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Cumann Matamaitice na hÉireann

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The aim of the *Bulletin* is to inform Society members, and the mathematical community at large, about the activities of the Society and about items of general mathematical interest. It appears twice each year. The *Bulletin* is published online free of charge.

The *Bulletin* seeks articles written in an expository style and likely to be of interest to the members of the Society and the wider mathematical community. We encourage informative surveys, biographical and historical articles, short research articles, classroom notes, book reviews and letters. All areas of mathematics will be considered, pure and applied, old and new.

Correspondence concerning books for review in the *Bulletin* should be directed to

`mailto://reviews.ims@gmail.com`

All other correspondence concerning the *Bulletin* should be sent, in the first instance, by e-mail to the Editor at

`mailto://ims.bulletin@gmail.com`

and only if not possible in electronic form to the address

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Submission instructions for authors, back issues of the *Bulletin,* and further information about the Irish Mathematical Society are available on the IMS website

`http://www.irishmathsoc.org/`

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Irish Mathematical Society Bulletin 94, Winter 2024

CONTENTS

EDITORIAL

It is an honour to have been appointed Editor of the Society's Bulletin at the 2024 Annual General Meeting in Belfast. I wish first and foremost to echo the appreciation of the Society, as expressed at its AGM, for Professor Tony O'Farrell's trojan work as Editor over the past fourteen years or so. I've surely been given some big shoes to fill. While Tony never made much of it, or even hinted at the work involved, believe me when I say that producing an issue of this Bulletin is a non-trivial task. The scripts that Tony has written to automate the various steps in the process is impressive. Having learnt the ropes with Tony's assistance, and with the help of David Malone and of Michael Mackey, this next issue of the Bulletin has miraculously seen the light of day.

Another who must have our heartfelt thanks is Ian Short (The Open University) who has produced the Problem Page for as long as Tony has been editor. Thank you most sincerely, Ian, for your service to the Society. Beginning with this issue, J.P. McCarthy (Munster Technological University) has kindly agreed to take up the mantle of the Problem Page. Please support J.P. by suggesting problems and solutions for future issues: `imsproblems@gmail.com`.

The continued success of the Bulletin rests squarely on the quality of submissions. Thankfully, we have several excellent articles in the current issue. It was agreed at the December 2023 IMS Committee Meeting that occasional interviews with members of our mathematical community would be a welcome addition to the Bulletin. Here we have a fascinating interview with Tony O'Farrell conducted by Pauline Mellon (University College Dublin). Also in this issue is a paper on Threshold Concepts in Undergraduate Teaching by Sinéad Breen and Ann O'Shea. As I understand it, a threshold concept is one that, when mastered, opens the door to the next level of understanding and facility with one's subject. Another example in a mathematical context, over and above those discussed in the article in question, might be 'linear independence of vectors': students who understand and internalise this concept can progress in their study of linear algebra and perhaps then on to functional analysis, whereas those who do not succeed in crossing this threshold, or climbing this ladder, are restricted to a lower conceptual level. Also noteworthy in this issue is an enlightening obituary of Petros Serghiou Florides written by Paul D. McNicholas (McMaster University).

I hope that you will enjoy reading these and the many other interesting articles in this issue. Remember that, for a limited time and beginning as soon as possible after the online publication of this Bulletin, a printed (grayscale, not full-colour) and bound copy may be ordered online on a print-on-demand basis at a minimal price[1].

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E-mail address: `ims.bulletin@gmail.com`

[1]Go to `www.lulu.com` and search for *Irish Mathematical Society Bulletin*.

LINKS FOR POSTGRADUATE STUDY

The following are the links provided by Irish Schools for prospective research students in Mathematics:

DCU: mailto://maths@dcu.ie
TUD: mailto://chris.hills@tudublin.ie
ATU: mailto://leo.creedon@atu.ie
MTU: http://mathematics.mtu.ie/datascience
UG: mailto://james.cruickshank@universityofgalway.ie
MU: mailto://mathsstatspg@mu.ie
QUB:
https://www.qub.ac.uk/schools/SchoolofMathematicsandPhysics/Research/culture-environment/PostgraduateResearch/
TCD: http://www.maths.tcd.ie/postgraduate/
UCC: https://www.ucc.ie/en/matsci/study-maths/postgraduate/#d.en.1274864
UCD: mailto://nuria.garcia@ucd.ie
UL: mailto://sarah.mitchell@ul.ie

The remaining schools with Ph.D. programmes in Mathematics are invited to send their preferred link to the editor.

E-mail address: ims.bulletin@gmail.com

NOTICES FROM THE SOCIETY

Officers and Committee Members 2024

President	Dr Leo Creedon	ATU
Vice-President	Dr Rachel Quinlan	UG
Secretary	Dr Derek Kitson	MIC
Treasurer	Dr Cónall Kelly	UCC

Assoc. Prof. C. Boyd, Dr T. Carroll, Dr R. Flatley, Dr R. Gaburro, Dr T. Huettemann, Prof. A. O'Shea, Assoc. Prof. H. Šmigoc, Dr N. Snigireva.

Officers and Committee Members 2025

President	Dr Rachel Quinlan	UG
Vice-President	Prof. David Malone	MU
Secretary	Dr Derek Kitson	MIC
Treasurer	Dr Cónall Kelly	UCC

Assoc. Prof. C. Boyd, Dr R. Flatley, Dr R. Gaburro, Dr T. Huettemann,
Dr P. Ó Catháin, Prof. A. O'Shea, Assoc. Prof. H. Šmigoc, Dr N. Snigireva.

Local Representatives

Belfast	QUB	Prof. M. Mathieu
Carlow	SETU	Dr D. Ó Sé
Cork	MTU	Dr J. P. McCarthy
	UCC	Dr S. Wills
Dublin	DIAS	Prof. T. Dorlas
	TUD, City	Dr D. Mackey
	TUD, Tallaght	Dr C. Stack
	DCU	Prof. B. Nolan
	TCD	Prof. K. Soodhalter
	UCD	Dr R. Levene
Dundalk	DKIT	Mr Seamus Bellew
Galway	UG	Dr J. Cruickshank
Limerick	MIC	Dr B. Kreussler
	UL	Dr Romina Gaburro
Maynooth	MU	Prof. S. Buckley
Sligo	ATU	Dr L. Creedon
Tralee	MTU	Prof. B. Guilfoyle
Waterford	SETU	Dr P. Kirwan

Applying for I.M.S. Membership

(1) The Irish Mathematical Society has reciprocity agreements with the American Mathematical Society, the Deutsche Mathematiker Vereinigung, the Irish Mathematics Teachers' Association, the London Mathematical Society, the Moscow Mathematical Society, the New Zealand Mathematical Society and the Real Sociedad Matemática Española.

(2) The current subscription fees are given below:

Institutional member	€250
Ordinary member	€40
Lifetime member	€400
Student member	€20
DMV, IMTA, NZMS, MMS or RSME reciprocity member	€20
AMS reciprocity member	$25
LMS reciprocity member (paying in Euro)	€20
LMS reciprocity member (paying in Sterling)	£20

(3) The subscription fees listed above should be paid in euro by means of electronic transfer, a cheque drawn on a bank in the Irish Republic, or an international money-order.

The subscription fee for ordinary membership can also be paid in a currency other than euro using a cheque drawn on a foreign bank according to the following schedule:

If paid in United States currency then the subscription fee is US$40.
If paid in sterling then the subscription is £30.
If paid in any other currency then the subscription fee is the amount in that currency equivalent to US$40.

The amounts given in the table above have been set for the current year to allow for bank charges and possible changes in exchange rates.

(4) Any member with a bank account in the Irish Republic may pay his or her subscription by a bank standing order using the form supplied by the Society.

(5) Any ordinary member who has reached the age of 65 years and has been a fully paid up member for the previous five years may pay at the student membership rate.

(6) Those members who have reached 75 years of age, and who have been members in good financial standing with the Society for the previous 15 years, are entitled upon notification to the Treasurer to have their subscription rate reduced to €0.

(7) Subscriptions normally fall due on 1 February each year.

(8) Cheques should be made payable to the Irish Mathematical Society.

(9) Any application for membership must be presented to the Committee of the I.M.S. before it can be accepted. This Committee meets three times each year.

(10) Please send the completed application form, available at

`https://www.irishmathsoc.org/business/imsapplicn_2024.pdf`

with one year's subscription, either by post or by email, to:

Dr Cónall Kelly
School of Mathematical Sciences
Western Gateway Building
University College Cork
Cork, T12 XF62, Ireland
`subscriptions.ims@gmail.com`

PRESIDENT'S REPORT 2024

Committee changes: From January 2025, the IMS has a new President, Rachel Quinlan, and a new Vice President, Professor David Malone. Padraig Ó Catháin (DCU) is a new Committee member of the IMS, having been elected at the AGM in Belfast on August 30, 2024. Congratulations also to Romina Gaburro and Professor Ann O'Shea who were re-elected to the Committee of the IMS at the AGM. The day before, at the IMS Committee meeting of August 29, Professor Peter Lynch (UCD) was co-opted to the Committee for 2024 to (among other things) assist with preparations for the 50th Anniversary of the Society in 2026. Tom Carroll's term as Committee member came to an end, but fortunately he has agreed to take over as Editor of the Bulletin of the IMS and was formally appointed to this role at the Society's AGM in Belfast on August 30, 2024. This summer, Professor Tony O'Farrell wrote in the Bulletin of the IMS that he is stepping down as Editor of the Bulletin. Thanks to Tony for many years of service to the Society, including decades of service as committee member, and until this year as Editor of the Bulletin of the IMS. Tony has served as Secretary and President of the IMS, as well as Editor of the Bulletin for the last 14 years. Thank you Tony.

New members of the Society are always welcome, and I am pleased that we have had over 40 new members join the IMS in the last two years. I offer a particular welcome to Professor Boris Botvinnik of the University of Oregon. Professor Botvinnik was nominated for honorary membership of the Society by Professors David Wraith and Ann O'Shea, and myself. Professor Botvinnik was previously a Distinguished Visiting Scholar at Maynooth University and maintains strong links with the Department of Mathematics and Statistics in Maynooth. Professor Botvinnik was elected an Honorary Member of the Irish Mathematical Society at the AGM on 30$^{\text{th}}$ August 2024.

IMS Bulletin: Thanks to Tony O'Farrell for his many years of service to the IMS and in particular for his many years as Editor of the Bulletin of the IMS. Thanks also to Tom Carroll for agreeing to take up this role as Editor and to the Editorial Board of the Bulletin for their continuing work. The Bulletin is freely available online on the Society's homepage and, in printed form, from lulu.com. Institutional members now receive a complimentary printed copy of each issue.

IMS meetings: The ICEDIM Women in Mathematics Day 2024 took place on Friday 10 May 2024. This one-day conference was organised by the Irish Committee for Equality, Diversity, and Inclusion in Mathematics (ICEDIM) and took place in the Department of Mathematics and Statistics and the Mathematics Application Consortium for Science and Industry (MACSI) at University of Limerick: Link to Meeting Website.

The Society's annual 'September Meeting' was held at Queen's University Belfast on 29$^{\text{th}}$ and 30$^{\text{th}}$ August 2024. The meeting was well organised by Martin Mathieu, Thomas Hüttemann and Salissou Moutari. Tributes were paid to Professors Martin Mathieu and Tony O'Farrell at the meeting. At the AGM members elected new members of the Committee of the IMS, approved an honorary IMS membership, and voted on a proposed rule change on fees for IMS members aged 75+. The report on the 2024 IMS Annual Meeting and the draft Minutes of the 2024 IMS AGM are available elsewhere in the Bulletin. The IMS is very grateful to Martin Mathieu for many years of service to the society. Martin retired soon after the IMS meeting in Belfast. The next Annual Meeting of the IMS will take place in Maynooth University on 28$^{\text{th}}$ and 29$^{\text{th}}$ August 2025.

Other activities and news: The European Mathematical Society and International Mathematical Union issue calls for nominations for several prizes - see both EMS and this link for details. Members are welcome to nominate individuals or to contact Committee members of the IMS to suggest institutional nominations. I attended the Meeting of Presidents of the European Mathematical Society which took place online on 17^{th} May. In July the IMS nominated two IMS members to be members of the EMS Young Academy - EMYA. Róisín Neururer (UCD) continues as chair of EMYA. I participated in the European Mathematical Society Council meeting in Granada on 13^{th} and 14^{th} July. The agenda, draft minutes and papers presented are available here: `https://euromathsoc.org/Council2024`

Much of the work of the IMS is done by the Editorial Board of the Bulletin and by the two subcommittees of the IMS: The Irish Committee for Mathematics Education (ICME) and The Irish Committee for Equality, Diversity and Inclusion in Mathematics (ICEDIM). The IMS Committee meets in person three times per year: it met on 18^{th} December 2023 at Queen's University Belfast (the day of the IMS Christmas Lecture delivered by Prof. Chu); on 9^{th} May 2024 at University of Limerick (the day before the ICEDIM Women in Mathematics Day); on 29^{th} August 2024 at Queen's University Belfast (the day before the AGM); and on 16^{th} December 2024 at Academy House, Dublin.

In December 2023, I received correspondence from the Residents' Association for Dominic Street in Dublin regarding the possible demolition of the birthplace of William Rowan Hamilton. It was determined that the property in question was not the actual birthplace. A new plaque was placed on the birthplace on Dominic St. in Dublin on 16^{th} October 2024. I was pleased to attend this unveiling organized by Dublin City Council, as well as other Hamilton Day events in Dublin including the Hamilton Walk and events at the RIA (the Hamilton Lecture and the Hamilton Prize ceremony).

As President I was pleased to accept an invitation to the Irish Mathematics Teachers' Association 60^{th} Anniversary dinner on 23^{rd} February.

The Astronomical Society of Ireland (ASI) contacted the Society in February regarding the Research and Innovation Bill 2024 which was due to be considered by the DFHERIS Oireachtas Committee (the Oireachtas Committee on Education, Further and Higher Education, Research, Innovation and Science). The ASI had written a letter to the Oireachtas Committee highlighting a need for greater emphasis on funding for fundamental research in the Bill. The IMS Committee agreed that the IMS should add its voice to the discussion, and I sent a letter on 27^{th} February 2024, to the Oireachtas Committee on behalf of the Society.

I attended The Irish Mathematical Trust Awards Ceremony in UCC on 25^{th} May, and presented awards to secondary school teachers, as well as presenting the Fergus Gaines Cup to Fionn Kimber O'Shea. See this link for an account of the ceremony.

Each year mathematical conferences in Ireland are supported by the IMS and I attended one of these - the 22^{nd} Galway Topology Colloquium on 4^{th} and 5^{th} June. I also took the opportunity to promote the IMS at seminars in Ireland, at events abroad (in Spain, Poland and Romania), and at RIA events.

Congratulations to Professor Jerome Sheahan of University of Galway who was conferred with an Honorary Doctor of Science at a ceremony in August 2024 at University of Galway.

It has been a great pleasure to serve the IMS over the last seven years, as a committee member and then as Vice President and President. I am very grateful to all the volunteers who worked so hard for the Society, especially Michael Mackey for maintaining the IMS website; to Clifford Gilmore on social media; to Tom Carroll, Tony O'Farrell and the editorial team at the Bulletin; to the IMS Committee, especially the Treasurer Conall Kelly and the Secretary Derek Kitson; to the chair Ann O'Shea and the members of the Irish Committee for Maths Education; and to the chair Romina Gaburro and the members of the Irish Committee for Equality, Diversity and Inclusion in Mathematics.

I wish the best of luck to the new President and Vice President, to the new IMS Committee member and the new Editor of the Bulletin, as well as the many other volunteers who do very important work in building a welcoming and inclusive community for mathematical research, education and applications in Ireland.

Leo Creedon
December 2024
E-mail address: `president@irishmathsoc.org and leo.creedon@atu.ie`

Draft minutes of the Irish Mathematical Society Annual General Meeting held on 30th August 2024 at Queen's University Belfast

Present: C. Boyd, P. Browne, S. Buckley, M. Bustamante, A. Carnevale, T. Carroll, L. Creedon, R. Flatley, B. Goldsmith, J. Grannell, R. Hill, T. Huettemann, C. Kelly, D. Kitson, B. Kreussler, E. Lingham, P. Lynch, D. Mackey, M. Mackey, J. Maglione, D. Malone, M. Manolaki, M. Mathieu, M. McAuley, P. Mellon, F. Murphy, P. Ó Catháin, A. O'Shea, G. Pfeiffer, K. Pfeiffer, R. Quinlan, T. Rossmann, H. Šmigoc, R. Smith, N. Snigireva, K. Wendland, D. Wraith.

Apologies: R. Gaburro, J.P. McCarthy, A. O'Farrell, S. O'Rourke.

(1) **Agenda / Conflicts of interest**
The agenda was accepted and no conflicts of interest were declared.

(2) **Minutes**
The minutes of the AGM held on 1st September 2023 at University of Limerick were accepted.

(3) **Matters Arising**
None.

(4) **Correspondence**
See President's report.

(5) **President's Report**
L. Creedon gave an overview of the Society's activities and correspondences in the past year. A full report will appear in the Bulletin. M. Mathieu has completed his term on the Committee and was praised for his long service to the Society having served as President, Editor of the Bulletin and committee member. A. O'Farrell has stepped down as Editor of the Bulletin after 14 years and received a round of applause in appreciation for his exceptional service to the Society. Peter Lynch (UCD) has been co-opted to the Committee to assist with preparations for the 50th Anniversary of the Society in 2026. The local organisers of the 2024 Annual Meeting, M. Mathieu, T. Huettemann and S. Moutari, were thanked for their efforts in creating an engaging and convivial meeting.

(6) **New members**
12 new membership applications were approved since the last AGM. The new members are: Milton Assunção, Rory Buckley, Joseph Dillon, Evan Keane, Mark Lyttle, Joshua Maglione, Michael McCauley, Salissou Moutari, Rory O'Brien, Maria Ryan, Eric Scala and Milena Venkova.

(7) **Nominations for honorary membership**
Professor Boris Botvinnik (University of Oregon) was nominated for honorary membership of the Society by A. O'Shea, D. Wraith and L. Creedon. Professor Botvinnik was previously a Distinguished Visiting Scholar at Maynooth University and maintains strong links with the Department of Mathematics and Statistics in Maynooth. The nomination was approved by the meeting.

(8) **Treasurer's Report**
Accounts for 2023 were presented. There was a significant increase in Subscriptions, mainly due to lifetime membership applications, and there was no cost to the Society for the 2023 IMS Annual Meeting. The Committee opted to move surplus funds to savings certificates. For 2024, EMS subscription costs are expected to increase. No shortfall is expected. Members were reminded that charitable donations can be made to the Society through the IMS website. The report was approved.

(9) **Conference support fund**
The following workshops were supported this year:
- •9th Conference on Research in Mathematics Education (DCU): October 2023.
- •Groups in Galway (University of Galway): May 2024
- •Topology Colloquium (University of Galway): June 2024
- •CETL-MSOR 2024 Conference (UL): August 2024

Applications to the fund were encouraged.

(10) **Bulletin**
A. O'Farrell has stepped down as Editor of the Bulletin and T. Carroll has been appointed as the new Editor. Submissions to the Bulletin are encouraged. E. Lingham is handling book reviews. Suggestions for books to review, and for reviewers, are encouraged.

(11) **Report from Irish Committee for Mathematics Education (ICME)**
A. O'Shea reported on ICME activities during the year. A full report will be published on the IMS website. The ICME membership consists of A. O'Shea (Maynooth, Chair), J. Crowley (MTU), R. Flatley (MIC), J. Grannell (UCC), M. Hanley (UCD), K. Pfeiffer (Galway) and R. Quinlan (Galway). The ICME ran a webinar series highlighting mathematics education articles of interest to the community. The talks attracted attendees from 12 universities across Ireland and the UK. Suggestions for future topics are welcomed. A report on second level textbook quality is being finalised. ICME plan to contact publishers, authors and teachers regarding the report. A report on the 2023 Leaving Certificate Exam Paper I has been completed and will be sent to the State Examinations Commission. The Irish Universities Association (IUA) has contacted universities requesting nominations for a review of the second level Mathematics syllabus. The Society has received a request from MTU to gather opinions from IMS members on what should be included in the review. A. Twohill (DCU) attended the General Assembly of the International Commission for Mathematical Instruction (ICMI) in Sydney in July.

(12) **Report from Irish Committee for Equality, Diversity and Inclusion in Mathematics (ICEDIM)**
N. Snigireva reported on activities during the year on behalf of ICEDIM chair R. Gaburro. There are currently eight members of ICEDIM: A. Carnevale (University of Galway), T. Carroll (University College Cork), R. Gaburro (University of Limerick), N. Madden (University of Galway), D. Mackey (TU Dublin), D. Malone (Maynooth University), N. Snigireva (University of Galway) and H. Šmigoc (University College Dublin). The ICEDIM online seminar series commenced in Spring with speakers Pauline Mellon (UCD), Arundhathi Krishnan (MIC), Victoria Sánchez Muñoz (Galway) and Ashley Sheil (MTU). The ICEDIM Women in Mathematics Day 2024 took place on 10th May at University of Limerick as part of the international May12 celebrations. The speakers were Natalia Kopteva (UL), Mariia Kiiko (EGMO), Eabhnat Ní Fhloinn (DCU), Rachel Quinlan (Galway), Sinéad Ryan (TCD). R. Gaburro attended the European Women in Mathematics (EWM) event on 14th July in Seville. An EDI statement for the Society is in preparation. M. Mackey (UCD) was thanked for creating the ICEDIM webpage.

