtracting the last row from the other rows transforms V_m into the following one:

$$V'_m = [v'_{i,j}]_{1 \leq i, j \leq m} : v'_{i,j} = \begin{cases} \frac{(1 - Ty_j^2)(x_i - x_m)}{(1 + x_i y_j)(1 + x_m y_j)}, & 1 \leq i < m; \\ \frac{x_m + Ty_j}{1 + x_m y_j}, & i = m. \end{cases}$$

By extracting the common row factor $x_i - x_m$ for $1 \leq i < m$ and the common column factor $\frac{1 - Ty_j^2}{1 + x_my_j}$ for $1 \leq j \leq m$, we find the following determinant equality

$$\det V_m = \det V'_m = \det V''_m \times \prod_{i=1}^{m-1} (x_i - x_m) \prod_{j=1}^m \frac{1 - Ty_j^2}{1 + x_my_j}, \quad (5)$$

where the matrix V''_m is explicitly given by

$$V''_m = [v''_{i,j}]_{1 \leq i,j \leq m} : v''_{i,j} = \begin{cases} \frac{1}{1 + x_i y_j}, & 1 \leq i < m; \\ \frac{x_m + Ty_j}{1 - Ty_j^2}, & i = m. \end{cases}$$

Expanding the determinant along the last row leads us to the equality

$$\det V''_m = \sum_{k=1}^m (-1)^{m+k} \frac{x_m + Ty_k}{1 - Ty_k^2} \det V_m[m, k], \quad (6)$$

where $V_m[m, k]$ is the sub-matrix of V''_m with the m th row and the k th column being crossed out. By making replacements $x_i \rightarrow x_i^{-1}$, we can reformulate Cauchy's double alternant as

$$\det_{1 \leq i,j \leq m} \left[\frac{1}{1 + x_i y_j} \right] = \frac{\prod_{1 \leq i < j \leq m} (x_j - x_i)(y_i - y_j)}{\prod_{1 \leq i,j \leq m} (1 + x_i y_j)}.$$

Then we can evaluate $\det V_m[m, k]$ by the following product expression

$$\begin{aligned} \det V_m[m, k] &= \frac{\prod_{1 \leq i < j < m} (x_j - x_i) \prod_{1 \leq i < j \leq m, (i,j \neq k)} (y_i - y_j)}{\prod_{1 \leq i < m} \prod_{1 \leq j \leq m, (j \neq k)} (1 + x_i y_j)} \\ &= (-1)^{m-k} \frac{\prod_{i=1}^m (1 + x_i y_k)}{\prod_{i \neq m} (x_i - x_m)} \prod_{j \neq k} \frac{1 + x_m y_j}{y_k - y_j} \frac{\prod_{1 \leq i < j \leq m} (x_j - x_i)(y_i - y_j)}{\prod_{1 \leq i,j \leq m} (1 + x_i y_j)}. \end{aligned}$$

Substituting this into (6) yields that

$$\det V''_m = \frac{\prod_{1 \leq i < j \leq m} (x_j - x_i)(y_i - y_j)}{\prod_{1 \leq i,j \leq m} (1 + x_i y_j)} \frac{\prod_{j=1}^m (1 + x_m y_j)}{\prod_{i \neq m} (x_i - x_m)} \sum_{k=1}^m \frac{x_m + Ty_k}{1 - Ty_k^2} \frac{\prod_{i \neq m} (1 + x_i y_k)}{\prod_{j \neq k} (y_k - y_j)}.$$

By decomposing into partial fractions

$$\frac{1}{1 - Ty^2} = \frac{1}{2\sqrt{T}} \times \left\{ \frac{1}{y + \sqrt{T^{-1}}} - \frac{1}{y - \sqrt{T^{-1}}} \right\},$$

we can evaluate the rightmost sum again by divided differences

$$\begin{aligned}
& \sum_{k=1}^m \frac{x_m + Ty_k}{1 - Ty_k^2} \frac{\prod_{i \neq m} (1 + x_i y_k)}{\prod_{j \neq k} (y_k - y_j)} \\
&= \Delta[y_1, y_2, \dots, y_m] \left\{ \frac{x_m + Ty}{1 - Ty^2} \prod_{i \neq m} (x_i + y) \right\} \\
&= \frac{1}{2\sqrt{T}} \Delta[y_1, y_2, \dots, y_m] \left\{ \frac{x_m - \sqrt{T}}{y + \sqrt{T^{-1}}} \prod_{i \neq m} (1 - x/\sqrt{T}) \right\} \\
&\quad - \frac{1}{2\sqrt{T}} \Delta[y_1, y_2, \dots, y_m] \left\{ \frac{x_m + \sqrt{T}}{y - \sqrt{T^{-1}}} \prod_{i \neq m} (1 + x/\sqrt{T}) \right\} \\
&= \frac{1}{2} \prod_{i=1}^m \frac{x_i + \sqrt{T}}{1 - y_i \sqrt{T}} + \frac{1}{2} \prod_{i=1}^m \frac{x_i - \sqrt{T}}{1 + y_i \sqrt{T}}.
\end{aligned}$$

Consequently, we derive the closed form expression

$$\begin{aligned}
\det V_m'' &= \frac{\prod_{1 \leq i < j \leq m} (x_j - x_i)(y_i - y_j) \prod_{j=1}^m (1 + x_m y_j)}{\prod_{1 \leq i, j \leq m} (1 + x_i y_j) \prod_{i \neq m} (x_i - x_m)} \\
&\quad \times \frac{1}{2} \left\{ \prod_{i=1}^m \frac{x_i + \sqrt{T}}{1 - y_i \sqrt{T}} + \prod_{i=1}^m \frac{x_i - \sqrt{T}}{1 + y_i \sqrt{T}} \right\}.
\end{aligned}$$

Finally, substituting this into (5) and then simplifying the resulting expression, we find that

$$\begin{aligned}
\det V_m &= \frac{\prod_{1 \leq i < j \leq m} (x_j - x_i)(y_i - y_j)}{\prod_{1 \leq i, j \leq m} (1 + x_i y_j)} \\
&\quad \times \frac{1}{2} \left\{ \prod_{i=1}^m (x_i + \sqrt{T})(1 + y_i \sqrt{T}) + \prod_{i=1}^m (x_i - \sqrt{T})(1 - y_i \sqrt{T}) \right\}.
\end{aligned}$$

This completes the proof of Theorem 2. \square

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