(13) **Proposed rule change for members aged 75+**
The following rule change was discussed and approved by the meeting: *Those members who have reached 75 years of age, and who have been members in good financial standing with the Society for the previous 15 years, are entitled upon notification to the Treasurer to have their subscription rate reduced to €0.*

(14) **Elections**
The current terms of the following committee members came to an end this year: Tom Carroll; Leo Creedon; Romina Gaburro; Rachel Quinlan; Ann O'Shea. T. Carroll and L. Creedon reached the end of three consecutive terms and were consequently not eligible for re-election to the committee. The remaining committee members were eligible for re-election.

The following nominations were received and election to these positions was approved by the meeting:

Candidate	**Role**	**Nominated by**	**Seconded by**
Rachel Quinlan	President	Leo Creedon	Cónall Kelly
David Malone	Vice President	Leo Creedon	Cónall Kelly
Romina Gaburro	Member	Derek Kitson	Rachel Quinlan
Padraig Ó Catháin	Member	Leo Creedon	Derek Kitson
Ann O'Shea	Member	Derek Kitson	Rachel Quinlan

(15) **AOB**
A prize for best poster was presented at the closing of the Annual Meeting. The prize was sponsored by SIAM UKIE.

Derek Kitson (MIC)
derek.kitson@mic.ul.ie

IMS Annual Scientific Meeting 2024
Queen's University Belfast
29 – 30 August 2024

The third *Annual Scientific Meeting of the Irish Mathematical Society* to be held in the 21st century at Queen's University Belfast took place on Thursday 29th and Friday 30th August 2024 in the Mathematical Sciences Research Centre. The meeting joined the *British Mathematical Colloquium* in April 2004 and, in September 2014, it was followed by an *International Workshop on Operator Theory*. The local organising team in 2024 consisted of Thomas Hüttemann, Martin Mathieu and Salissou Moutari.

Financial support was gratefully obtained from the Irish Mathematical Society, the School of Mathematics and Physics of Queen's University, the Mathematical Sciences Research Centre as well as from the UKIE Section of SIAM for the best poster prize.

Four plenary lectures were delivered by Prof. Martin Bridson, FRS (Oxford) on *Soap films, snowflake discs and annuli: the geometry of decision problems in group theory*; Prof. Miguel Bustamante (UCD) on *Open problems on the dynamics of nonlinear resonant wave systems: from FPUT recurrence to gravity water waves, atmospheric waves and millennia-long solar cycles*; Prof. Claire Gormley (UCD) on *Apposite statistical models for network data*; and Prof. Silvia Sabatini (Cologne) on *Positive monotone symplectic manifolds with symmetries.*

The life and work of the late Professor Seán Dineen was remembered in a joint presentation by Pauline Mellon (UCD) and Ray Ryan (UG).

The Society's AGM was held at midday on 30th August and a joint dinner, which doubled-up as a retirement dinner for Prof. Martin Mathieu, was held the previous evening.

Nearly all of the approximately 60 participants at IMS2024.

Twelve further 25 minute contributed talks and a poster session completed the scientific programme. Contributed talks were as follows:

- Patrick Browne (TUS):
  Erdös–Ko–Rado type problems in root systems.
- Padraig Ó Catháin (DCU):
  Monomial representations and complex Hadamard matrices.
- Oisín Flynn-Connolly (Université Sorbonne Paris Nord)
  Higher invariants in homotopy theory.

- Brendan Guilfoyle (MTU):
  Zeros of polynomials and isolated umbilic points.
- Fintan Hegarty (UG):
  Mathematics for content-based language learning.
- Gabor Kiss (QUB):
  Timely Testing and Treatment in Gonorrhoea Control: Insights from Mathematical Modelling.
- Peter Lynch and Michael Mackey (UCD):
  Counting Sets with Surnatural Numbers.
- Michael McAuley (TU Dublin):
  Geometry of Gaussian fields.
- Joshua Maglione (UG):
  Igusa zeta functions and hyperplane arrangements.
- Andrew D. Smith (UCD):
  Spirals in Spaces of Holomorphic Functions.
- Richard J. Smith (UCD):
  The extreme point problem in Lipschitz-free spaces.
- Yinshen Xu and Miguel D. Bustamante (UCD):
  Singularity of bounded vortex-stretching fluid under rotational symmetry.

The UKIE Section of SIAM sponsored a prize of €100 for the best poster, which was awarded to Joseph Dillon (Nashville). The full list of poster presentations is:

- Elife Cetintas (Wuppertal):
  The term 'structure' in mathematical discourse from 1889 to 1942. A bibliometric study by using the Jahrbuch über die Fortschritte der Mathematik.
- Joseph Dillon (Nashville):
  Symmetry and the Riemann zeta function.
- Fergal Murphy (UCC):
  Invariant Polynomials in Harmonic Analysis.
- Zgisis Sakellaris (UCC):
  Near-resonant approximation for the rotating stratified Boussinesq system.

A full record of the meeting is available at `http://ims2024.martinmathieu.net/IMS2024-programme-booklet-final.pdf`

Abstracts of Invited Talks:

Soap films, snowflake discs and annuli: the geometry of decision problems in group theory

Martin Bridson
University of Oxford

Plateau's Problem, rooted in the study of soap films, concerns the nature of discs and minimal surfaces with a given boundary loop. The shimmering appeal of such questions contrasts sharply with the typical reaction to the study of complexity and decision problems in group theory. In this talk, I shall explain how insights of Gromov forged powerful links between these two seemingly disparate pursuits. I shall explain some highlights of the resulting surge of activity, with emphasis on 2- and 3-dimensional spaces and the novel geometries that came to light through the study of Word Problems for groups. I will end by sketching the state of the art concerning the less-understood theory of annuli in geometry and Conjugacy Problems in group theory.

Open problems on the dynamics of nonlinear resonant wave systems: from FPUT recurrence to gravity water waves, atmospheric waves and millennia-long solar cycles

Miguel Bustamante

University College Dublin

In this talk I will present a survey of my research on the dynamics of nonlinear wave systems in the context of wave-wave resonances and their role in solving open problems such as: the Fermi-Pasta-Ulam-Tsingou recurrence, the experimental search for resonances in gravity water waves and atmospheric planetary waves, and the explanation of millennia-long solar cycles.

I will show how, by navigating the boundaries between hyperchaos and integrability, this research is connected with the phenomenon of phase synchronisation in networks and with the theory of integrable systems. Also, I will explain how this work was impacted by areas of 'pure' mathematics, such as number theory, in the search for exact resonances in nonlinear wave systems.

Apposite statistical models for network data

Claire Gormley

University College Dublin

Interactions between entities (*e.g.*, social actors, regions of the brain, phones, countries) are frequently represented using network data. These interactions take a variety of forms, *e.g.*, they may be binary or count, directed or undirected. Additionally, there may be very many or very few entities and they typically form interactions in heterogeneous ways.

Statistical models are useful for modelling such network data as, *e.g.*, they allow us to learn about the processes generating the network data, about patterns within them and/or to predict future network data. Latent position models are apposite and widely used statistical models for network data. Latent position models assume each entity is positioned in a latent space and the likelihood of interactions between entities depends on their relative positioning in the latent space.

This talk will outline some challenges in latent position modelling and propose potential solutions. For example, inferring the dimension of the latent space is difficult and, for simplicity, two dimensions are often used. Here a Bayesian nonparametric framework is employed, inducing shrinkage of the variance of the latent positions across higher dimensions, providing automatic inference on the latent space dimension. Interactions can take different forms, here addressed by developing apposite logistic and Poisson models, for binary and count valued interactions respectively. Heterogeneity within entities is addressed through a mixture modelling framework, providing a clustering of entities. Inference for such latent position models is computationally expensive; here utilising novel surrogate proposal distributions within an Markov chain Monte Carlo (MCMC) algorithm, and a variational inference approach for large networks, are proposed.

These latent position model developments are explored through simulation studies, and practical utility is illustrated through application to real network datasets. Open source software assists with implementation of the developed modelling tools.

This is joint work with Dr Xian Yao Gwee and Dr Michael Fop (University College Dublin).

Positive monotone symplectic manifolds with symmetries

Silvia Sabatini
University of Cologne

Positive monotone symplectic manifolds are the symplectic analogues of Fano varieties, namely they are compact symplectic manifolds for which the first Chern class equals the cohomology class of the symplectic form. In dimension 6, if the positive monotone symplectic manifold is acted on by a circle in a Hamiltonian way, a conjecture of Fine and Panov asserts that it is diffeomorphic to a Fano variety.

In this talk I will report on recent classification results of positive monotone symplectic manifolds endowed with some special Hamiltonian actions of a torus, showing some evidence that they are indeed (homotopy equivalent/homeomorphic/diffeomorphic to) Fano varieties.

Report by Prof. emer. Martin Mathieu (QUB)
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Reports of Sponsored Meetings

Reports received of sponsored meetings held in 2024:

Groups in Galway 2024
16–17 May 2024, University of Galway

The 2024 instalment of the series of meetings 'Groups in Galway' took place at the University of Galway on 16–17 May 2024. The meeting was organised by Angela Carnevale, Joshua Maglione and Tobias Rossmann. It was supported by Athena SWAN Ireland, by the de Brún Centre for Mathematics, by the Irish Mathematical Society and by the Office of the Registrar and Deputy President of the University of Galway. There were 8 invited speakers and over 30 participants.

The talks covered a wide range of topics in contemporary group theory and related fields and were spread over three sessions.

The invited speakers were:

(1) Anna Giordano Bruno (University of Udine)
(2) Alex Evetts (University of Manchester)
(3) Itay Glazer (University of Oxford)
(4) Waltraud Lederle (Dresden University of Technology)
(5) Mireille Soergel (Max Planck Institute for Mathematics in the Sciences)
(6) Mima Stanojkovski (University of Trento)
(7) Matteo Vannacci (University of the Basque Country)
(8) Andoni Zozaya (University of Ljubljana)

Titles and abstracts:

- Anna Giordano Bruno: *A brief history and recent advances in the theory of characterized subgroups of the circle group.*
  A subgroup H of the circle group $\mathbb{T}$ is said to be *characterized* by a sequence of integers $\mathbf{u} = (u_n)_{n\in\mathbb{N}}$ if $H = \{x \in \mathbb{T} : u_n x \to 0\}$. The first part of the talk discusses characterized subgroups of $\mathbb{T}$ and their relevance in several areas of Mathematics where the behaviour of the sequence $(u_n x)_{n\in\mathbb{N}}$ as above is studied, such as Topological Algebra (topologically torsion elements and characterized

subgroups), Harmonic Analysis (sets of convergence of trigonometric series, thin sets) and Number Theory (uniform distribution of sequences).

Recently, generalizations of the notion of a characterized subgroup of $\mathbb{T}$ were introduced, based on weaker notions of convergence, starting from statistical convergence and ending with $\mathcal{I}$-convergence for an ideal $\mathcal{I}$ of $\mathbb{N}$, due to Cartan. A sequence $(y_n)_{n\in\mathbb{N}}$ in $\mathbb{T}$ is said to *$\mathcal{I}$-converge* to a point $y \in \mathbb{T}$, denoted by $y_n \xrightarrow{\mathcal{I}} y$, if $\{n \in \mathbb{N} : y_n \notin U\} \in \mathcal{I}$ for every neighborhood U of y in $\mathbb{T}$. A subgroup H of the circle group $\mathbb{T}$ is said to be *$\mathcal{I}$-characterized* with respect to $\mathcal{I}$ by a sequence of integers $\mathbf{u} = (u_n)_{n\in\mathbb{N}}$ if

$$H = \{x \in \mathbb{T} : u_n x \xrightarrow{\mathcal{I}} 0\}.$$

The second part of the presentation proposes an overview on the results obtained on this new kind of characterized subgroups, with special emphasis on $\mathcal{I}$-characterized subgroups of $\mathbb{T}$.
Based on a joint work with D. Dikranjan, R. Di Santo and H. Weber.

- Alex Evetts: *Twisted conjugacy growth of virtually nilpotent groups.*
  The conjugacy growth function of a finitely generated group is a variation of the standard growth function, counting the number of conjugacy classes intersecting the n-ball in the Cayley graph. The asymptotic behaviour is not a commensurability invariant in general, but the conjugacy growth of finite extensions can be understood via the twisted conjugacy growth function, counting automorphism-twisted conjugacy classes. I will discuss what is known about the asymptotic and formal power series behaviour of (twisted) conjugacy growth, in particular some relatively recent results for certain groups of polynomial growth (i.e. virtually nilpotent groups).

- Itay Glazer: *Fourier and small ball estimates for word maps on unitary groups.*
  Let $w(x, y)$ be a word in a free group. For any group G, w induces a word map $w : G^2 \to G$. For example, the commutator word $w = xyx^{-1}y^{-1}$ induces the commutator map. If G is finite, one can ask what is the probability that $w(g, h)$ is equal to the identity element e, for a pair (g, h) of elements in G, chosen independently at random. In the setting of finite simple groups, Larsen and Shalev showed there exists $\epsilon(w) > 0$ (depending only on w), such that the probability that $w(g, h) = e$ is smaller than $|G|^{-\epsilon(w)}$, whenever G is large enough (depending on w). In this talk, I will discuss analogous questions for compact groups, with a focus on the family of unitary groups; For example, given a word w, and given two independent Haar-random $n \times n$ unitary matrices A and B, what is the probability that $w(A, B)$ is contained in a small ball around the identity matrix?
  Based on a joint work with Nir Avni and Michael Larsen.

- Waltraud Lederle: *Boomerang subgroups.*
  Given a locally compact group, its set of closed subgroups can be endowed with a compact, Hausdorff topology. With this topology, it is called the Chabauty space of the group. Every group acts on its Chabauty space via conjugation. This action has connections to rigidity theory, Margulis' normal subgroup theorem and measure preserving actions of the group via so-called Invariant Random Subgroups (IRS). I will give a gentle introduction into Chabauty spaces and IRS and state a few classical results. I will define boomerang subgroups and explain how special cases of the classical results can be proven via them.
  Based on joint work with Yair Glasner.

- Mireille Soergel: *Dyer groups: Coxeter groups, right-angled Artin groups and more.*
  Dyer groups are a family encompassing both Coxeter groups and right-angled Artin groups. Each of these two classes of groups have natural piecewise Euclidean CAT(0) spaces associated to them: the Davis-Moussong complex for Coxeter groups and the Salvetti complex for right-angled Artin groups. In this talk I will introduce Dyer groups and give some of their properties.

- Mima Stanojkovski: *Studying p-groups via their Pfaffians: isomorphism testing and the PORC conjecture.*
  Given a field K, to each alternating $n \times n$ matrix of linear forms in $K[y_1, \ldots, y_d]$ one can associate a group scheme G over K. In particular, when K is the field of rationals and F is the field of p elements, the F-points $\mathrm{G}(F)$ of G form a group of order p^{n+d} and so, as p varies, one obtains an infinite family of p-groups from G. In this talk, I will present joint work with Josh Maglione and Christopher Voll, as well as ongoing work with Eamonn O'Brien, on the geometric study of automorphisms and isomorphism types of groups associated to small values of the parameters n and d. I will also explain the implications of our work in connection to claims made around Higman's famous PORC conjecture.

- Matteo Vannacci: *Profinite groups of finite probabilistic virtual rank.*
  A profinite group G carries naturally the structure of a probability space, namely with respect to its normalised Haar measure. We study the probability $Q(G, k)$ that k Haar-random elements generate an open subgroup in the profinite group G. In particular, in this talk I will introduce the probabilistic virtual rank $\mathrm{pvr}(G)$ of G; that is, the smallest k such that $Q(G, k) = 1$. We will discuss some key theorems and open problems about random generation in profinite groups, with a view toward finite direct products of hereditarily just infinite profinite groups. Classic examples of the latter type of groups are semisimple algebraic groups over non-archimedean local fields. This is joint work with Benjamin Klopsch and Davide Veronelli.

- Andoni Zozaya: *Linearity of compact analytic groups over domains of characteristic zero.*
  A p-adic analytic group is a topological group that is endowed with an analytic manifold structure over $\mathbb{Z}_p$, the ring of p-adic integers. This definition can be extended by considering the manifold structure over more general pro-p domains, such as the power series rings $\mathbb{Z}_p[[t_1, \ldots, t_m]]$ or $\mathbb{F}_p[[t_1, \ldots, t_m]]$ (where $\mathbb{F}_p$ denotes the finite field of p elements).

  Lazard established already in the 1960s that compact p-adic analytic groups are linear, as they can be embedded as a closed subgroup within the group of invertible matrices over $\mathbb{Z}_p$. Nonetheless, the question of the linearity of analytic groups over more general domains remains unsolved.

  In this talk, we shed some light to this question by proving that when the coefficient ring is of characteristic zero, every compact analytic group is linear. We will provide background on the problem and outline the strategy of our argument. Joint with M. Casals-Ruiz.

The conference website is to be found at `https://groupsingalway.github.io/posts/GiG2024`.

Report by Angela Carnevale, University of Galway
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22nd Galway Topology Colloquium
4–5 June 2024, University of Galway

The 22nd Galway Topology Colloquium took place at the University of Galway on 4–5 June 2024. It was organised by Aisling McCluskey and Nina Snigireva (both from University of Galway).

The Colloquium Series was established in 1997 by Aisling McCluskey and Paul Gartside and rotates annually between the centres of topological research in Ireland and the UK (apart from the Covid period). The colloquia provide postgraduates and early career researchers in topology, as well as seasoned academics, with the opportunity to share their research in a friendly, informal and supportive environment. Topology is interpreted broadly and includes set-theoretic topology, algebraic topology, continua theory, topological dynamics, as well as cross fertilisation between topology and category theory, order theory, metric space theory, and analysis.

There were a total of seven invited speakers:

- Paul Bankston (Marquette University, US)
  Betweenness and Equidistance in Hyperspaces.
- K. P. Hart (TU Delft)
  Many subalgebras of $\mathcal{P}(\omega)/fin$: A tale of mass murder and mayhem.
- John C. Mayer (University of Alabama at Birminghmam, US)
  Complex Dynamics: Polynomials, Julia Sets, Parameter Spaces, and Laminations.
- Anca Mustata (University College Cork)
  Families of manifolds with large symmetry groups.
- Richard Smith (University College Dublin)
  de Leeuw representations of functionals on Lipschitz spaces.
- Filip Strobin (Łódź University of Technology, Poland)
  Rate of convergence in the deterministic version of the chaos game algorithm.
- Stephen Watson (York University, Toronto)
  On the existence of Nash equilibrium.

The following speakers also contributed talks::

- Daron Anderson
  Non-Block Points in Hereditarily Decomposable Continua.

- Christopher Boyd (University College Dublin)
  Order Continuous Polynomials.
- Robin Knight (University of Oxford)
- Simo S. Mthethwa (University of KwaZulu-Natal, South Africa)
  A few points in pointfree topology.

Abstracts of Invited Talks:

- Paul Bankston: *Betweenness and Equidistance in Hyperspaces.*
  We explore what it means when one compact set lies between - or is equidistant from - two others, in the context of metric spaces. We are also interested in notions of convexity that arise from these considerations.

- Klaas Pieter Hart: *Many subalgebras of $\mathcal{P}(\omega)/fin$: A tale of mass murder and mayhem.*
  In answer to a question on MathOverflow we show that the Boolean algebra $\mathcal{P}(\omega)/fin$ contains a family $\{\mathcal{B}_X : X \subseteq \mathfrak{c}\}$ of subalgebras with the property that $X \subseteq Y$ implies $\mathcal{B}_Y$ is a subalgebra of $\mathcal{B}_X$ and if $X \not\subseteq Y$ then $\mathcal{B}_Y$ is not embeddable into $\mathcal{B}_X$. The proof proceeds by Stone duality and the construction of a suitable family of separable zero-dimensional compact spaces.

- John Mayer: *Complex Dynamics: Polynomials, Julia Sets, Parameter Spaces, and Laminations.*
  Laminations are a combinatorial and topological way to study connected Julia sets of polynomials. While each locally connected Julia set has a corresponding lamination, laminations also give information about the structure of the parameter space of degree $d \geq 2$ polynomials with connected Julia sets. A d-invariant lamination of the unit disc consists of a closed collection of chords, called leaves, which meet at most at their endpoints, and which is forward and backward invariant under the angle-d-tupling map on the unit circle. Of particular interest are leaves in a lamination which are periodic, return for the first time by the identity, and whose endpoints are in different orbits. Such leaves play an important and understood role in the parameter space of quadratic polynomials and in the parameter spaces of unicritical higher degree polynomials, but more study is needed in the more general case of multiple criticality. Here we focus on the first case where there are open questions about the laminations: the angle-tripling map corresponding to degree 3 polynomials with connected Julia set.
  Coauthors: Brittany E. Burdette and Thomas C. Sirna.

- Anca Mustata: *Families of manifolds with large symmetry groups*
  In this talk we discuss families of complex projective varieties with relatively large groups of symmetry, which can be found as moduli spaces of objects in highly symmetric complex projective hypersurfaces. We discuss special families of $(n-3)$-dimensional complex varieties whose automorphism groups lie inside the $(n+1)$-th symmetric group. A particular case is the Wiman-Edge pencil of genus 6 complex projective curves. First found in a paper in 1895 by A. Wiman, its modular interpretation was first found by Ph. Candelas, X. de la Ossa, B. van Geemen, D. van Straten in 2012 and explained by Zagier (2014) and I. Dolgachev, B. Farb, E. Looijenga (2018), who proved that every smooth projective curve of genus 6 endowed with a faithful A5-action is equivariantly isomorphic with a member of this pencil.

- Richard Smith: *de Leeuw representations of functionals on Lipschitz spaces.*
Let $\mathrm{Lip}_0(M)$ be the Banach space of Lipschitz functions on a complete metric space (M,d) that vanish at a point $0 \in M$. This has an isometric predual $\mathcal{F}(M) \subset \mathrm{Lip}_0(M)^*$, called the Lipschitz-free (hereafter free) space over M. Free spaces are at the interface between functional analysis, metric geometry and optimal transport theory. They are the canonical way to express metric spaces in functional analytic terms, analogously to how compact Hausdorff spaces can be expressed using $C(K)$-spaces.

  We still have a quite poor understanding of the spaces $\mathcal{F}(M)$ and (even more so) their biduals $\mathrm{Lip}_0(M)^*$. Their structure can be probed using the 'de Leeuw transform', which yields representations of each functional on the Lipschitz space $\mathrm{Lip}_0(M)$ in the form of (non-unique) measures on the Stone-Čech compactification $\beta\widetilde{M}$ of $\widetilde{M} := \{(x,y) \in M \times M \,:\, x \neq y\}$.

  In this talk we introduce the above and show how topological concepts such as the uniform compactification and 'Lipschitz realcompactification' of (M,d), can be used to study de Leuuw representations of elements of $\mathcal{F}(M)$ and $\mathrm{Lip}_0(M)^*$ and thus shed light on the structure of these spaces. Along the way we introduce a 'metric bidual' of (M,d), whose relationship with (M,d) is analogous to the relationship between a Banach space and its bidual.

  This is joint work with Ramón Aliaga (Universitat Politècnica de València) and Eva Pernecká (Czech Technical University, Prague).

- Filip Strobin: *Rate of convergence in the deterministic version of the chaos game algorithm.*
The validity of the classical chaos game algorithm for generating images of attractors of contractive iterated function systems can be explained by the fact that, with probability 1, a randomly chosen sequence from a given finite alphabet is disjunctive, meaning that it contains all finite words from that alphabet as its subwords. In particular, given a disjunctive sequence, the generated orbit will approximate the attractor. During my talk I will explain how to measure the rate of convergence of orbits to the attractors and show that additional properties of disjunctive sequences give some control over that rate. On the other hand, I will show that a typical (in the sense of Baire's category and even porosity) disjunctive sequence does not give any control over the rate of convergence. Finally, I will present the result which shows that the situation can be completely different from the probabilistic point of view - in some cases, with probability 1, the rate of convergence of a randomly chosen driver is controlled by the dimension of the invariant measure.

  Results related to the deterministic chaos game is joint work with Krzysztof Leśniak and Nina Snigireva, and can be found in K. Leśniak, N. Snigireva, F. Strobin, *Topological prevalence of variable speed of convergence in the deterministic chaos game*, Rev. Real Acad. Cienc. Exactas Fis. Nat. Ser. A-Mat. 118, 157 (2024) and in K. Leśniak, N. Snigireva, F. Strobin, *Rate of convergence in the disjunctive chaos game algorithm*, Chaos 32 (2022), no. 1, Paper No. 013110.

  Results related to the probabilistic chaos game can be found in B. Bárány, N. Jurga, I. Kolossváry, *On the convergence rate of the chaos game*, Int. Math. Res. Not. 2023 (2023), no. 5, 4456-4500.

- Stephen Watson: *On the existence of Nash equilibrium.*
Nash equilibrium is regarded as one of the most important notions in Game Theory. The concept dates back to at least Cournout. However, its current

formalization is due to Nash, whose original proof, given in 1950, relies on Kakutani's fixed point theorem. One year later, Nash gave a different proof, which uses Brouwer's fixed point theorem.

The self-contained proof here makes no use of fixed point theorems. Our proof can be split in two parts. The first part introduces two new notions: root function and distributed equilibrium. A root function is a map from the set of mixed strategy profiles to the set of pure strategy profiles. A distributed equilibrium is a subset of mixed strategy profiles that generalizes Nash equilibrium. In the second part, elaborating an argument used by McLennan and Tourky, we show that arbitrarily small distributed equilibria always exist. By means of compactness, we obtain the existence of a Nash equilibrium.

Joint work with D. Carpentiere.

The conference website is to be found at `https://maths.nuigalway.ie/galwaytopology/`.

Report by Aisling McCluskey and Nina Snigireva, University of Galway
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CETL-MSOR 2024
28–30 August 2024, University of Limerick

CETL-MSOR 2024 was hosted in the University of Limerick on the 28th, 29th and 30th of August this year. This annual conference is a meeting of practitioners of the teaching, learning and support of mathematics, statistics and operations research in higher education. This was only the second time this conference was held outside the UK. The conference themes this year were:

- Linking research and practice in mathematics and statistics education in Higher Education – opportunities and challenges;
- Teaching mathematics for mathematics specialist and non-mathematics specialist groups;
- The changing nature of mathematics and statistics learning support;
- Exploring the affective domain in third level mathematics and statistics education.

Eighty delegates from the UK, the US, Ireland and Eastern Europe were in attendance. As part of the conference schedule, the delegates were given a tour of UL Glucksman

Library which homes a fascinating array of ancient and very rare mathematical texts. Professor Kenneth Stanton, the executive dean of the Faculty of Science and Engineering, officially opened the conference.

Pictured below are our keynotes speakers with Dr Olivia Fitzmaurice, Chair of the 2024 CETL-MSOR conference organisation committee.

Dr Eabhnat Ní Fhloinn (DCU), Dr Olivia Fitzmaurice (UL - conference chair), Dr Joe Kyle (Birmingham University), Dr Rafael De Andrade Moralis (Maynooth University).

Our keynote speakers were:

Dr Eabhnat Ní Fhloinn (Dublin City University):
Mathematics Learning Support in Ireland: Do we know it Inside Out?
Abstract: It is over 20 years since the first Mathematics Learning Support (MLS) Centre opened in the University of Limerick, where we now find ourselves celebrating CETL-MSOR 2024. During this time, MLS has expanded and become viewed as a mainstream support service in many Higher Education Institutes. The Irish Mathematics Learning Support Network (IMLSN) has played a pivotal role in this development, and in bringing together practitioners and researchers from around the country, much as other similar networks have done in the UK, Scotland and Germany. In this talk, we consider the historical challenges faced by MLS in Ireland, look at what we learned from these, and explore any new challenges facing us in the coming years. We ask the question - after more than twenty years of MLS in Ireland, do we know it inside out?

Dr Rafael De Andrade Moralis (Maynooth University):
Notes and Tricks for Teaching Statistics using Music and Magic.
Abstract: In this talk, I will share my recent experience using musical parodies and magic tricks to teach different statistical concepts. I will draw parallels between my lecturing experience in Brazil and in Ireland, and discuss how I use general pedagogy and active methodologies to encourage student participation. I will also discuss successful approaches, as well as other approaches still under development. I will showcase some of these activities in the context of explaining the concepts of conditional probability, p-values, and hypothesis tests. Finally, I will present the tools and equipment I currently use to produce music videos to teach statistics and give tips on what I think has helped improve the quality of the materials I have been producing.

Dr Joe Kyle (Birmingham University):
Beyond the Grave Morrice.
Dr Kyle gave the closing plenary presentation in which he discussed developments in mathematics education, mainly the use of AI, and concluded with insights gained over the course of conference.
Abstract: Casting a glance backwards as well as looking into the future (as far as that is possible), this talk will take upon itself the task of responding to and reacting to developments reported this year at the Limerick conference. As we struggle to harness the power of generative AI (or is it we who are being harnessed?) we look back to tried and tested axioms that may guide us on the new adventures ahead. And, as problem-solving is at the heart of mathematics, and problems are at the heart of problem-solving, there may be the odd puzzle to keep us all awake.

Dr Ciarán Mac an Bhaird was the recipient of 2024 international award 'The Lawson-Croft Award for Outstanding Achievement in Mathematics and Statistics Support'. Dr Mac an Bhaird is pictured with Professor Michael Grove, Deputy Pro-Vice-Chancellor, University of Birmingham who made the announcement at the conference.

We would sincerely like the thank our sponsors who ensured the conference was a success: EPI*STEM, the National Centre for STEM Education; the Centre for Transformative Learning (UL); The President's Office (UL); The Department of Mathematics and Statistics (UL); and the Irish Mathematical Society (IMS).

The Local Organisation Committee comprised: Dr Olivia Fitzmaurice - Chairperson, Dr Richard Walsh, Dr Aoife Guerin, Dr Patrick Johnson, Dr Niamh O'Meara, Prof. John O'Donoghue.

Report by Olivia Fitzmaurice, University of Limerick
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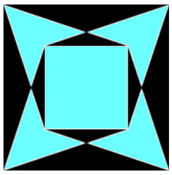

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Petros Serghiou Florides 1937–2023

PAUL D. MCNICHOLAS

FIGURE 1. Petros Florides at his inauguration as Pro-Chancellor of Trinity College Dublin, in the Provost's House, on 4th November 2010.

Petros Florides was born in Lapithos, Cyprus on 16th February 1937. He was the fourth of five children, and youngest son, born to parents Serghios Florides and Panayiota Florides (née Hadjiphotiou). His aptitude for mathematics was recognized by one of his teachers, prompting a move to London, England, in February 1954, where he was subsequently joined by his mother and, later, by his father and his sister Nitsa. There he studied at Northern Polytechnic culminating with the award of a BSc (Special) in Mathematics in 1958 from the University of London.

Key words and phrases. Petros Florides, Obituary.
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The author greatly appreciates the very helpful suggestions and points of information provided by Brien Nolan of Dublin City University. The author is grateful to Arthur White of the School of Computer Science and Statistics, and to Tristan McLoughlin and John Stalker of the School of Mathematics, at Trinity College Dublin for their assistance with a number of details. The author extends special thanks to Despina, Serghios, Andros, and Constantinos Florides.

Petros continued to doctoral studies at Royal Holloway College, where he worked under the guidance of Sir William Hunter McCrea FRS. He submitted his thesis entitled *Problems in Relativity Theory and Relativistic Cosmology* on 29th September 1960, and was awarded a PhD a few weeks later at just 23 years old. From there, Petros moved to Dublin to work as a research scholar with John L. Synge FRS in the School of Theoretical Physics at the Dublin Institute for Advanced Studies (DIAS). He wasted no time, with his position at DIAS starting on 1st October 1960, just two days after he submitted his PhD thesis! While still working at DIAS, Petros took a part-time lecturer position at University College Dublin (UCD) in 1961, where he was responsible for courses on relativity theory. In 1962, he accepted a permanent position as a lecturer in applied mathematics at Trinity College Dublin. Shortly thereafter, in 1963, he was elected a Fellow of the Royal Astronomical Society.

Petros held both McCrea and Synge in great esteem. He often spoke fondly of them and their time together. He continued to work with Synge as he settled into his position at Trinity College Dublin and for several years thereafter. Not only was Synge the reason Petros moved to Dublin, but he remained a friend until his death in 1995. Petros was also close to Synge's daughter, Cathleen Synge Morawetz, and enjoyed a dear and lifelong friendship with McCrea that extended beyond both men to their respective families. On the subject of friendships, the late, great Irish poet and novelist Brendan Kennelly was one of the first people Petros met when he came to Dublin. They both lived in Trinity College Dublin and quickly struck up a friendship, going out regularly together and reciting poetry – Kennelly in English and Petros in Greek. They, too, were lifelong friends.

Petros capitalized on Synge's connection to Trinity College Dublin to establish a prize and a lecture in his honour in 1992. The J.L. Synge Public Lectures and the J.L. Synge Prize in Mathematics are still given in alternate years. The former is a very popular event at Trinity College Dublin, and Petros took great pleasure in hosting these lectures. He used the occasion to recall to his audience Synge's life and achievements. It is a tribute both to Synge's lasting influence on general relativity as well as to Petros' connections in the international community and his powers of persuasion that the list of speakers at this biennial event is a roll call of some of the most outstanding figures in the area over the last 50 years, including: Sir Hermann Bondi FRS, who delivered the inaugural J.L. Synge Public Lecture in 1992; Nobel laureate Sir Roger Penrose FRS (1996); Roy Kerr FRS (2008); Sir Martin Rees FRS (2012); and Dame Jocelyn Bell Burnell FRS (2014).

At the invitation of the Royal Society, Petros wrote a wonderful biographical memoir of Synge, published in 2008 [17], where he takes great care to pull information from many and varied sources, including his own earlier work [16]. He also wrote respective obituaries for Synge [15] and McCrea [19]. In this memoir [17], Petros recounts McCrea's praise for Synge as a lecturer:

> The greatest living lecturer in mathematics lives in Dublin. . . Every lecture he gives is the superb performance of a master—or ought I say maestro?

Of these words, Petros writes [17]:

> It may be added that the word maestro is in no way misplaced.

This is one of very many indications of the importance Petros placed on the art and practice of lecturing.

Petros was, for four decades, one of the most well known, best liked, and highest regarded lecturers at Trinity College Dublin. Every lecture was a well-rehearsed performance. His lectures were carefully planned, so much so that he developed a vast repository of beautiful lecture notes. In the lecture theatre, he was superb. He dressed immaculately, even wearing black tie on rare occasions in preparation for an imminent College event, and spoke in a gentle, mellifluous voice with a Greek accent. His writing on the blackboard was exquisite, his notation was memorably clever, and he drew magnificent diagrams to help students grasp concepts. In fact, he could draw an essentially perfect circle on the blackboard with seemingly little effort. He solved example problems with a slickness that gave students something to aspire to. And he did all this with a smile and incredible enthusiasm, with the latter often causing lectures to run slightly over time.

Even in the face of unexpected questions or comments from students, he was relentlessly kind. For example, when a student wondered about his pronunciation of the word analogous (a-nal-o-goose), Petros just smiled and reminded the student that it was a Greek word. He made complicated ideas seem disarmingly straightforward and brought an elegance to mathematical methods that could easily have been made to seem clunky. On the occasions when he was discussing material that had an inherent elegance, he elevated it to a level of class that seemed almost out of place in a mathematics lecture. McCrea's choice of the word maestro in praise of Synge would be in no way misplaced when describing Petros as a lecturer. Few, if any, were better and none cared more about their students.

Petros worked at Trinity College Dublin for 40 years, retiring as a Senior Fellow in 2002, having been elected to Fellowship in 1971. He served Trinity College Dublin in many ways, including a significant stint as Warden of Trinity Hall (1989–1996) when he helped resist efforts to sell Halls [3], and service on many College committees including the Board, the Academic Council, and the Central Fellowship Committee. During his years as a Senior Fellow Emeritus, Petros remained active in College life. He was elected Pro-Chancellor in 2010 and greatly enjoyed presiding over Commencements in that capacity. At his inauguration ceremony, which was held in the Provost's House, then-provost John Hegarty welcomed Petros to the role as follows [27]:

> It is with the greatest pleasure that I welcome a colleague with a tremendous record of teaching, scholarship and contribution, a colleague with such a tremendous record of service to the College and wider community, to the Pro-Chancellorship of the University of Dublin.

As part of his speech, Petros in turn reflected on the importance of presiding over Commencements noting [3]:

> . . . the strong bond that always existed between me and my students, and my sincere and deep empathy with them, will enable me to enhance this experience and make it a memorable one.

Petros was not exaggerating when he spoke of the 'strong bond' he had with his students, and he certainly made Commencements memorable with his grace, kindness, and the marrying of Latin words with his Greek accent. He greatly enjoyed occasions when he was complimented on the latter. In fact, when writing to Senators in an Election Message ahead of a Pro-Chancellorship election, Petros outlined the most important duty of the role, as he saw it, and alluded to 'Latin with a touch of Greek pronunciation':

> The most important and frequent duty of a Pro-Chancellor is to officiate at *Commencements*, the degree-conferring ceremonies. These are performed in Latin and they are, undoubtedly, one of the most solemn public functions of the university. I believe that I can fulfill this particular duty very well, bringing '*a warm, genuine, human touch to the office, along with an easy and natural dignity, and all that is good*', to quote the unsolicited observations of a respected colleague in a recent letter of

> support for my candidature. With regard to Latin, I regret to say that I have had no formal education in this subject. But this did not prevent me from discharging, in 1976/7, the duties of Junior Proctor eloquently, thanks to a little coaching by the late Dr D.E.W. Wormell, the Regius Professor of Latin at the time. Indeed, as was then said, Latin with a touch of Greek pronunciation can sound quite beautiful!

In addition to being a maestro in the lecturer theatre and a great servant to Trinity College Dublin, Petros was a renowned researcher. His interests focused on Einstein's theory of relativity, venturing now and again into the fields of cosmology and astrophysics. Starting with his PhD work with McCrea, Petros made several important contributions to the problem of energy and its localization. For example, he was the first to show that the charge of a spherically symmetric (charged) system contributes to the gravitational mass of the system an amount which is exactly the mass-equivalent of the electric energy of the system ([4], [7], [9]). In another important contribution, he showed that the Tolman and Møller mass-energy formulae in general relativity, which for forty years had been considered completely independent and unrelated, are in fact completely equivalent [13].

His major work with Synge concerns the formulation of approximate methods for the solution of the Einstein field equations ([2], [20], [21], [22], [24]). Extensive applications for these methods are detailed in [5], [6], [20], and [25]. From these papers, it emerged that a rotating sphere [25] and a rotating spheriod [6] are possible sources of the Kerr (exterior) solution. Notably, in the aforementioned approximation methods, the exterior and interior (inside the matter) fields are calculated simultaneously. Thus, [25] and [6] also provide (approximate) Kerr solutions. Petros also obtained a number of interior exact solutions for the Einstein and Einstein-Maxwell equations ([7], [9], [23]). Of particular interest, perhaps, is the 'new interior Schwarzschild solution' [23], sometimes referred to as the Florides solution. It is, by far, the simplest interior solution and is characterized by the complete absence of radial stresses; physically, it represents the field of an 'Einstein cluster'.

His work on the Robertson-Walker metrics, and their generalizations, is perhaps more important in differential geometry than in cosmology ([8], [10], [11]). This work establishes the rather unexpected result that, independently of dimensionality and signature, the necessary and sufficient condition for a Robertson-Walker metric to be expressible in time-independent form is for the Robertson-Walker manifold to be of constant curvature. In later work, Petros was concerned with the formulation of a model for steadily rotating prolate galaxies ([12], [14]).

Petros supervised a number of research students during his time at Trinity College Dublin. Among these were Phelim Boyle (PhD, 1969), who went on to do seminal work in mathematical finance, introducing Monte Carlo methods in option pricing [1], Richard Jones (PhD, 1970), and Brendan Guilfoyle (MSc, 1991), who is now on the faculty at Munster Technological University (MTU) Tralee. Guilfoyle went on to do a PhD (1997) under Karen Uhlenbeck at the University of Texas at Austin, and continues an active research career in differential geometry and geometric analysis, with much of his work bearing the clear stamp of a relativist.

Petros went to great efforts to communicate important scientific ideas to the general public. The J.L. Synge Public Lectures were a wonderful example of this. In addition to organizing public lectures, Petros also delivered them expertly. His public lectures on the life and work of Albert Einstein FRS were a particular favourite for many. Petros' standing in the general relativity community and his powers of persuasion have been central to the success of the J.L. Synge Public Lectures. These attributes also came to the fore when Malcolm MacCallum, secretary of the International Society on General

Relativity and Gravitation, invited the local general relativity community to consider hosting the Society's 17th International Conference. Petros subsequently chaired the local organizing committee for this conference, which took place in July 2004, joined by Guilfoyle (MTU), Peter Hogan (UCD), Brien Nolan (Dublin City University), Niall Ó Murchadha (University College Cork), and Adrian Ottewill (UCD). His abilities were vital to the success of the conference, not least in securing the ideal venue (the RDS) at excellent rates and SFI funding for the conference, and in convincing his good friend and former Trinity College Dublin colleague, President Mary MacAleese, to preside over the opening ceremony. What would normally have been a meeting of interest almost exclusively to its participants gained much greater exposure when, a few weeks before the conference took place, Stephen Hawking FRS announced he had solved the much-debated Black Hole Information Paradox. At late notice, a slot on the schedule was found, and much media brouhaha ensued. Petros chaired the session in which Hawking presented his results to an audience of some 600 physicists, dozens of journalists, and a handful of others; ultimately, his PhD student Christophe Galfard delivered the presentation and Hawking subsequently fielded questions, with Kip Thorne moderating the Q&A session. Petros put his inimitable stamp on the session in his introduction, noting that while Einstein maintained nothing travels faster than the speed of light, this hypothesis had been invalidated by the speed at which Hawking's announcement had spread around the world!

Petros remained active in research long after he retired from lecturing. As recently as 2013, he published a preprint about what he called 'the midwife' [18]. The abstract Petros wrote for this preprint is a wonderful summary:

> Long before the general theory of relativity was finally formulated in 1916, arguments based entirely on Einstein's equivalence principle predicted the well known phenomenon of the gravitational red shift. Precisely the same arguments are widely being used today to derive the same phenomenon. Accordingly, it is often claimed that the observed gravitational red shift is a verification of the equivalence principle rather than a verification of the full theory of general relativity. Here we show that, contrary to these claims, the arguments based on the equivalence principle are false and that *only the full theory of general relativity can correctly and unambiguously predict the gravitational red shift.*

Petros was a scientist who thought very deeply about theories and problems. His 2013 preprint [18] is a perfect illustration of this: not only does he question arguments of the most famous scientist of the 20th century, but he proposes an elegant alternative.

Besides his stature as an eminent lecturer and scientist, Petros was a human being of the very highest quality – the kind of person who was not only great, but who stood out as great even when compared to other great people. No matter how hopeless or dire a situation seemed, Petros managed to come up with a smile as well as good advice. He was both wise and compassionate: a wonderful combination. From offering a stressed-out student a cigar, when smoking was allowed in faculty offices, to sitting with a struggling student over coffee, he was consistently and unerringly kind. The presence of such a brilliant and yet kind man had a notable impact on his department as well as the broader community at Trinity College Dublin. Charles Mollan, who wrote extensively about the lives and contributions of eminent Irish scientists, appreciated the broad impact of Petros' presence [26]:

> . . . Florides, having moved there [DIAS] and then transferred to the Department of Mathematics in Trinity College, spent almost the whole of his academic life in Dublin, to the great advantage of the country and his many grateful students and friends.

Petros was extremely proud of his Greek Cypriot heritage and, despite having left as a teenager, he carried his love of Cyprus with him throughout his life. He was a founding member, past-president, and active patron of the Irish-Hellenic Society. He served as a member of the preparatory committee for the University of Cyprus — the first public university in the Republic of Cyprus — and as chair of the selection committee for the early cohorts of academic members of the Department of Mathematics and Statistics. He was also a founding member of the Hellenic Society on Relativity, Gravitation and Cosmology.

Above all else, Petros loved his family and the time he spent with them. He married Despina in 1967, and they had three sons: Serghios, Andros, and Constantinos. Petros loved literature, music, and the arts, but he was especially fond of music and poetry. Fittingly, Serghios read one of his favourite poems, 'Ithaca' by Cavafy, at his funeral service. Petros was a fine violinist and often treated his household to classical music, Hungarian gypsy dances, and Greek and Cypriot songs. As Andros pointed out in his eulogy:

> His claim to fame was that he once played the violin with a famous physicist named Lanczos who in turn had once played the violin with Einstein, bringing him another step closer to his hero.

Petros was, at heart, a deeply loving and passionate person. In his copy of Synge's *Kandleman's Krim* [28], which he acquired in January 1962, Petros underlined some sentences here and there. One such sentence should strike a note with those who had the great good fortunate to know him:

> For what is life but a passionate pilgrimage?

Petros died on 30$^{\text{th}}$ October 2023 in Athens with Despina by his side. Petros is survived by Despina, Serghios, Andros, Constantinos, and his grandchildren Anna, Enzo, Sofia, and Alexia. His funeral service was held at Trinity College Chapel on 7$^{\text{th}}$ November 2023. Petros' son Andros delivered a moving eulogy, which concluded:

> I remember being asked countless times as a kid what I wanted to be when I grew up. I never had a clear answer back then. Today, I know exactly what I want to be: like him.

Something we could all aspire to.

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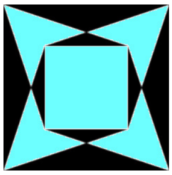

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An Interview with Professor Anthony G. O'Farrell

PAULINE MELLON

Abstract. This article consists of an interview with Prof. Anthony G. O'Farrell charting some of his mathematical and life's journey over the past eight decades. It begins with his mathematical awakening in primary school in 1950s Ireland, through secondary school in Templemore and Drimnagh Castle, his undergraduate days in UCD via a Met Cadetship with the Met Office, and his postgraduate work in the US after receiving the NUI Travelling Studentship. It explores his return to take the first Professorship to ever be offered to a lay person in St. Patrick's College, Maynooth at the age of just 27 years. It touches on his subsequent 50 plus years of research in mathematics, in which he remains active to the current day, and his overall contribution to Irish mathematical life.

1. Introduction

I recorded an informal interview with Tony O'Farrell at his home on October 4$^{\text{th}}$ 2024. I edited the transcript of this interview, which Tony then reviewed and edited. I subsequently edited the final version of the document.

PM: *The first question is to ask you about your early schooling, in particular, when did you realise that you were interested in mathematics?*

AOF: Well I remember a few things. I started school in Roscrea for a few months and then we moved over to Templemore and I went to the Convent of Mercy in Templemore. I remember when this nun – now I don't remember her name – showed us how you could add numbers expressed in decimal form – you could add 21 and 33 by just adding the digits up. The light bulb went off when I saw that. That was a non-obvious kind of a trick, which was going to work for adding these big numbers together, and I thought that is pretty good, and it is!

Actually the algorithms that we learned for adding and multiplying decimal numbers were a major advance – they are old but they are effective. I used to do them for fun. Once I learned about multiplication, I would write down a random 10 digit number and another random 10 digit number and then multiply them together so I would have a big long page like this of all the stuff set out. I enjoyed those kind of things.

The next piece of mathematics that I remember that was interesting was the quadratic equation. In 3rd or 4th class we used to graph quadratics – we had graph paper – we had copies with squares on them. You would plot some points on the quadratic and you joined them up by hand.

I remember Rory Geoghegan telling me – his big brother was in 5th or 6th class – that in 5th or 6th class they did something else. They didn't just plot those things but they explained why they looked like that, so that struck me as an interesting thought. Then there was one day – it was either in 5th or 6th – it is funny because I talked to

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FIGURE 1. Tony O'Farrell

Brother Skehan about this afterwards in more recent times. Brother Skehan was only about fifteen years older than me – but to me he was a very old man to look at. He was teaching us everything, of course, because it was primary school.

One day he explained how you could start with the quadratic equation $ax^2+bx+c=0$, take the c over to the other side, complete the square so that you got

$$\left(x+\frac{b}{2a}\right)^2$$

equal to something and then take the square roots of the two sides and subtract the $\frac{b}{2a}$ from the other side and you get this formula and it went step by step down the board. I just thought, God, this is an entirely different level of operation, this is something interesting.

I was talking to him years afterwards and I told him this. I said that made a big impression on me and he said: "You know, I remember that too, because it was one of those long days – and I was just feeling the way you sometimes feel, and so I thought, ah, I will just do it!"; so he just did this and he said: "I could tell that you were paying attention – the rest of them were all over the place". The usual thing – it was a double class because there were three school rooms in the school, one for second class – there was no first class – you went straight into 2nd class – one for 3rd and 4th and one for 5th and 6th. This was 5th and 6th so there would have been maybe 70 kids across the room in the two classes.

So he was telling one group about this – probably the 6th – but I remember the event, the logic of the thing. You were just doing this and it was like a steel trap – it was just beautiful. That made an impression.

You know, I still find youngsters that graduate from college who don't understand this. They can't tell you why the quadratic equation is solved by $\dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. I have actually set that exercise to the 3rd years one time, to prove that if, and only if, this, and you would think that they could do that, but no.

How did we get onto that? You asked me about mathematics that I remembered.

PM: *The age at which you first noticed – but I think from your answer it is quite clear. It was starting with addition, although, if you are noticing that you were interested already at the point where you were doing addition – that is quite young indeed.*

AOF: I could see there was a trick to it, that it was interesting. I clearly had a talent for mathematics. I remember the nun setting us a board-full of problems and sitting down. I just did them all and folded my arms and she was surprised because everybody else was still slaving away for the rest of the hour but I had finished them in a minute or two. It was clear there was a difference in terms of how I could perform.

PM: *Did you also go to secondary school in Templemore?*

AOF: I started school in Templemore and then went up to Dublin to Drimnagh Castle – we lived in Walkinstown and I went over to Drimnagh Castle.

I met Brother McGrath the first time – my mother took me over to see if I could get into the school and he took me in and gave me an exam – some maths to do. He must have spoken to me as well to see if I spoke Irish.

You see I had gone to a school in Templemore that by a kind of fluke operated entirely in Irish – I thought that was the way the world was but actually it hadn't been going for very long and I don't suppose it continued but that was the situation then. As we walked in through the gate hand-in-hand, somebody said we have to speak Irish here and we did. For the five years that we were there we just spoke in Irish.

Anyway Brother McGrath put me into the A class – there was an A class and a B class – so that meant that I did the honours courses in various things. I did science subjects instead of doing commercial subjects and so on for the Leaving Cert. He was a big influence on my life. I had Brother Guilfoyle as well – taught me maths, Latin and Spanish – he was good – but Brother McGrath was exceptional. I published a book [4] in homage to him.

PM: *I saw that listed actually on your list of publications* [5].

AOF: He taught us religion and physics as well in much the same way. He had you buy a notebook and then he dictated. He had the diagrams on the board and wrote some formulae and things and we copied them into our notebooks; so I had three of these 120 page notebooks full of this material. He had it down to a fine art.

I believe he taught Latin as well but I didn't have him for Latin and I believe he was outstanding. He taught in the CBS in Ennis before coming up to Dublin and he had Flannan Markham down there, and Fr Markham ended up as a teacher in Maynooth – on the seminarist course in Maynooth. He told me once – Brother McGrath said: "Flannan Markham was the best all round student I ever had". He said: "You were the best at mathematics but Flannan Markham was the best all-rounder".

PM: *Which other science subjects did you do?*

AOF: We did Maths, Applied Maths, Physics and Chemistry for the Leaving Certificate and the manipulation was enjoyable. For example, proving trigonometric identities – we just were aces at proving trigonometric identities because he gave us hundreds of them to work through. It was an enjoyable kind of an exercise to develop that skill of handling these functions and the identities.

PM: *When you were thinking about going to university, had you already decided at second level that it would be mathematics you would study?*

AOF: It was kind of random in a way. I initially assumed that I would be an engine driver – something like that – and then I thought I would be a tradesman, like my father and grandfathers, so I was heading in that kind of direction.

At some stage a friend of ours, of my parents, said: "You know, they are recruiting people into this programme in Kevin Street which is really good for getting to be an electrician – you should send him down". So I went down to Kevin Street one evening and there were dozens of people there and they were being advised by the staff in this big room. So when it came to my turn I met Tom Ambrose – yes, Tom Ambrose was there – and he was actually acquainted with my Uncle Billy in Roscrea. In any case, Tom Ambrose was there and he just asked me a few questions. I explained to him what I was doing and where I was – at that stage I was about 15 – and he said: "Go back and stay in school and finish!" He said: "Don't dream of doing this, go back to school and stay there!"

I was interested always in understanding everything, so I wanted to understand the hard stuff. I was reading about things. I wasn't spending much time on what was being done in school but I was using the library. I was reading about Biology, I was reading about Physics and Chemistry and so on and I was kind of surprised when they actually started doing some of the stuff in school that I had been reading about. I was kind of pleased, like the atomic theory and so on when that came up. It was interesting that they were actually going to do that in school because I hadn't been depending on that.

Then I got to figure things out – what was hard. What was hard at that point were these new things in 20th century physics like quantum theory and relativity – which were clearly difficult things and I wanted to understand those.

I would say that was my most important motivation at that point when I was coming up to the Leaving Cert because I wanted to understand the hard stuff in physics, which is basically mathematical physics, I guess.

The Christian Brothers were good at getting people jobs – that is what they figured they were there for really – to train young people so they could get steady jobs and that mostly meant the civil service or teaching or whatever. So they sent you in to do the Junior X examination.

PM: *Which is a civil service exam?*

AOF: For entry to the civil service. So we did that – you wrote essays in Irish and English and you did something else – I don't remember – and then you got allocated a place. You were 360 or '400 and something' or whatever on the list and so you would get that offer in due course and that would be the start of a step where you come in at a junior level of the civil service and you would work your way up to secretary of a department or something – inevitable steps.

They also sent me down to do the Met Office exam – the Met Office had invented this concept of the Met Cadet. They had two positions every year and they had their own maths exam – it was just a maths exam – but they set it themselves. You did this and then they interviewed. They called the people who did well to interview and, if you were successful, you got one of these Met Cadetships.

What that did was that it paid you a salary, you were employed as a Met Cadet, you were paid the salary of a Met Assistant – someone who reads the instruments – that kind of salary – about £500 a year and your fees were paid to go to college and do a degree in physics related subjects. You could do Physics or you could do Mathematical Science, in any of the universities. So I was offered that.

The other possibilities were scholarships from the Council – we lived in the County rather than the City. The city had a number – I think they had about 48 scholarships – some number like this of scholarships – I think it was a multiple of 12 – but there were fewer council scholarships for college. Then there were entrance scholarships as well that a person could get. You paid for education, and you paid for secondary school. My parents had to pay £10 a year for me to go to secondary school, which was about a week's wages.

Most of the people who went to college were from a professional upper middle class kind of family. I think about 2% of the intake in UCD when I went in there came from the class of people who worked in any way at all with their hands, including tradesmen, and unskilled workers and so on. All you had to do was get 5 passes in the Leaving Cert, including Irish and Latin, and you got in. If you had matriculated, you could do whatever you wanted, that was the setup. There was one guy – a friend of mine – who like some people had trouble with Latin so he went to Caffrey's College for a year to pass the Latin in the matric – so he joined us when we were in second year – he arrived into first year. I think he took about six years to get through Pharmacy after that.

PM: *So during your college years you were a Met Cadet?*

AOF: I was working for the Met service from '64 to '68 – and I was a Met Cadet for the three years in college. That meant that I was on their time when I went to college. It also meant that if I won scholarships as I went through – which I did – I didn't get them. Actually I missed out on a substantial amount of prizes and scholarships as I went through. Then when I finished I had to work for them for ten years – that was the deal. We can come back to that.

PM: *Let's go back to your time in UCD. Did you go in to study Mathematical Science?*

AOF: That is right. With Maths and Maths Physics – they were the interesting things – you had to do four subjects so I did Physics and Chemistry. The Physics and Chemistry were just the same as the Leaving Cert – they were trivial – there was nothing new in those at all – so you just ignored those for the year and concentrated on the Maths and Maths Physics, which were hard and interesting. Then at the end of that year you could stop doing Physics and Chemistry and continue with Maths and Maths Physics.

Somewhere along the way – I think in about second year – I decided that the Maths was more interesting than the (Maths) Physics because it was being done better. There were very good people – David Judge was a wonderful teacher. I met an awful lot of mathematics first in the Maths Physics classes because they were all technique, so the techniques for differential equations and they would be using Hilbert spaces operators, things like this. But without any rigour, they wouldn't worry about it but they needed to use them so they used them. Things were proceeding more slowly in mathematics but they were proceeding carefully.

And after a while you get tired of this other way of looking at things – of doing things – and you wanted to do it right, and that became preferred as far as I was concerned.

Anyway, my job was to do the degree and then work for the Met service and I was quite prepared to do that. So I was interested in the (Maths) Physics all along and interested in the Maths. I had good people. Timoney's father – Richard M. Timoney's father, Dick Timoney he was called – we didn't call him Dick Timoney of course – he was teaching me, and Gormley and Ingram – Ingram was a Jesuit – and Franklin – this would be the Franklin before any Franklin you would know – it was the elder Franklin – he was David Franklin also but he taught Statistics – they were an interesting bunch of people.

PM: *What about contemporaries – did you have an interesting group of fellow Maths Science students?*

AOF: The class was small. Gormley did a thing called 'clearing first honours' at the beginning. So the first day of first honours you had about 50 people in the room – they were standing at the back and so on – and then for two weeks he just covered the board in some random hard story. In that case it was about identities – trigonometric identities – using De Moivre's theorem and so on – and that whittled it down then to about 20. They carried on with the 20 and then he wouldn't let about 10 of those continue into second year, so we ended up with a small class in second year. I learned recently that the Department of Education were pretty annoyed about the fact that he produced so few honours maths graduates [3]. The other universities graduated many more people with honours maths who became maths teachers, but Gormley was focused on matching the Cambridge tripos regime, and few of the survivors went into teaching.

Who was with me? Jimmy Fay was my best friend there – he lives in Canada now. He went into IT. Jerry Lynam also ended up in IT. Jerry went to America. There was John Bradley, who did a doctorate afterwards with John Miller in numerical and then he ended up working for the ESRI. He was a Professor with the ESRI. He used to do these economic forecasts, I guess they were – for the whole national economy anyway, programmed in Fortran.

Who else was there? There were clerics. Seamus Mooney was a bright fellow but he was a Holy Ghost to start with and then, when he stopped being a Holy Ghost, he went over to California. I met him again – he was doing a PhD in Economics in UCLA – so I was on his committee there for a while for a preliminary examination. He went to work for a financial company. He started working for WG Grace, I think they were called. He was actually in the World Trade tower when it was hit and walked out. He died a few years ago.

There was Seamus Hegarty, who was an Oblate novice out there in Stillorgan and he ended up as an education Professor over in England. He ran a large national centre for educational research in London.

There was Mike Norris, who went on to run the dot-ie internet domain. He helped me in the early nineties when I was administrating domains here for maths and computer science.

PM: *I know that you spent a year working in the Met service after your degree in UCD. Do you want to tell me how it progressed from there or what changes happened during that year?*

AOF: Well I was interested in the subject and they were exciting times in meteorology because people were starting to use computers. There was a method that we had – a manual method that we had – for forecasting, which was reasonably good – at that time the numerical methods were not superior but they were going to be superior – that was clear – and so this was an interesting kind of a development.

I got interested in computing because Ingram taught us how to program in Fortran when I was in second year. I had some facility with that. So I was following these developments and reading the literature. We had a subscription in the Met service to the Quarterly Journal of the Royal Meteorological Society and there were interesting developments taking place in terms of modelling the atmosphere and so on. I realised that actually the Met service at the time wasn't terribly interested in this kind of thing. They wanted me to forecast but I looked around me and all the other guys were actually in this thing for the money – it was a job – they were interesting people but their satisfaction was after work. They finished work and they went off. One guy was writing a doctorate in Philosophy, another guy was into Drama – this kind of stuff going on in their lives. I was 20 and my life – that wasn't for me – it had to be everything. It had to be what I was doing... I might do this or I might just join a religious order and become a monk or something. But what I was doing all day had to be the thing. This wasn't working for me so I resigned. I said: "I don't want to do this anymore, what do I have to do?"

So I went up and met Transport and Power's Establishments Branch and we agreed that I would pay them back all the wages they had paid me when I was a student – refund them all that – so there was a bill for this.

PM: *Sounds like it could have been substantial?*

AOF: It was. It was actually less than I had foregone in scholarships coming through on the way up, but never mind, that was the deal.

So I agreed with them that I would pay this over three years and I did that. That was the way I got out of the Met service and I went back to college.

Gormley who was Professor of Maths in UCD was sympathetic to this of course. If I had started straight into the Masters (the year before) I had the scholarship – £550 or something for the year – but they weren't going to give me that a year later so Gormley gave me some tutorial work to do, correcting work and so on, and Frank Anderson took me on as an advisor in the Computer Centre, so that I could get some income from that and I did the Masters.

Then at the end of the Masters I did the studentship as well – that is what you did – you took the studentship exam – and I went off to America and I started the doctorate. So a couple of years into that I paid off the Met service and we were flying.

PM: *I know that you went to Brown University. How did you end up going there?*

AOF: Again, I didn't understand much about the world. Maurice Kennedy was the Registrar and he was in the Maths Department – he had studied at Caltech – and he was teaching the Real Analysis Masters course, I think it was, which I didn't take. I took Complex Analysis and Algebra – you took two subjects for the Masters.

So Gormley taught Real Analysis off the programme – it wasn't officially part of the programme at all – on Saturday mornings because there was a man called McKenna – Joe McKenna I think it was. There was a fellow he liked anyway who had gone to work for Irish Life or something – he had gone to work in industry – and had to stop studying. Just for his sake, he put on this Real Analysis course on Saturday mornings; so three hours every Saturday morning. I went to that and Liam O'Callaghan and Joe and we read Hewitt and Stromberg.

Gormley also taught us German one evening a week for a while – he taught Liam O'Callaghan and myself German – bought us these books – I still have it, *Deutsche Sprachlehre für Ausländer*, there somewhere – enough German so we could get by.

Maurice Kennedy – I never had Maurice Kennedy as a teacher – called me in to his office and he said: "Go to America, because it is better – the graduate study system is better in America". He said: "Go there and they have this setup where you can get support, you will have courses and you will be properly prepared". So I said: "Fine, I will do that".

When I was doing the Masters I had two subjects. For Algebra I had mainly Tom Laffey – Tom Laffey is a fantastic teacher, as you know, but for Complex Analysis – I had already had a whole year of complex analysis from Gormley – we had this guy, Ernie Schlesinger, who was a student of Ahlfors and came on sabbatical for the year and so he taught the Complex course. This was really a second year of Complex Analysis on top of the other one so we learned a great deal of Complex Analysis from Ernie. I asked him – I said: "I like this stuff – where do I go? What is the best thing?". So he advised me which schools I might apply to and so I made a list of five and applied to them.

One of the things I was concerned about was I had heard these stories about American graduate schools that there was this terrible competition that went on. People were viciously competitive, the students were competitive with one another and it was hard to get supervisors to take you on, and so on and so forth. I was concerned about that and so that dictated what happened next.

Harvard was very sniffy. They had this thing where you had to apply to apply and I didn't like that. That seemed to me to be pushing it a bit, so I had a few places – Michigan was one of them as well. Ernie had suggested places that would do things like Complex Analysis. Now he did say actually that things had moved on so mostly we talk about function spaces rather than functions – that is the centre of interest – so you might be thinking of looking at that – so he had suggested looking at these different places. Maryland was one of them and Brown and Michigan.

So it boiled down to it that Brown just did a better marketing job as far as I was concerned in that Schlesinger had a friend there – Robert Accola – who was another student of Ahlfors and he sort of wrote a "Dear Bob" letter to him.

I also got a letter back from the Foreign Student Officer, Mrs Burnight, saying that she had visited Ireland and she had had a very nice time. She was there the year before and the weather had been lovely. I wrote back and said I remember that week – that was great. She got the joke and invited me to drop by when I arrived. It was just personal and it seemed nice so I thought that sounded like a place that I could work in and enjoy and it was. It was terrific.

There were about 4500 under-graduates and 1500 graduate students in the place and there was a tremendous atmosphere really because there were these people, who were doing all these other subjects, that were interesting to talk to and they had a programme of Colloquia – people who would come to visit – they were all superstars that came to visit.

Off the top of my head, there was Lang, Grothendieck, Deligne, Douglas – Jessie Douglas – Hörmander, Segal, various top complex analysts would come by.

PM: *A very stimulating environment then?*

AOF: Yes. There were about 30 staff and they were world class people and they all came to coffee at 4 o'clock in the afternoon every day so you got to meet them. If you were stuck on something or you wanted to hear about something, you could go and you could ask them.

There was a separate division of Applied Mathematics. Wendell Fleming had a position in both of them. You know the way, once you have separate departments, they start fighting each other but there was a connection. I went over there to learn APL and to listen to Lorenz – the guy who talked about chaos. He had discovered the chaotic behaviour of the weather system – discovered chaos I guess – and I went over there to listen to him talking about the impossibility of forecasting over a long time.

On the Mathematics staff there were Katsumi Nomizu, Allan Clark, Jonathan Lubin, Paul Baum, Alan Landman, Gayn Winter, Robert Ferguson, Tom Banchoff, Michael Rosen, Bruno Harris, Yuji Ito, etc, lots of talent and variety.

Back along Hayman had been there – before my time – and Tamarkin – these kind of people. They had benefitted from what happened with what the Germans did, what Hitler did; so they had all these people. They were a very strong group in Functional Analysis and Banach algebras – that was the reason I was going there in the first place – that is why Schlesinger picked that particular school. So you had John Wermer and Andrew Browder and Brian Cole and Barnet Weinstock, who was in several complex variables, and Eva Kallin.

You see, at UCD I had been introduced to algebra by Fergus Gaines in the first place. He was the first guy who told me the definition of a group. I thought it was great. As I say I had a year of algebra then for the Masters with Fergus and Tom – that was all good stuff. It was skewed; there was a lot of stuff about finite groups and that but as far as rings went... You see the influence there was from non-commutative rings particularly, so Herstein and Kaplansky, etc. – Fergus had been a student of Olga Taussky-Todd – and they hadn't told me anything about commutative algebra and recent algebraic geometry, so that was a revelation.

The American system is great – you get to take these graduate courses. They required you for the prelims – they had to do prelims after the first year – they required you to do Algebraic Topology because that kind of stuff – homology and cohomology and so on – was really established as an important thing.

So in my first year then I didn't take the courses on Complex and Algebra that they were going to examine for the prelims because I had done that stuff but I did the Algebraic Topology with Bruno Harris and I did the Real Analysis course with Herbert Federer as well which was fascinating. Basically he was teaching it from his book on geometric measure theory [2].

I took a reading course with Barny Weinstock on several complex variables and so that was when I started learning about sheaves and the Cartan theory – the Oka-Cartan stuff but then there was a lot of that in the air around the place, that way of looking at things.

I listened to Landman and Fulton on Algebraic Geometry, Wermer on Potential Theory, Ito on Ergodic Theory, and Accola and A.O.L. Atkin on modular functions.

I took the PDE course twice, once from Federer and once from Walter Strauss. Walter Strauss was from out of Courant – he was that kind of background – a very standard Courant-Hilbert kind of approach to things.

Federer liked to always get down to the fundamentals so there is a point in the theory of hypoellipticity you need to actually use some stuff which is nothing like analysis, which is the Seidenberg-Tarski theory that goes behind that.

The basic idea is you want to know if you have some kind of a semi-algebraic set then the way it grows as you go out towards infinity is controlled by a power, and so, to prove that you have got to use the Euclidean Algorithm in the non-Euclidean setting of more polynomials over several variables. Federer went right through all that stuff. He took a couple of weeks just doing Seidenberg-Tarski and explaining all that stuff and it was almost like logic more than anything else. Quite different from Walter's course. I mean they would have overlapped on basic things like Cauchy-Kovalevskaya and so on. They diverged quite a bit then in terms of where they went with that.

The other thing I got from Federer was he introduced me to Hausdorff measures, which was a fantastic idea.

I was taking a reading course with Wermer on, well whatever he wanted really, but he gave me a paper by Gamelin and Garnett about Dirichlet algebras. I guess he had it to referee – I am not sure – it was a pre-print from Theodore Gamelin and John Garnett about Dirichlet algebras and they were using capacities to do that.

Simultaneously then I am learning about Hausdorff measures from Federer, so I figure, ok, here is what we will do. These capacities must have some kind of a dimension – a relationship to Hausdorff dimension – so we will try and figure out what that is. So I figured out that the analytic capacity should break at dimension one, and I cobbled together a proof. So I took this into Wermer and I said: "Look, I think this is what happens", and it turned out this was a known thing, which had been proven by Dolzhenko already but it was the start of my interest in capacities that came about at that point.

PM: *You ended up doing your thesis with Brian Cole?*

AOF: First of all I wanted to work with Federer because I liked him but he had three students in various stages – he was just too preoccupied – so it wasn't going to work. This was the beginning of second year.

I am looking at who else I would work with and Brian was giving a course and when I heard about it there was just one student that was going to be there, Richard Basener, and it was on Rudin's book on Function Theory in Polydiscs – they were working their way through that – so I joined that. I listened in and liked the way he operated; so I asked him if I could work with him and so he took me on. He was good. He was a good listener for a start, eccentric in terms of work time – he was nocturnal – so he stayed up all night working and he would come in about lunch time with a flask of coffee.

I met him once a week for the whole afternoon or whatever – we would talk for the afternoon – and he would stay there all night then after that – sleep in the morning – but he was very good. He taught me a lot – we had a lot of interesting conversations – and he listened to what I had to say. He did make some suggestions but I was really pursuing lines of interest to myself. I had a thought about a way things could be done. At some point I formed this idea that the capacities... you see Gamelin and Garnett had used *two* capacities. They had used the Ahlfors capacity and they had used the continuous analytic capacity as well; so I realised that this was flexible. There is a mantra that there is a capacity for every problem and so I figured, ok, let's systematise that.

I was influenced by reading Constance Reid's biography of Hilbert at the time. The way Hilbert operated – he sort of axiomatised and systematised things – so I had this idea that this is how you do things. Also it was in the era of Bourbaki so people did all these kinds of things like that.

I approached this then by trying to make it like that. So I would say we will have categories and we will have functors and there will be functors from a problem area which would be one kind of a category to a capacity which would be something else and the capacity would capture the thing.

So it was a cosmic scheme and basically it is a sound enough scheme and there is a lot of work to be done on it still but I was pursuing that circle of ideas. I could see that there were connections into complex analysis but you could use it also for PDEs – for elliptic PDEs as well – and you mixed it with function spaces.

So basically for an operator and for a function space you are going to have a capacity – from the combination of these two you are going to have an associated capacity and the capacity somehow captures most of what you need to know about that function space and operator. Then you would, if you were comparing two operators or comparing two spaces, looking at approximation problems or removable singularities problems or whatever, that the capacity would be the thing that you could use to combat that. So that was the cosmic scheme.

Brian suggested something to me – he suggested that I look at the Hausdorff measures associated with the Gleason distance. The Gleason distance is just the metric of the dual space of a uniform algebra. So you could look at that metric on the maximal ideal space and that made a metric space out of it and he asked "what could you say about that?" I decided that actually you couldn't say a whole lot about it but it did eventually lead me some years later to a result where I looked at the variation of the Hausdorff content as a function of the dimension. So I have a paper about that – I don't know if there is anybody else who has written about that – but it is a paper, in the JLMS I think.

You see when you actually go to apply these capacities to problems what turns out is that the Hausdorff measure is not the most suitable thing because the Hausdorff measure is very often infinite on bounded sets. It is better to have something which is finite on bounded sets; so the content does that. It is the size infinity approximating Hausdorff measure in Federer's terms and that stays finite on bounded sets.

If you look at that, since it is finite for every dimension, you have a function of the dimension there and you could ask about the variation of that function of dimension and its continuity properties from right and from left and so on; so I investigated that.

I was able to use some lemmas and things that I had worked out to approach Brian's problem to get a result there, because what you are doing is all about coverings – the contents are defined in terms of coverings – so if you are looking at the variation – varying the dimension – then you have a family of coverings. You want to extract from the family of coverings some kind of a covering at the end which will do something so you are looking at that kind of a problem where you are extracting a convergent sequence in some way from a family of coverings. What you don't want is that they all end up being points or something at the end. That was the problem that we hit when I tried to do the thing with the Gleason distance – they tended to end up being points – but, if you just did the straightforward question it was ok.

PM: *After your PhD you spent some time in the States before you came back to Ireland?*

AOF: Well I was interested in these analytic capacities and capacities generally and expertise in those – where was it? There were people in Russia, there were people in Sweden and there were people in Los Angeles – that was the universe as far as that went. You had Vitushkin and Melnikov – these kind of people in Moscow – you had Carlesen and people around him, Hedberg, in Uppsala and you had the school in Los Angeles. Oh and there were people in Indiana as well – Thomas Bagby in Indiana had worked on the capacities for L^p spaces – L^p analytic spaces – so I was interested in those.

I tried to go to those places. I applied for positions in – I didn't apply to Moscow – but I applied to Uppsala (Mittag Leffler) and to Indiana and Los Angeles. I was particularly keen to talk to the guys in Los Angeles, Gamelin and Garnett, because I had put a lot of energy into studying their work. So it worked out that I got the job in Los Angeles. I had a visa which said I had to go back home after finishing so I had to explain to the embassy that I needed to go for a couple of years to Los Angeles instead.

I met a friend afterwards who said: "How did you do that?" I told him and he tried to do the same thing, but he was in English Lit, and he wanted to go to Bates College and they said: "No, you can't do that". I am not sure what swung it but I suspect it might have been when I said that the expertise on this is in Los Angeles and Moscow.

PM: *The fear of Moscow maybe!*

AOF: It might have been crucial – I don't know – but there was no problem. They just listened to me and they said: "Fine", so I got the permission to get the visa for an extra couple of years and I went there. I was never intending to stay in any case. I was planning to come back home but I figured I needed to talk to people because the guys in Brown didn't really do these capacities. They were very, very strong on the functional analysis and the abstract end of functional analysis and Banach algebras and so on but they were not into the hard analysis of capacities which is kind of a black art – a lot of people shied away from it because of that – but these guys, Gamelin and Garnett, they were really stuck into that so it was great to be able to talk to those people.

I was always planning to come back and so I was building up. I was photocopying things that I couldn't read then but I figured I would read later when I got back. When I was at UCLA I was copying away anything like that and planning to ship it all home because I was coming back here where there would be nothing. None of this stuff would be readily available – I would be sending for inter-library loans for weeks. That was the scheme. It was good, they had an excellent programme there of visitors. It was a bigger school with I think about 70 or 100 staff, a very good library. They had a very good library because something to do with... I think it might have had to do with the atomic bomb programme or something.

You see UCLA was one of the first places in on the ground floor of numerical computing and they had kind of lost that when I got there. They were trying to re-establish it – they were trying to hire Garabedian for example because of that and he was playing them around because he wanted to improve his position somewhere else – but they had a super library – just a maths department library – very well equipped.

Richard Arens was there. He would have been the grand old man of that material. Sario was there of course as well, but Sario didn't come in very much. Then there were people like Redheffer and these guys – Coddington and Redheffer in differential equations. They had visitors – Takahashi – these C^*-algebra people came around to visit – as well – Costant – so I learned a bit about that end of stuff as well.

The Banach algebra meetings that went on every two years for ever – they started that. That is where I met Garth Dales in my second year and the first of those meetings was at UCLA. Sandy Grabiner and Joe Stampli came and Bill Badé came down from Berkeley. I shared an office with his student, Fred Dashiell at UCLA.

PM: *The whole transition then from the US back to Ireland and your position in Maynooth – there has to be an interesting story there?*

AOF: Well, as I say, I was always intending to come back here. Now I did make applications around the place just to see what would happen. You will observe I am moving west, right, so I had gone to the east coast and west coast – I think it is a Celtic tendency to travel west – so I did actually get a job offer from Hawaii at one stage as well and that was kind of attractive. I think it was that going westward. There was a guy there called H.S. Bear – I guess he did Gleason parts – so that was tempting. I had this interest in warm places, sunny beaches and things like that; so Hawaii was attractive from that point of view.

I had offers from several places in the States as well but I would have had to get some special kind of visa to do that but I suppose it would have been possible. There were these H visas or something which would do it if they made a case for you.

Basically I turned them all down because they are out of synch with Ireland, so you get the offers in America and you have to respond before things are settled in Ireland so I just decided to burn my boats. I applied for a Department of Education – the Department of Education had one post-doctoral fellowship in mathematics which was designed exactly for people like me. You could come back to Ireland and you could hang around until a job turned up because the jobs were few and far between and so I was pretty sure I would get that.

Then Trevor West wrote. Trevor West would land in – he knew people everywhere – and say "I am staying with you today", or whatever, that kind of a way. So Trevor West arrived in, introduced himself and stayed with us and he took me under his wing. He wrote to me and said: "Apply for the job in Maynooth".

Oh I applied for a job in Cork as well. They called me to interview. That was when I was at Brown – when I was finishing up at Brown I applied for a job in Cork – Gormley told me to apply there. They called me to interview but I decided not to go – I decided I would go to the other place.

Timoney – Timoney's father wrote to me as well – Gormley was dead at that stage – and he wrote and said "You should apply for the job in Maynooth, because you won't get it but you will be in a strong position if you are shortlisted for the next statutory lectureship that comes up in UCD." Trevor wrote to me and said: "Apply for the job in Maynooth and put me down as a reference" so I did that and then I was called to interview – what was I – 27?

As far as I was concerned I had won because I had been given a free trip home – that was progress – that was all I could expect to get out of it – so I wasn't at all worried about it. There was no pressure. So I came home – I was home for a week or so – and I took the bus 66 out to Maynooth and went for the interview. Trevor marked my cards as well. Trevor had a football team and I had to go and play football with his football team up in Alexandra in Ballinteer. He said: "You are a professional – just talk about your stuff".

Of course I had very fixed – formed – opinions about what should be done, like what I would do differently if I were doing my own education over again, what people should know about and what was important in mathematics and so on. I had views about this – perhaps premature – but anyway they were views.

So I went down and enjoyed myself – it was fine. There was a board with A.J. McConnell. It was the first time they had open competitions. It used to be that the bishops would get together and decide who would be appointed. Way back in the 19th century they used to have concursus – candidates would come – the whole college community would come and listen – they would have this debate, where people would throw questions but more recently the bishops just decided what happened.

Maynooth had just opened up to the world at large – that was a good thing. When I was away in America, I thought Galway would do, because the wind comes in from the ocean, and the air is better in Galway; but Maynooth is upwind of Dublin as well so I figured that was ok; so it was an opportunity.

They had this idea – they had a board with a non-voting chairman.

PM: *That was A.J. McConnell?*

AOF: That was A.J. McConnell, and he was Provost at Trinity at that time and he was impressed that I had burned my boats. At this stage I had refused all offers to go elsewhere. I had this thing in my pocket now – the scholarship.

PM: *You had been offered that?*

AOF: I had been offered that and my mother was scandalised. She said: "After all you have done that is all they are going to pay you?", but nevertheless I was coming back with that regardless. That probably had some influence. They were going to get me – I was definitely coming. Tom Fee was the President and he was mostly interested in whether I spoke Irish or not, and was Irish – I think that was important to him as well. So we had a little chat in the first official... and that was fine. I'd kind of kept it up. I bought a record in Donegal Irish on my way out in Shannon and played it obsessively when I was over there. I didn't know any Donegal Irish when I went out to America but I had learned it by the time I came back.

The rest of them then – they are all dead – there was John Lewis, Gerry McGreevy and Joe Spelman so Director of the School of Theoretical Physics at DIAS, Professor of Maths Physics, Professor of Physics and David Simms.

So we just had this conversation. The next day I gave a seminar over in UCD just to tell them about stuff and David whispered to me that I had been recommended for the job. I wrote a little article about this.

PM: *I didn't spot that actually, as I would have been interested.*

AOF: It is called "An Chéad Ollamh Tuata" – you will find it on my website [5].

I just told this story about the interview process because it showed how they were just feeling their way in Maynooth at that stage.

PM: *When you came to Maynooth, it would have been a small department?*

AOF: There were just two other permanent staff – David Walsh and Richard Watson – who were lecturers in the department. They had a temporary – a one-year – position the year before. The previous Professor was J.J. McMahon and he left the priesthood and resigned his position. He was gone off to Nigeria. He spent a couple of years in Nigeria and then he was hired in Limerick after that. He died not long afterwards. I met him at the Institute symposium maybe once or twice but then he got cancer and died. He was, apparently, quite an eccentric character. There are some amusing obits of him in the Bulletin.

The two other guys were both hired in the early '70s – I think, David about '72. David was a student of Finbarr Holland. They hired a few people in '72. Richard was actually in UCD in a temporary position the year I did the Masters. He had gone through Maynooth seminary and did the Divinity degree as well. Then he went to England to study after that. He studied in Warwick and Swansea and came back and he was appointed in Maynooth; so I had those two guys.

Initially it was a bit challenging, in that I looked at the programme and decided it needed beefing up. The system had been to run things in such a way that for most departments just a man and a boy would have been fine, because you had a first year general course and then you had an extra hour for the honours – and that was the first year honours course – and then you had a cyclic second and third year general course and some extra hours for the honours course on top of that. You might have five lectures a week to give, you see, for that and four for the other; so it is a nine hour teaching load. So with two people it is quite comfortable with that – they can play golf in the afternoon, that kind of thing.

Whereas I looked at our programme. We had first of all an entirely separate pass and honours, so straight away we have got a heavier load than they have – number one. Number two, there wasn't enough being done – I didn't like the cyclic thing. I said we have got to get rid of the cyclic thing. They have got to do second year and then third year; so first of all we have got more to do in second and third year because we have four plus six – I am raising it from five to six – and then we have to double that, so that is 20 hours plus the other. The other departments – the experimental departments – had no honours degree at that stage. The only honours degrees that we had in scientific areas were the mathematics ones with mathematical physics, but they were keen to do that. As a first step to that, they wanted to have a fourth year on top of their general degree and so they introduced this course where there would be a fourth year.

We had to provide a fourth year with that and, of course, we couldn't use the one we already had because it was too hard for them. Even in fourth year they weren't going to be ready for our third year course so we had to do a special course for them. That was another four hours on top, so it soon totted up to 41 hours – we always ran a taught Masters course as well.

Now, unwinding the cyclic thing – that took a couple of years – and we did get a staff member. They had a kind of a tradition that, if you got a new Professor, he got a new staff member and then they thought he was happy after that; so we hired Dave Redmond.

So we had four staff to teach the 41 hours so that was roughly ten or eleven hours each – that was the deal. By modern standards, this is a lot but that is the way we worked. It got the thing going so that it was credible.

PM: *At that stage, it was probably very comparable with the other universities though?*

AOF: I think Gormley and Timoney and those guys were working about 14 hours anyway, until the three young staff came when I was, I think, in second year or third year. Fergus Gaines and Tom Laffey and David Tipple came together and that was something that Jeremiah Hogan did – he was the President. He had this idea of what you call College Lecturers – it was an invention.

What they had up to that point were statutory positions – there were Professors and statutory lecturers – and perhaps temporary people – but he invented the 'college lecturer' and he allocated three of them to mathematics. Before that they just had Gormley, Timoney, Maurice Kennedy, Stephen O'Brien and Franklin – that is the whole lot. They had to do the science degree, the arts degree, the engineers – the engineers did four years of mathematics back in those days – I am not sure what else they had to do that was separate. Maybe architecture and there was something else that they were at.

PM: *Probably commerce or business students as well?*

AOF: They had BComms – that is correct. They were flat out. I mean they were crippled – buried under a load of work – at a killing kind of a pace. It is a pity really in retrospect. I mean Gormley had potential, he had written about half a dozen papers. He obviously had a lot of talent and he had a lot of stuff that he could have done. He wrote to me in America – he was thinking about the stuff that Loomis had done. When we read Hewitt and Stromberg he was pursuing the thoughts that he picked up there and pushing out on that but I don't believe that was ever published. His published papers are all from earlier on. He has a paper on quaternion linear fractional transformations with applications to special relativity. His doctoral thesis was on differential geometry.

PM: *Over the next decades then, there were obviously lots of changes in all of the universities but, in particular, in Maynooth. Do you want to summarise the trajectory there, from arriving into a department with just two other staff to the makings of a modern department of mathematics?*

AOF: Well, I had to be rude to everybody until we got enough staff to work six hours. I figured that was the target as far as I was concerned. If people could have six hours then I expected them to have time to do research as well as that. I didn't think it was reasonable that they should have more than that and they didn't generally understand that so I had to be rude to everybody for a good while and I think it was the 90's before we got to that level.

PM: *That is a good number of years of being rude to people!*

AOF: Yes. I can remember these various incidents – one time I referred to the academic staffing committee – of course it used to be more democratic – the academic council would actually decide this stuff – nowadays administration decides all this kind of stuff – but the council would really determine the policy.

The Council had a committee of course – an academic staffing committee – that they would refer this to and they would draw up a proposal for what we would do in the way of hiring. I described them one time as a "cabal"; which is a term technically speaking where you are referring to a black mass kind of group, people who were doing something devilish. Matt O'Donnell was very annoyed about that and insisted that I withdraw "cabal". Anyway you had to go at them. Eventually it got to a level where I thought it was reasonable.

As I say the youngsters nowadays are used to a different world and would find this excessive perhaps but, when I was in America, the standards were that, there were teaching institutions – there were two year colleges and four year colleges – and there were universities – and in these different places expectations were different. Twelve hours was regarded as a teaching load for someone who was just teaching and, for someone who was expected to get some research done, six hours was regarded as the standard so I took that as a basis. I certainly found that I could work like that myself and it was ok.

I suppose people put up with my bluntness partly because I was energetic and helpful: I looked after the purchase and installation of the first computer in College, set up and ran the Computer Centre for a while, took over and organised the university timetable, ran Computer Science for a while, volunteered a lot, etc.

PM: *Do you want to say anything, Tony, about your private life and everything going on there at the same time? At some point you obviously got married and had a family. At what stage did that happen?*

AOF: As I say, I sort of thought I might become a monk, early on, and that lasted until I met Lise. It was always in the back of my mind as a possibility, until I decided that I wanted to marry her. I remember that was a bit of a crisis for me in the sense that it was a shock to find that actually that was going to happen.

PM: *That, by definition, is a very romantic whole change of heart to some extent! I mean, being a monk and being married...*

AOF: I thought women were wonderful and interesting to talk to and so on but I wasn't really definite about what I was going to do about that until I met and fell in love with Lise. I remember in the first month or so – the first few weeks after that – being somewhat disoriented, because I was re-orienting my direction and getting used to it. So she was a huge influence in lots of way. She introduced me to choral singing, for example. Probably I had a much bigger social circle, a richer social life than I would have had otherwise. I learned a lot of things from mixing with Lise – Lise brought me into a whole world of possibilities. She is perpendicular to mathematics [6].

PM: *Did you meet Lise in the United States?*

AOF: Yes. When I went over in September – you see I did the Masters exam, the studentship exam, one week and went to America the next week and the results didn't come in for another month or whatever. So when I got over there, I was supposed to go up to the Foreign Student Office to introduce myself so I did, and there she was. She was the President of the foreign students association – the International Association.

The foreign student officer asked her to come in that week and meet the incoming foreign students so, when I arrived up there, there she was sitting out in front of the office. They had given her a desk out in the concourse in front of the foreign student office. She was wearing this very fetching dress – she had bought these dresses in Africa I think or somewhere – they were very nice. They were short and they had patterns on them and things – African kind of patterns.

We hit it off because – well, she will tell you the story – but, she told me what she was President of and I asked her where she came from and she said she was from Seychelles and I knew about Seychelles. She was used to all these people who would say: "Where is that?" I can't tell you the number of times I have heard Lise say – she had a little spiel – "they are a group of islands about 1000 miles off the east coast of Africa", and they would never know. But Ireland has this missionary diaspora so we have missionary orders who go all over the place and they have magazines – I still have subscriptions to the Africa Magazine and these kinds of things – so these came into the house. So sooner or later somebody writes two pages about Seychelles.

There was a Franciscan Capuchin magazine that came in and so I had read this – because I read everything that was available – and so I knew the basics about Seychelles when I met her and I was pleased because I was interested in hot foreign places as well – Paradise and all that sort of thing. Gordon, the guy (British General) who was killed in Khartoum (1885), thought that the Vallee de Mai in Praslin was the Garden of Eden, he had this theory that that was it. If you know about Seychelles, it is famous, Vallee de Mai. So, anyway, that was the start of a beautiful business. We got – and we weren't quick about this – engaged officially in '71 and married in '72. She graduated in '71. In '72 I was still a graduate student – I graduated in '73. Now I had written the thesis. Basically I wrote the thesis – I figured out the stuff that was in my thesis — in my second year – it is a four year programme – the expectation was that you are going to be there for four years. The way I am thinking is I am going back to Ireland and it is going to be a wilderness; so I was in no hurry to go back. I needed to work on my inner fat and have that with me when I went back.

Figure 2. Lise O'Farrell

I had the results – I wrote them up in the first half or so of the third year – but I continued taking courses, learning more mathematics and using the library and I started writing papers outside my thesis as well; so that was what I was up to.

Actually when I applied for that job in Cork – it was in my third year when I think about it – Gormley wrote and said: "Apply for this job in Cork", and he said: "Put it on a page and be brief"– that was the advice, right, and so I did that – a one page application.

Tadhg Ó Ciardha invited me to interview and I made arrangements that I could have gone then as well. I transferred the credit from my year in UCD to Brown so that I could skip a year if I wanted to but I decided, no, I am going to stay and just use the time and use the facilities here.

You see they had this library – the place is there since 1764 – and they had every book. Henry Pohlmann looked after the library and it was brilliant, they had every reference to everything you wanted – it was always down there in the library – so I was game for that.

Now Lise, she came home with me in ‘71 to visit and we made arrangements to get married in ‘72. Then we came back here and got married in summer ‘72 and then we went back to Providence. She was working in the public schools in Rhode Island for that year – she qualified as a teacher for the state of Rhode Island – she had the appropriate licence or whatever – so she was working there. She was actually supporting us – I mean she was the main bread-winner – I had a scholarship from the department which was fine but she was making more than I was. Then at the end of that year I graduated and we moved to Los Angeles and we went to Seychelles in the summer of ‘73 to visit her parents and family. On the way back in Ireland we discovered she was pregnant. We had planned to drive across the States but we had to skip that and fly straight.

We spent two years in Los Angeles and then we came here. We have been basically here except for some leaves ever since – it is 50 years next year.

We spent a term I suppose really in Connecticut one time from August – we went in the summer and stayed until Christmas. I had a year’s sabbatical one time in ‘85/‘86 and we travelled around. We went over to England for a while, to Cambridge, then IHES and then Israel and back to Cambridge. Other than that, we have been here the whole time.

PM: *You have seen a lot of changes, obviously in Ireland, but in the universities and, certainly, within mathematics within the universities over those 50 years. Too many changes to summarise?*

AOF: What can I say? The big change, of course, is the internet which has revolutionised the practicalities of doing mathematics, because it has made all this stuff available – it used to take two or three weeks to get something on inter-library loan – it mostly had to come from Boston Spa. You always had to work on several lines of enquiry in parallel because it could block on something where you just needed to get hold of something before you could continue and then you just have to wait for it to come; whereas now you can get instant access.

Communication – I mean I used to say that what we needed were research grants for phones. Bill Ziemer had a research grant for phone calls in Indiana when I went to see him at the time they offered me the job. That was very sensible because long distance telephone calls were expensive and that was the main thing – you could just communicate with people and ease that.

They never got that really – that it was worth giving people grants for. Communication is now – since email really got going properly – fantastic. The difference that has made! So in terms of the practicalities of doing work, that is the big change.

Administration – I don’t think we should even go there – the corporate business that has taken over is just sickening to look at really. Among other things, I was the Jimmy Hoffa of Maynooth for a while. There was some iteration or other of the national wage agreements which was introducing something which looked as if it was going to be a disaster. So Vincent Comerford drew this to my attention and I agreed that I would have a go, so I became the person in charge of the IFUT branch here in Maynooth – for a while I was doing that – so I was dealing with these guys in the personnel office and the President and so on. It is just soul destroying looking at the way they operate but I think it will pass. I have hope.

PM: *Really, you do think it will pass?*

AOF: I mean the academic enterprise – this whole idea of the way we do things has worked for thousands of years – basically it is the same scheme – this managerialism and what not is currently triumphant but we will see the back of these guys eventually. They can't stop us doing what we love, that is really what it boils down to. We would actually do this for nothing – literally at the moment I don't have to – they will pay me the pension regardless anyway – we would do it for nothing.

Jerry Brown was the Governor of California when I was there – he was a Democrat I think – and they gave a pay rise at one time to all the public servants in California except the university Professors. He said: "They don't need it – those guys have *psychic wages*".

People were outraged but he was right – it is the best job in the world. In fact, it is not a job. When I was young you had this idea that people did that – they got jobs – but what is a job? I had a job in the summer of my 5th year delivering frilly garments around Dublin on a bicycle and I realised that it was just slavery. I realised that because one day there was nothing to do – I was waiting for something to do – and I pulled out a book and started reading the book – and my boss said: "What are you doing?" And I said: "Oh, I am reading the book while I am waiting,"and he said: "You can't read". So I would put the book down and do nothing, right, until he had something for me to do.

PM: *Even though there was nothing for you to do, they owned the time.*

AOF: They owned the time, so a slave. I decided, ok, this is not what I want – it is not the way. This life – this university academic life – is an example of a kind of life where you do what you want, you pursue your particular passion. The religious life is the same, you just do the one thing and focus on that and do it well.

There always have been people who did it – it was not fair – those people were rich. Nowadays I think we detect talent better than we used to do – so we do have a better chance of picking up if a young person has talent than we did before. There was a lot that went to waste and we have removed barriers. I don't know what you feel about this but I think it is probably better for girls – for women – growing up. I think they have better chance if they have potential to have that recognised and used and channelled. In general that is the case. We have removed obstacles to talent.

You see a lot of people around the town – a lot of older people than me – they had interests that they were not able to pursue. Like the fellow who used to fix my car, I remember when he was coming up to retirement he said: "I would love to do some third level". Never got the chance and he was thinking of doing something when he retired and he would pursue that kind of an interest. You meet a lot of these people – you used to meet a lot of these people – even the old shopkeeper down in Rosslare who had just done the old Inter-Cert. There was an old Inter-Cert where they did things like Latin roots – that was a subject – but he thought about philosophy and had an interest. It is good that nowadays you can pursue these interests – you are more likely to be able to pursue them.

Our species is a kind of fluke. I noticed there you had a question about AI down at the bottom.

PM: *I did, yes.*

AOF: They talk about this AI catastrophe which is when the machines – the intelligent machines – take over us. I think the AI catastrophe already happened – it happened to God because he made us and we rejected him – we took over.

We are such a fluke – we are on this little skin on the surface of the globe and we are the only – certainly the only one left – thing like us, in that we have this capacity for reasoning – we have this intelligence – and what did they do with it? Like in the 19th century, you take these people who essentially, among other things, had the capabilities of a general purpose Turing machine and you get them to work on one specialised task – they could be programmed of course for any task – but you get them to run a cotton mill or something or a loom – something like this – and you pay them to do that for 12 hours a day six days a week and they are wage slaves.

Now we can certainly produce all we need on earth with about 20% of our time and our labour so we should have an enormous amount of free time and labour to do other things after we have fed ourselves and clothed ourselves and looked after ourselves. That is what humanity is for – is good for.

PM: *Have you been involved in the second level maths curriculum in Ireland?*

AOF: Yes of course. I regard it as a duty of being Head of Mathematics here to take an interest in that, and so yes, at various times. There is a book [3] by Susan Mac Donald, just published by Logic Press, which tells the story of the geometry disaster – required reading I think for how not to run an Education Department.

So I was involved in various ways. One is that I was active in support of Paddy Barry and other people in relation to this geometry curriculum and the whole disaster around that. I was on the NCCA syllabus committee that brought in the Project Maths syllabus more recently.

Way back in the '70s Sean Ashe and I got involved – he was very interested in the fact that the lower Leaving Certificate course that they had then had a very high failure rate – both of them had a high failure rate – and he felt that the weak students were not been catered for and something needed to be done about that and so I was involved in that as well. I would have talked to the IMTA a couple of times and also just generally trying to contribute.

I have a paper about geometry there – school geometry – in the IMTA newsletter. I have a paper with Paddy Barry about the geometry as well and I wrote the geometry document for the syllabus. I had help but I was the principal author of that geometry document that forms part of the syllabus – "Geometry for Post-Primary School Mathematics". Ian Short did the diagrams for me and Stefan Bechtluft-Sachs gave me a hand with it as well – but I put that together. That is what I call a Level 2 account of the geometry. That is to say you have a fully rigorous level, which I call Level 1. For this geometry programme – that is Paddy Barry's geometry book [1]. This is a Level 2 account which is supposed to be a somewhat simplified version of that and suitable as a foundation. Then they needed text books which would be Level 3 – largely absent still. There is no Level 2 account of the rest of the programme in schools either; so there is a lot of work that needs to be done.

PM: *There is a review going on at the moment.*

AOF: The IMS Education committee are looking at text books at the present time – Ann O'Shea is chairing that. We have no system of validation or approval of text books – other places do. If you read Feynman's book, for example – he was involved in the California State one for physics – I think for mathematics as well, back along. They had this setup where somebody says there are no mistakes in this book and it is suitable to use – that is the minimum – and we don't have that.

The whole thing needs a lot of work. Or they take something off the shelf that is working somewhere else and is properly done, and translate it if necessary into English and use it. We ended up with this setup where it made no sense and it turned a whole generation of people off geometry – everybody hates geometry – just a disaster.

PM: *I know you retired in 2012 so that is before Project maths was fully rolled out. Do you have any views on how Project Maths has functioned since then?*

AOF: I don't know very much about what has been happening but I know a little bit and I am sort of worried about whether we will make it back to some sort of stable situation or not. I think huge damage was done because, as I say, a whole generation including all the presently active teachers, were cheated really of an education in geometry. It is very hard to work back from that situation.

I am worried about implementation, mistakes in the text books – mess everywhere – but we can but hope.

PM: *You played a founding role in the Irish Mathematical Society?*

AOF: Well I was around when it started alright – that is true – and I think the initiative was – was it Trevor, Finbarr, Tom Laffey and those guys who got it going originally – but I was around the place alright. I was Secretary and President at different times.

I think I had a hand in getting the September meeting off the ground – the idea of a scientific meeting. The Institute used to have these biennial meetings and everybody came. Like the BMC – everybody in Britain used to go to the BMC – at one time you would have about 600 people going. Now it has become the kind of thing where people come who are interested in the particular topics they are going to talk about. I suppose that is a general phenomenon – people don't like these broad meetings – a pity.

The Institute's symposia – everybody who was active in mathematics and mathematical physics in the country used to come and then they stopped doing that, when John Lewis finished, I guess. There was a gap there and I thought the society should do that.

So we had this idea of the scientific meeting – the annual scientific meeting – so they have been useful and good I think. Some very good folk came. Fred Almgren came from Princeton. I think Wermer came to one of those as well.

PM: *I have something here to remind me to ask you about the Hamilton Walk, which seems to have developed its own life now.*

AOF: Well that is a thing where you have something concrete and tactile that you can associate with mathematics. That was a thing. Hadamard has this book 'The Psychology of Invention in the Mathematical Field' where he talks about the way you work. I mean how do you do mathematics well – you take something and you think about it and you think about it hard for a while. Then maybe you don't – maybe you just let it sort of stew for a while – and then at some point or other your subconscious works away once you have got it started and then sooner or later the thing pops up into the forefront with some kind of progress.

There are a number of events that are recorded – we have reports of this – and one of them is the Archimedes in the bath business. Then there is the story of Poincaré stepping off a bus and inventing Fuchsian groups or something. And then we have this Hamilton story and the bridge – this moment when he suddenly realised that this is how it should be done.

The bath has gone and the bus has gone but we have the bridge still so we have a thing – a spot – and it does draw people. We have arrived at the bridge and found other people there, independently arriving from other places, so it is a thing. I thought that was good.

I did take an honours class – I had a small honours class years and years ago. I had a headache and I put them in the car and took them over there one time and then the thing started regularly. I believe it was 1990 that it started being an annual thing.

I went over to Dunsink and I talked to Ian Elliott – God rest him – who was the astronomer in charge over there – he wasn't the Director – and he was very helpful. He pointed out and explained to me about the geography of things. We had to pick a route. It was either down Dunsink lane or across the field, so we went for across the field – off road – actually there were a number of obstacles down Dunsink lane in those days – and mapped all that out.

So I wrote something and sent it around the Maths Departments to tell them about it and the thing kicked off. David Simms and Nigel Buttimore turned up on the first day as well and we had a walk – the rest is history. It is nice.

It gave a focus. Maths Week then – they decided to build it on the 16th – to have the week embracing the 16th – and DEPFA Bank came along looking to support mathematics and so it was natural to fold that into Hamilton day as well and have it be a day for mathematics.

PM: *I did have that question, which you have partially answered, about recent developments in AI. We are seeing in the universities now massive effects on how we can do assessments. Do you have any views on that?*

AOF: Well, I am not in any way an expert in AI. I have had a look at these generative things that are available and they are very limited in what they can actually do; so we are a very long way from real AI in any kind of Turing sense – nothing like that. I wrote some little note about it in the last editorial I wrote for the Bulletin, where I just described some experiment there and I have had another look or two since. I think it is well well short of actual artificial intelligence and it is over-rated in terms of expectations.

The thing you mention is right that the students can copy things – it was already beginning to be a problem anyway that they were copying things and it has undermined the whole idea of having continuous assessment in tutorials. We didn't have any of that when I was young. We didn't have any CA. In fact, there were very few examinations and we gave something up when we started having all this CA – we gave up the business where a student was just left alone.

When I went to college I spent a great deal of my time learning about music and philosophy, ancient classics – all this sort of stuff – and that didn't matter in terms of assessment. I could just do the examinations at the end. The final examination was at the end of the summer for both the Bachelors and Masters degrees; so you had a great deal of time to pursue other interests and then do this exam. Whereas (now) they are being monitored and regimented a lot – forced to do things as they go along and that is the downside.

I introduced all the tutorials and things when I came here because I thought it helped the students; so we set up this whole elaborate system now which is there for tutorials and set homeworks and all the rest of it. It was well-meaning but whether it is really better or not I don't know.

I mean I am not convinced about education. I think you can really divide the populace into the people that can't be taught and the people that don't need to be taught, they teach themselves; so the most we can do is point people in the right direction. Standing over them – I don't know – and this CA system has always been open to abuse – it has always been possible to copy and it has become worse and worse.

PM: *How would you describe your research style?*

AOF: Well what I do really is move out from things that I understand to the things that I don't and there is a nice principle which is that you are never very far away from the unknown stuff – the frontier is very close to anything at all. So I think the way I have gone about it is to try to understand things – to try to get right down and understand them and as soon as you study anything in sufficient detail, you will become aware of problems – things that you don't know that are around about that. And then I just operate by writing down any thoughts that I have and keep them. I have stacks of paper in there and I have notebooks and things and I just move around these things then and tackle them as I can.

There is sort of a chain of connected things that I would have worked my way around [5], like capacities connected with length and area, rectifiability, dimension, algebraic structures, algebras, modules, groups, Hausdorff measures, kernels, Suslin sets, Polish spaces, extension problems, polynomial and rational hulls, function spaces, derivations, categories, exceptional sets, removable singularities – all kinds of problems about singularities – and then from singularities for holomorphic things to elliptic things, approximation problems. I always liked approximation problems.

It is a matter of taste what you do and what you like. And I always liked this business of approximation. The first approximation theorem I learned was the Bernstein approximation theorem – Gormley taught us that. We had first a raw observation that if you take a Taylor series that it usually doesn't represent the function. It might but it usually doesn't so then what do you do? The Bernstein thing was the first example of a theorem that says, well ok – it is actually Weierstrass's theorem – that you can approximate any continuous function with polynomials on an interval. Weierstrass's own proof of that is one way – Bernstein's proof is a different way – a limited way in some sense because it has this positivity thing that it goes with – but I liked that kind of thing.

So a theorem that says that we can approximate all these by rational functions in these circumstances or we can approximate by functions holomorphic in a larger set or whatever it is. We can take away the singularities and still approximate – I like all those kind of questions. I would regard those as sufficiently interesting by themselves.

Garth Dales asked me – he and A.M. Davie had these spaces of infinitely differentiable functions on an interval, where they made Banach algebras out of them by having some kind of a growth condition and a suitable norm condition that gave some multiplicative norm, and he wanted to know if the polynomials were dense in those algebras. That's enough for me – if you asked me are the polynomials dense in that – I like that question – and so I thought about that until we solved it. It took a long time before we saw what to do but eventually the penny dropped and we saw a way to do it.

Then you have got this business of special sets – Cantor sets and sets of convergence. Then potential theory, singular integrals, measurability and the whole business then of functional analysis, algebras of functions.

Browder taught me a functional analysis course in my first year at Brown and he did the Gelfand theory of commutative Banach algebras and that is just so beautiful. What he is doing is he is taking the stuff the algebraic geometers have done and he is transferring it then to this context of uniform algebras but you get things like Wiener's theorem – that a convergent Fourier series which doesn't vanish also has a reciprocal which is convergent, and which is provable in three lines if you look at it the right way but it is a hard thing to prove if you try to do it by brute force. I thought that was very attractive.

The use of abstract techniques from functional analysis, there was this proof that Wermer showed of the Stone-Weierstrass theorem. This proof says just look at the extreme points of the unit ball of the annihilator and they have got to be points so you win. It is just coming from this basic fact of the compactness of the unit ball of the dual and the fact that weak-star compact convex sets have extreme points – I love that. So I would have used that kind of approach tackling a problem about sums of algebras.

I notice things. When you are younger, you notice things and I tucked them away for future reference.

I like it when things mix together as well. I would be on the look-out for connections. The other thing is that it never hurts to know things and it is always a good idea to keep learning. My favourite results took years to obtain, and the final step usually followed digesting some new theory.

PM: *Which are your favourite results?*

AOF: I have a soft spot for the generalised Walsh-Lebesgue Theorem (research paper 7 [5]). That was the first result I really liked. It has a kind of polished perfection. Simple statement, deep proof, involving Caratheodory's beautiful theory of prime ends and Glicksberg's generalised F.&M. Riesz Theorem, on top of the original Walsh-Lebesgue. I proved it not long after moving to UCLA, and it was my first *Meisterstück*: I felt I was no longer a journeyman, but finally a master craftsman, like my father and grandfathers. It is published in a minor journal, because Wermer had just been appointed to their editorial board and asked me if he could give it to them, but it is good enough for anywhere.

Hard-won and beautiful were the theorem with Marshall on sums of algebras (paper 28), the theorem with Preskenis on the algebra generated by two plane homeomorphisms (paper 41), and the theorem with Alejandro Sanabria-Garcia on De Paepe's disc (paper 63). The papers on capacities, on derivations, and on reversibility represent stages in ongoing campaigns, open-ended results, and have attracted other workers abroad, but there is a particular satisfaction when you kill off a topic with a single blow, and I have a good few papers that do that. Some other efforts that have attracted little notice have potential for development. A young person could do worse than study my least-cited papers.

PM: *Thank you very much indeed for your time Tony.*

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Two Extensions of Cauchy's Double Alternant

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ABSTRACT. Two extensions of Cauchy's double alternant are evaluated in closed form that may serve also as parametric generalizations of the remarkable determinant identity of a skew–symmetric matrix discovered by Schur (1911) and its multiplicative counterpart due to Laksov–Lascoux–Thorup (1989).

1. Introduction and Outline

There exist numerous determinant identities in the literature (cf. [9, 13]). For example, the determinants of Vandermonde and Cauchy's "double alternant"

$$\Lambda_m = \det_{1\le i,j\le m}\left[x_i^{m-j}\right] = \prod_{1\le i<j\le m}(x_i - x_j),$$

$$\det_{1\le i,j\le m}\left[\frac{1}{x_i+y_j}\right] = \frac{\prod_{1\le i<j\le m}(x_i-x_j)(y_i-y_j)}{\prod_{1\le i,j\le m}(x_i+y_j)};$$

play an important role in symmetric functions and group characters (cf. [5, 8, 10]). In general, a matrix $T(x_1, x_2, \cdots, x_m)$ of order $m \times n$ in m variables is called an alternant (cf. [9, §321]) when the elements of the first row of T are all functions of variable x_1, the elements of the second row the like functons of x_2, and so on. For example

$$\begin{bmatrix} e^x, \sin x, \cos x \\ e^y, \sin y, \cos y \\ e^z, \sin z, \cos z \end{bmatrix}.$$

Likewise, a matrix $\mathbb{T}(x_1, x_2, \cdots, x_m; y_1, y_2, \cdots, y_n)$ is a double alternant if $\mathbb{T}$ is an alternant respect to both rows in variables $\{x_1, x_2, \cdots, x_m\}$ and columns in variables $\{y_1, y_2, \cdots, y_n\}$. Suppose that $f(x, y)$ is a bivariate function, we have the following general double alternant

$$\Big[f(x_i, y_j)\Big]_{\substack{1\le i\le m\\ 1\le j\le n}} = \begin{bmatrix} f(x_1,y_1), & f(x_1,y_2), & \ddots, & f(x_1,y_n) \\ f(x_2,y_1), & f(x_2,y_2), & \ddots, & f(x_2,y_n) \\ \vdots & \vdots & \cdots & \vdots \\ f(x_m,y_1), & f(x_m,y_2), & \ddots, & f(x_m,y_n) \end{bmatrix}.$$

There exist several generalizations (cf. [1, 3, 6]) of the determinants for Cauchy's double alternant. By employing the calculus of divided differences, the second author [2] evaluated determinants for a large class of variants of Cauchy's double alternant. As a complements to the results appearing in [2], we shall examine, in this little article, the

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determinants of two particular matrices. Let T and $\{x_k, y_k\}_{1\le k\le m}$ be indeterminates. Define two matrices by

$$\mathrm{U}_m = \big[u_{i,j}\big]_{1\le i,j\le m} : u_{i,j} = \frac{x_i+Ty_j}{x_i+y_j}, \tag{1}$$

$$\mathrm{V}_m = \big[v_{i,j}\big]_{1\le i,j\le m} : v_{i,j} = \frac{x_i+Ty_j}{1+x_iy_j}. \tag{2}$$

We shall prove the following two surprisingly elegant determinant identities.

Theorem 1 ($m\in\mathbb{N}$).

$$\det \mathrm{U}_m = \frac{\prod_{1\le i<j\le m}(x_i-x_j)(y_i-y_j)}{(1-T)^{1-m}\prod_{1\le i,j\le m}(x_i+y_j)} \times \Big\{\prod_{i=1}^m x_i - T\prod_{i=1}^m(-y_i)\Big\}.$$

Theorem 2 ($m\in\mathbb{N}$).

$$\det \mathrm{V}_m = \frac{\prod_{1\le i<j\le m}(x_j-x_i)(y_i-y_j)}{\prod_{1\le i,j\le m}(1+x_iy_j)} \times \frac{1}{2}\Big\{\prod_{i=1}^m(x_i+\sqrt{T})(1+y_i\sqrt{T}) + \prod_{i=1}^m(x_i-\sqrt{T})(1-y_i\sqrt{T})\Big\}.$$

When $T=0$, the corresponding identities in both theorems are equivalent to the Cauchy double alternant. For $T=-1$ and even $m=2n$, these identities reduce, in the case $x_k=y_k$ for all k, to the following remarkable Pfaffian formulae discovered by Schur [11] and Laksov–Lascoux–Thorup [7] (see also [12]), respectively:

$$\det_{1\le i,j\le 2n}\left[\frac{x_i-x_j}{x_i+x_j}\right] = \prod_{1\le i<j\le 2n}\left(\frac{x_i-x_j}{x_i+x_j}\right)^2,$$

$$\det_{1\le i,j\le 2n}\left[\frac{x_i-x_j}{1+x_ix_j}\right] = \prod_{1\le i<j\le 2n}\left(\frac{x_i-x_j}{1+x_ix_j}\right)^2.$$

2. Proof of Theorem 1

For the matrix U_m, by subtracting the last row from the other rows, we can check that the resulting matrix becomes

$$\mathrm{U}'_m = \big[u'_{i,j}\big]_{1\le i,j\le m} : \; u'_{i,j} = \begin{cases} \dfrac{(1-T)(x_i-x_m)y_j}{(x_i+y_j)(x_m+y_j)}, & 1\le i<m; \\ \dfrac{x_m+Ty_j}{x_m+y_j}, & i=m. \end{cases}$$

By extracting the common row factor $(1-T)(x_i-x_m)$ for $1\le i<m$ and the common column factor $\dfrac{y_j}{x_m+y_j}$ for $1\le j\le m$, we find the following determinant equality

$$\det \mathrm{U}_m = \det \mathrm{U}'_m = \det \mathrm{U}''_m \times (1-T)^{m-1}\prod_{i=1}^{m-1}(x_i-x_m)\prod_{j=1}^m \frac{y_j}{x_m+y_j}, \tag{3}$$

where the matrix U''_m is given by

$$\mathrm{U}''_m = \big[u''_{i,j}\big]_{1\le i,j\le m} : u''_{i,j} = \begin{cases} \dfrac{1}{x_i+y_j}, & 1\le i<m; \\ \dfrac{x_m+Ty_j}{y_j}, & i=m. \end{cases}$$

Expanding the determinant along the last row leads us to the equality

$$\det \mathrm{U}''_m = \sum_{k=1}^{m} (-1)^{m+k} \frac{x_m + Ty_k}{y_k} \det \mathrm{U}''_m[m,k], \tag{4}$$

where $\mathrm{U}''_m[m,k]$ is the sub-matrix of U''_m with the mth row and the kth column being removed. By applying the Cauchy double alternant, we can evaluate

$$\begin{aligned}\det \mathrm{U}''_m[m,k] &= \frac{\prod_{1\le i<j<m}(x_i - x_j) \prod_{1\le i<j\le m\ (i,j\ne k)}(y_i - y_j)}{\prod_{1\le i<m} \prod_{1\le j\le m\ (j\ne k)}(x_i + y_j)} \\ &= (-1)^{m-k} \frac{\prod_{i=1}^{m}(x_i + y_k)}{\prod_{i\ne m}(x_m - x_i)} \prod_{j\ne k} \frac{x_m + y_j}{y_k - y_j} \frac{\prod_{1\le i<j\le m}(x_i - x_j)(y_i - y_j)}{\prod_{1\le i,j\le m}(x_i + y_j)}.\end{aligned}$$

Here and henceforth for simplicity, $\prod_{\ell\ne k}$ stands for the product with the index ℓ running from 1 to m except for $\ell = k$. Substituting this into (4) gives rise to the following expression

$$\det \mathrm{U}''_m = \frac{\prod_{1\le i<j\le m}(x_i - x_j)(y_i - y_j)}{\prod_{1\le i,j\le m}(x_i + y_j)} \frac{\prod_{j=1}^{m}(x_m + y_j)}{\prod_{\imath\ne m}(x_m - x_\imath)} \sum_{k=1}^{m} \frac{x_m + Ty_k}{y_k} \frac{\prod_{i\ne m}(x_i + y_k)}{\prod_{j\ne k}(y_k - y_j)}.$$

Denote by $\Delta[y_1, y_2, \cdots, y_m] f(y)$ the divided difference (cf. [2]) of the function $f(y)$ at the points $\{y_k\}_{k=1}^{m}$, which can be expressed by Newton's symmetric sum

$$\Delta[y_1, y_2, \cdots, y_m] f(y) = \sum_{k=1}^{m} \frac{f(y_k)}{\prod_{j\ne k}(y_k - y_j)}.$$

Then we can evaluate the last sum (cf. [4]) as

$$\begin{aligned}\sum_{k=1}^{m} \frac{x_m + Ty_k}{y_k} \frac{\prod_{i\ne m}(x_i + y_k)}{\prod_{j\ne k}(y_k - y_j)} &= \Delta[y_1, y_2, \cdots, y_m] \Big\{ \frac{x_m + Ty}{y} \prod_{i\ne m}(x_i + y) \Big\} \\ &= \Delta[y_1, y_2, \cdots, y_m] \Big\{ Ty^{m-1} + \frac{\prod_{i=1}^{m} x_i}{y} \Big\} \\ &= T - (-1)^m \prod_{i=1}^{m} \frac{x_i}{y_i},\end{aligned}$$

which leads us to the following simpler formula

$$\det \mathrm{U}''_m = \frac{\prod_{1\le i<j\le m}(x_i - x_j)(y_i - y_j)}{\prod_{1\le i,j\le m}(x_i + y_j)} \frac{\prod_{j=1}^{m}(x_m + y_j)}{\prod_{\imath\ne m}(x_m - x_\imath)} \Big\{ T - (-1)^m \prod_{i=1}^{m} \frac{x_i}{y_i} \Big\}.$$

Substituting this into (3) and then simplifying the resulting expression, we confirm the determinant identity stated in Theorem 1. □

3. Proof of Theorem 2

By following exactly the same procedure as done in the last section, we can explicitly evaluate the determinant for the matrix V_m. Subtracting the last row from the other rows transforms V_m into the following one:

$$\mathrm{V}'_m = \big[v'_{i,j}\big]_{1\le i,j\le m}: \ v'_{i,j} = \begin{cases} \dfrac{(1 - Ty_j^2)(x_i - x_m)}{(1 + x_i y_j)(1 + x_m y_j)}, & 1 \le i < m; \\ \dfrac{x_m + Ty_j}{1 + x_m y_j}, & i = m. \end{cases}$$

By extracting the common row factor $x_i - x_m$ for $1 \le i < m$ and the common column factor $\dfrac{1-Ty_j^2}{1+x_m y_j}$ for $1 \le j \le m$, we find the following determinant equality

$$\det \mathrm{V}_m = \det \mathrm{V}'_m = \det \mathrm{V}''_m \times \prod_{i=1}^{m-1}(x_i - x_m)\prod_{j=1}^{m}\frac{1-Ty_j^2}{1+x_m y_j}, \tag{5}$$

where the matrix V''_m is explicitly given by

$$\mathrm{V}''_m = \left[v''_{i,j}\right]_{1\le i,j\le m} : v''_{i,j} = \begin{cases} \dfrac{1}{1+x_i y_j}, & 1 \le i < m; \\ \dfrac{x_m + Ty_j}{1-Ty_j^2}, & i = m. \end{cases}$$

Expanding the determinant along the last row leads us to the equality

$$\det \mathrm{V}''_m = \sum_{k=1}^{m}(-1)^{m+k}\frac{x_m + Ty_k}{1-Ty_k^2}\det \mathrm{V}_m[m,k], \tag{6}$$

where $\mathrm{V}_m[m,k]$ is the sub-matrix of V''_m with the mth row and the kth column being crossed out. By making replacements $x_i \to x_i^{-1}$, we can reformulate Cauchy's double alternant as

$$\det_{1\le i,j\le m}\left[\frac{1}{1+x_i y_j}\right] = \frac{\prod_{1\le i<j\le m}(x_j - x_i)(y_i - y_j)}{\prod_{1\le i,j\le m}(1+x_i y_j)}.$$

Then we can evaluate $\det \mathrm{V}_m[m,k]$ by the following product expression

$$\begin{aligned}\det \mathrm{V}_m[m,k] &= \frac{\prod_{1\le i<j<m}(x_j - x_i)\prod_{1\le i<j\le m\ (i,j\ne k)}(y_i - y_j)}{\prod_{1\le i<m}\prod_{1\le j\le m\ (j\ne k)}(1+x_i y_j)} \\ &= (-1)^{m-k}\frac{\prod_{i=1}^{m}(1+x_i y_k)}{\prod_{i\ne m}(x_i - x_m)}\prod_{j\ne k}\frac{1+x_m y_j}{y_k - y_j}\frac{\prod_{1\le i<j\le m}(x_j - x_i)(y_i - y_j)}{\prod_{1\le i,j\le m}(1+x_i y_j)}.\end{aligned}$$

Substituting this into (6) yields that

$$\det \mathrm{V}''_m = \frac{\prod_{1\le i<j\le m}(x_j - x_i)(y_i - y_j)}{\prod_{1\le i,j\le m}(1+x_i y_j)}\frac{\prod_{j=1}^{m}(1+x_m y_j)}{\prod_{i\ne m}(x_i - x_m)}\sum_{k=1}^{m}\frac{x_m + Ty_k}{1-Ty_k^2}\frac{\prod_{i\ne m}(1+x_i y_k)}{\prod_{j\ne k}(y_k - y_j)}.$$

By decomposing into partial fractions

$$\frac{1}{1-Ty^2} = \frac{1}{2\sqrt{T}} \times \left\{\frac{1}{y+\sqrt{T^{-1}}} - \frac{1}{y-\sqrt{T^{-1}}}\right\},$$

we can evaluate the rightmost sum again by divided differences

$$\begin{aligned}
&\sum_{k=1}^{m}\frac{x_m+Ty_k}{1-Ty_k^2}\frac{\prod_{i\neq m}(1+x_iy_k)}{\prod_{j\neq k}(y_k-y_j)}\\
&=\Delta[y_1,y_2,\cdots,y_m]\Big\{\frac{x_m+Ty}{1-Ty^2}\prod_{i\neq m}(x_i+y)\Big\}\\
&=\frac{1}{2\sqrt{T}}\Delta[y_1,y_2,\cdots,y_m]\Big\{\frac{x_m-\sqrt{T}}{y+\sqrt{T^{-1}}}\prod_{i\neq m}(1-x/\sqrt{T})\Big\}\\
&-\frac{1}{2\sqrt{T}}\Delta[y_1,y_2,\cdots,y_m]\Big\{\frac{x_m+\sqrt{T}}{y-\sqrt{T^{-1}}}\prod_{i\neq m}(1+x/\sqrt{T})\Big\}\\
&=\frac{1}{2}\prod_{i=1}^{m}\frac{x_i+\sqrt{T}}{1-y_i\sqrt{T}}+\frac{1}{2}\prod_{i=1}^{m}\frac{x_i-\sqrt{T}}{1+y_i\sqrt{T}}.
\end{aligned}$$

Consequently, we derive the closed form expression

$$\begin{aligned}
\det \mathrm{V}''_m=&\frac{\prod_{1\leq i<j\leq m}(x_j-x_i)(y_i-y_j)}{\prod_{1\leq i,j\leq m}(1+x_iy_j)}\frac{\prod_{j=1}^{m}(1+x_my_j)}{\prod_{i\neq m}(x_i-x_m)}\\
&\times\frac{1}{2}\Big\{\prod_{i=1}^{m}\frac{x_i+\sqrt{T}}{1-y_i\sqrt{T}}+\prod_{i=1}^{m}\frac{x_i-\sqrt{T}}{1+y_i\sqrt{T}}\Big\}.
\end{aligned}$$

Finally, substituting this into (5) and then simplifying the resulting expression, we find that

$$\begin{aligned}
\det \mathrm{V}_m=&\frac{\prod_{1\leq i<j\leq m}(x_j-x_i)(y_i-y_j)}{\prod_{1\leq i,j\leq m}(1+x_iy_j)}\\
&\times\frac{1}{2}\Big\{\prod_{i=1}^{m}(x_i+\sqrt{T})(1+y_i\sqrt{T})+\prod_{i=1}^{m}(x_i-\sqrt{T})(1-y_i\sqrt{T})\Big\}.
\end{aligned}$$

This completes the proof of Theorem 2. □

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Threshold Concepts as an Anchor in Undergraduate Mathematics Teaching

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Abstract. In this short article, we consider the notion of Threshold Concepts and give some examples in mathematics. We will also explore the role of Threshold Concepts as potentially powerful transformative points in a student's learning of mathematics at University level and discuss the theoretical and practical approaches proposed by various educators to address such concepts at both modular and programme level.

1. Introduction

About 20 years ago, two economists (Erik Meyer and Ray Land) introduced the notion of a *Threshold Concept* while working on a project which aimed to enhance the teaching and learning in undergraduate courses. In a series of articles, [11], [12], [10], they defined the characteristics of such a concept and gave examples from various subjects including mathematics. Meyer and Land's core idea is that many academic disciplines have concepts that act as conceptual gateways or portals, and that while developing an understanding of these concepts students are led to engage in previously inaccessible ways of thinking [11]. These portals are places where students often 'get stuck', but when these concepts are fully mastered students are able to behave more like experts in the field [7], and crucially see the subject in a new way. Meyer and Land [11] gave the example of opportunity cost in economics to illustrate the idea. However one of their other examples, namely the ϵ - δ definition of the limit of a function, may resonate more with mathematicians. We will consider some examples of Threshold Concepts in the undergraduate mathematics syllabus shortly, but first let us define what we mean by this notion.

2. Characteristics of Threshold Concepts

Meyer and Land, [11], defined threshold concepts in terms of five characteristics. These characteristics are: transformative, irreversible, integrative, troublesome and bounded. The first of these characteristics has been alluded to above. A concept has this *transformative* character if an understanding of the concept not only changes the student's comprehension of the topic but their view of the subject. (You might stop to consider the impact of understanding the definition of a limit had on your own view of real analysis.) This transformation should also be *irreversible*, that is the change in perspective cannot easily be forgotten and so is usually permanent. This means that it is sometimes difficult for experts in the field to put themselves in the position of the students in their classes. The *integrative* nature of threshold concepts means that they allow students to make previously unseen connections between parts of the subject or

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links between 'isolated islands of knowledge' [13]. It is usually the case that Threshold Concepts are *troublesome* for students, indeed we would not expect such dramatic changes otherwise. Often the knowledge and understanding needed to master these concepts seems alien or counter-intuitive to novices, and may require a suspension of disbelief. Lastly, Meyer and Land added the *bounded* characteristic to their definition of a threshold concept. This characteristic conveys the idea that the understanding and use of a concept is specific to the particular discipline and may even define the boundaries of the discipline. Consider how the formal definition of a limit is a demarcation between calculus and analysis. Note that the five characteristics have some overlaps. For example, there is a deep connection between transformative, integrative and irreversible [7].

While the idea of a threshold concept is relatively new (introduced in 2003), some of its essential features have been described before in other ways. For example, in a 1990 AMS article on mathematics education, Thurston, the renowned mathematician and Fields medallist spoke about the 'compressibility' property of mathematics that arises once a concept or topic is completely understood:

> *Mathematics is amazingly compressible: you may struggle a long time, step by step, to work through some process or idea from several approaches. But once you understand it and have the mental perspective to see it as a whole, there is often a tremendous mental compression. You can file it away, recall it, quickly and completely when you need it, and use it as just one step in some other mental process. The insight that goes with this compression is one of the real joys of mathematics.* [19]

The struggle Thurston mentions speaks to the *troublesome* nature of particular mathematics concepts; the ability to see the process or concept 'as a whole', file it away and recover it efficiently at any future point, possibly just as one step in some other process, alludes to the *transformative, irreversible* and *integrative* nature of understanding such concepts. The fact that other researchers have come to similar conclusions or used related constructs to describe the learning of mathematical concepts lends weight to the characterisation and usefulness of a threshold concept for mathematics.

Since their introduction in 2003, the threshold concepts that have been discussed in different domains have adopted one of three distinct forms [4]. The first form is that of a key idea in a knowledge field (e.g. opportunity cost in economics); the second is a key skill for professional practice in the area (e.g. constructing researcher identity for a PhD student, procedural decomposition in computer programming); and the third, is a stance (e.g. learner-centredness in education).

Although the idea of a threshold concept is relatively new in education, it is an idea that has been embraced by many researchers and educationalists. A wealth of concepts, skills or stances have been identified as Threshold Concepts in the literature. Some early examples of these, from different disciplines, are: the laws of motion and heat transfer in physics, equal temperament in music, sampling distribution in statistics [11].

3. Examples and Non-Examples in Mathematics

Let us look in more detail at some examples from mathematics.

3.1. Limits. The $\epsilon - \delta$ definition of the limit of a function is often troublesome for students and many mathematicians will naturally think of it when they hear about threshold concepts and remember the problems that students encounter. It is *troublesome* for a number of reasons. Previous studies on the learning of limits have found, for instance, that pre-existing images and understandings students have of the phrases 'limit' and 'tends to' cause problems for them [3]. The structure of the definition itself is also often a stumbling block for students [17] because of the inequalities and quantifiers used in the definition and more specifically the combination of both 'there exists' and 'for all' quantifiers in the same mathematical statement. However, coming to an understanding of the limit definition can be argued to be both *transformative* and *irreversible*. One indication of this is that mathematicians often remember the precise moment when they reached a clear understanding of the formal definition of a limit and report it as a real 'aha' moment. Other studies have found that students' way of speaking about limits changes when they begin to understand the definition, providing further evidence that the concept is transformative and irreversible. In addition, the concept of a limit can be thought of as *bounded* as it sits on the boundary between calculus and analysis and it acts in effect as a gateway to mathematical analysis. It is also *integrative* for this reason as it links differential and integral calculus to each other and to mathematical analysis.

3.2. **Proving.** Proof is a key component of mathematics and proving a key competence of mathematicians. The act of proving has many functions including verification, explanation, communication, and systematization. Easdown [8] argues that the ability to not only understand but also to construct proofs is *transformative* for students – not only in terms of how they perceive old ideas but also how they receive new and exciting mathematical discoveries. Mastering the act of proving acts as a 'rite of passage' to membership of the mathematical community and is often accompanied by a 'road to Damascus' effect [8]. As such, it is *irreversible*.

However, proving can be *troublesome* for students due to their uncertainty about how to start and about what is, or is not, allowed when constructing a proof [22]. Insufficient knowledge of the rules of logic and different proving strategies can also cause difficulties. Not only this but the notion that mathematics is **de**ductive rather than **in**ductive like other sciences can be counter-intuitive for students. However, proving can be thought of as *integrative* in the sense that it allows various results to be organised into a system of axioms, concepts and theorems. Proof is *bounded* since its use and meaning is specific to the subject of mathematics, and indeed is one of the defining characteristics of the discipline itself.

3.3. **Non-examples.** To illustrate the difference between 'key' or 'core' concepts and skills and 'threshold concepts', consider the processes of the chain rule for differentiation and integration by parts. Students often find these processes troublesome, and it can be challenging for them to use the techniques correctly. However, while these processes may be key to the learning of calculus, there is no evidence that the mastering of these processes is transformative or irreversible.

For a further discussion of the threshold concepts of limits and proving, along with other threshold concepts in mathematics, see [1], [15], [22], [21].

4. Why are Threshold Concepts Important?

We have seen that threshold concepts are present in the undergraduate mathematics curriculum and that they present both difficulties and opportunities to students. It seems sensible then to pay attention to them when designing courses and to help students navigate the consequent blocked spaces "by, for example, redesigning activities and sequences, through scaffolding, through provision of support materials and technologies or new conceptual tools, through mentoring or peer collaboration" [10, p.62-63]. Traditionally, undergraduate mathematics syllabii have been described solely in terms of mathematical content (such as techniques, theorems, etc.) and this approach has been seen as sometimes leading to rote learning rather than deep understanding and as inhibiting mastery of the subject. Davies and Mangan assert that putting an emphasis on the threshold concepts in a course can help:

> *Threshold concepts offer potential help to lecturers in higher education who are grappling with ... students who struggle with underpinning theory and resort to verbatim learning of isolated aspects of the subject that they seem unable to use effectively.*[6, p.711]

Cousin [5] has some advice for lecturers. She asserts that threshold concepts should be seen as 'jewels in the curriculum' as they are central to students' efforts to master the subject. Because of this centrality and also the problems that these concepts pose for students, she advises that lecturers adopt a recursive approach; that is, that we should not assume that once we have 'covered' the material that students now understand it, rather we should revisit these topics multiple times during a course to give students a chance to view the concept in a variety of ways. The need for active student engagement with, and manipulation of, the conceptual material is also emphasised [10] in order to enable students to experience the 'ways of thinking and practising' that are expected of practitioners within a given discipline, and to facilitate them to join that community of practice. Timmermans and Meyer [20] reinforce this view and advocate that activities used by teachers should deliberately confront learners with the 'troublesomeness' of threshold concepts causing them to 'get stuck'.

The word *threshold* was chosen by Meyer and Land to convey the idea that these concepts act as gateways into an expert-like appreciation of a topic. Because of the nature of a threshold concept, and the difficulty it presents to students, it often seems that mastery of a threshold concept involves the occupation of a liminal space, that is, learners oscillate between old and new understandings and (hopefully) emerge transformed. (This is reminiscent of adolescence which can be seen as a liminal state between childhood and adulthood, where adolescents often oscillate between child-like and adult-like behaviour.) This transformation can entail letting go of an earlier, comfortable position to enter a sometimes disconcerting new territory [13]. Therefore we, as lecturers, should not only expect that students will experience confusion when struggling with these concepts but we should appreciate this confusion as an opportunity for developing deep understanding. Indeed, Cousin [5] recommends that lecturers support students while in this liminal space, and explain to them that the feeling of confusion is normal and even necessary. If knowledge is to have a transformative effect, it probably should be troublesome, but that does not mean that it should be overly stressful or anxiety-inducing for students [13]. Lecturers need to walk a fine line between allowing students to struggle for too long (in which case students may resort to rote-learning to succeed) and shielding them from difficult topics (and thus denying them some valuable learning opportunities).

Cousin suggests that a key component of good lecturing is the ability to listen to students and to understand what their misunderstandings are. This is difficult for experts in the field, since by the nature of threshold concepts, once you have grasped one it can be hard to remember what it was like not to understand it. It can therefore be very challenging for lecturers to see things from the students' point of view. We also need to be aware of students' mathematical backgrounds and how this might help or hinder their development of understanding. For example, it may be that students are hampered by some previous understanding or tacit knowledge (such as the notion that a limit is something that can never be reached), and lecturers may need to help students develop new intuitions and break previous rules. Moreover, it may be that students' prior educational experiences have taught them to value being correct and, thus, they may be avoiding crossing thresholds in a classroom or a discipline [9]. Giving them an understanding and appreciation of the importance of truly mastering a particular concept may be necessary to encourage them to willingly enter a liminal space.

5. From Theory to Practice

Many lecturers and educators have endeavoured to take on board the advice above in relation to the teaching and learning of threshold concepts. We are not aware of such studies pertaining to the teaching of mathematics in particular and so we will consider some findings below for subjects other than mathematics.

Olaniyi [14] focussed on incorporating a number of Meyer and Land's guidelines ([13]) for overcoming barriers to students' understanding of threshold concepts - namely, engaging students, providing for recursive and excursive learning journeys through a topic, and including peer assessment as a means of students sharing their difficulties and anxieties as they inhabit the liminal space. She chose a flipped classroom approach to fit her needs as it facilitates active learning which she asserts should be the cornerstone of any pedagogical approach adopted for teaching threshold concepts. Active learning is important because it encourages deep rather than superficial learning. Indeed, in the US, the Conference Board of the Mathematical Sciences (of which the AMS is a member) called for the incorporation of active learning into university mathematics classrooms [2]. Olaniyi carried out an action research study into her own implementation of a flipped classroom approach to teaching the threshold concept of thermodynamics in physics and reported positively on the results both in terms of improvements in students' understanding and their study skills.

Rodgers et al [16] adopted the perspective of foregrounding threshold concepts as 'jewels in the curriculum'. They describe an action research approach through which they systematically identified a set of five threshold concepts that are encountered by students on their occupational therapy programme. These threshold concepts were then used to underpin the redesign of their curriculum so that the concepts were encountered and re-encountered at various points in the programme. Benefits of this curriculum redesign included a more consistent approach to the threshold concepts from staff and a more coherent integration of concepts overall, making learning less confusing for students. For staff, an awareness of threshold concepts helped them to develop a whole-of-programme view and use this perspective when designing and structuring content, learning activities and assessment tasks. For students, the focus on threshold concepts and the process of making troublesome knowledge explicit helped to 'capture the essence of the programme' (p.552). It also facilitated their development of a professional identity.

One way of gathering information about student thinking is to set assessment tasks in which students are asked to explain or represent a concept in a new way, and to make connections to other parts of their knowledge. To this end, Scott, Peter & Harlow [18] advocate the construction and use of concept inventories in the teaching and learning of threshold concepts. A concept inventory is a set of questions designed to gauge the depth of students' conceptual understanding of a given topic. Such an inventory is pedagogically desirable and powerful due to a two-fold function: firstly, it affords the measurement of true conceptual understanding or correct thinking on the part of students; and secondly, it enables an evaluation of the effectiveness of instruction (or an increase in student understanding) through pre- and post-testing. Scott et al. [18] believe there is a natural marriage between the theory of threshold concepts and that of concept inventories, and they acted upon this by designing and validating a concept inventory for threshold concepts arising in electronics engineering.

6. Conclusion

To an expert, a threshold concept is an idea that gives shape and structure to their subject, but such concepts can be inaccessible to a novice. From the students' point of view, grappling with threshold concepts is often a rite of passage. If they succeed in their struggle and cross the threshold, learners may find it easier to gain entry into the community of practice in their subject [4]. There is also evidence that students who do not develop a good understanding of these concepts may end up resorting to a rote-learning approach [6] or even withdraw from the study of that subject [10]. If our goal is to help students to experience the compression that Thurston described, then it is important for the mathematical community both to identify threshold concepts in our undergraduate modules, and to think carefully about the teaching methods that we use in relation to teaching these concepts. There may also be a hidden benefit for us as teachers. Timmermans and Meyer [20] have observed that some teachers experience a transformation in their conceptualization of their own disciplines, their teaching and their understanding of their students' learning during the work of identifying threshold concepts. We hope that this article will provide some food for thought on this front.

A comprehensive bibliography on the Threshold Concept Framework is maintained at the website `https://www.ee.ucl.ac.uk/mflanaga/thresholds.html` which itself acts as a portal to this area of Threshold Concepts!

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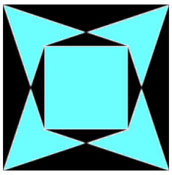

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Wiring Switches to Light Bulbs

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Abstract. Given n buttons and n bulbs so that the ith button toggles the ith bulb and at most two other bulbs, we compute the sharp lower bound on the number of bulbs that can be lit regardless of the action of the buttons.

1. Introduction

1.1. **Origins.** The following problem was posed in the 2008 Irish Intervarsity Mathematics Competition[1]:

> In a room there are 2008 bulbs and 2008 buttons, both sets numbered from 1 to 2008. For $1 \le i \le 2008$, pressing Button i changes the on/off status of Bulb i and one other bulb (the same other bulb each time). Assuming that all bulbs are initially off, prove that by pressing the appropriate combination of buttons we can simultaneously light at least 1340 of them. Prove also that in the previous statement, 1340 cannot be replaced by any larger number.

This problem, henceforth referred to as the *Prototype Problem*, can be generalized in a variety of ways:

(a) Most obviously, "2008" can be replaced by a general integer n.
(b) We can consider more general wirings W, where each button switches the on/off status of a (possibly non-constant) number of bulbs.
(c) We may consider initial configurations c where not all of the bulbs are off.
(d) We however insist that the numbers of buttons and bulbs are equal, and that Button i changes the on/off status of Bulb i, $1 \le i \le n$.

Figure 1 is a sketch of a typical wiring.

These problems are related to the type of problem known as MAX-XOR-SAT in Computer Science. We discuss this connection in more detail in Subsection 2.7 below. There may also be a connection to a meta-Fibonacci sequence related to A046699. See [1].

1.2. **Notation.** Before we continue, let us introduce a little notation. For a fixed wiring W, where the initial on/off configuration of the bulbs is given by c, let $M(W, c)$ be the maximum number of bulbs that can be lit by pressing any combination of the buttons.

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[1]Set by the first author. One proof he gave at that time established Theorem 1.1(b) below by exploiting the discrete dynamical systems associated to $\mu^*(n, 2)$ in a manner similar to the proof in Subsection 4.1.

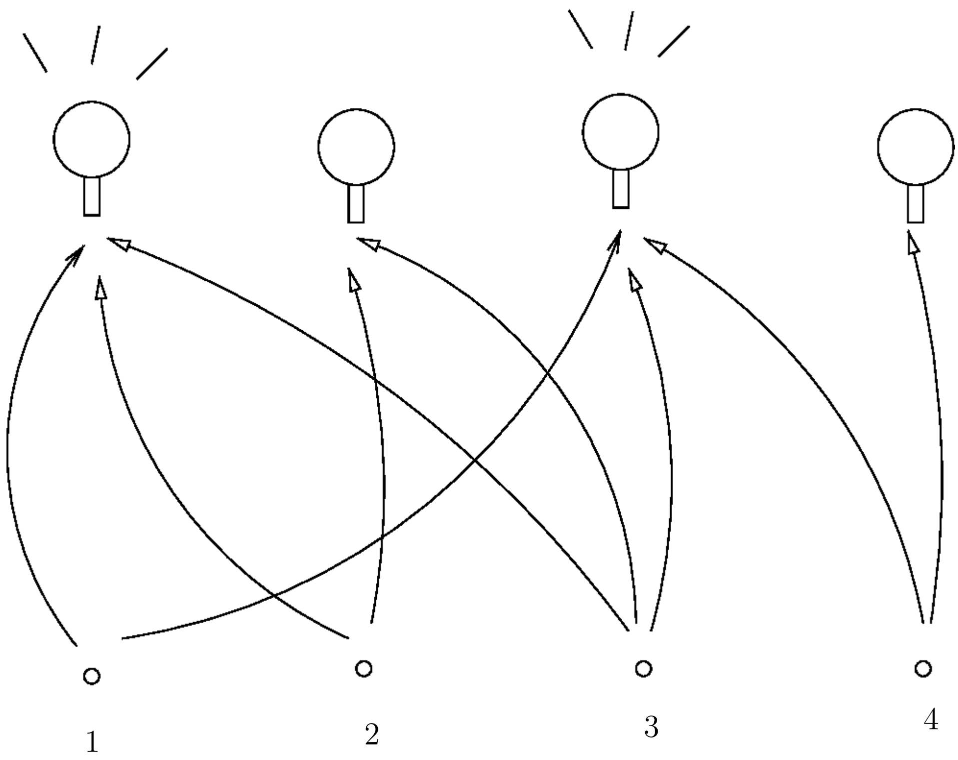

Figure 1. A Wiring

Suppose $n, m \geq 1$. Let $\mu(n,m)$ be the minimum value of $M(W,c)$ over all wirings W of n buttons and bulbs, where Button i is connected to *at most* m bulbs, including Bulb i, for each $1 \leq i \leq n$, and initially all bulbs are off (which we write as "$c = 0$"). If additionally $n \geq m$, let $\mu^*(n,m)$ be the minimum value of $M(W,c)$ over all wirings W of n buttons and bulbs, where Button i is connected to *exactly* m bulbs, including Bulb i, for each $1 \leq i \leq n$, and $c = 0$. Thus the Prototype Problem is to show that $\mu^*(2008, 2) = 1340$.

We define $\mu(n) = \mu(n,n)$, which trivially equals $\mu(n,m)$ for all $m > n$. Thus $\mu(n)$ is the minimum value of $M(W,0)$, over all wirings of the n buttons, subject only to condition (d) above.

We also define $\nu(n,m)$, $\nu^*(n,m)$, and $\nu(n)$ in a similar manner to $\mu(n,m)$, $\mu^*(n,m)$, and $\mu(n)$, respectively, except that we take the minima over all possible initial configurations c, rather than taking $c = 0$. In this article, we are mainly interested in $\mu(n,m)$ and $\mu^*(n,m)$, and we compute these functions for $m \leq 3$. However the more easily calculated ν-variants provide very useful explicit lower bounds (cf. Theorem 3.2 below).

1.3. **Results.** Our first theorem gives formulae for $\mu(n,2)$ and $\mu^*(n,2)$; note that $\mu(n,2) = \mu^*(n,2)$ except when $n \equiv 1 \mod 3$.

Theorem 1.1. *Let* $n \in \mathbb{N}$.

(a) $\mu(n,2) = \lceil 2n/3 \rceil$.

(b) *If* $n \geq 2$, *then* $\mu^*(n,2) = 2\lceil n/3 \rceil$ *is the least even integer not less than* $\mu(n,2)$.

Next we give formulae for $\mu(n,3)$ and $\mu^*(n,3)$.

Theorem 1.2. *Let* $n \in \mathbb{N}$.

(a) $\mu(n,3) = \mu(n,2)$.

(b) *If $n \geq 3$, then*

$$\mu^*(n,3) = \begin{cases} 4k-1, & n = 6k-3 \text{ for some } k \in \mathbb{N}, \\ \mu(n,3), & \text{otherwise.} \end{cases}$$

Note that $\mu^*(n,3) = \mu(n,3) + 1$ in the exceptional case $n = 6k-3$.

We shall discuss $\mu(n,m)$ and $\mu^*(n,m)$ in the case $m > 3$, (and the relationship to a meta-Fibonacci sequence) in a separate article [1]. Let us simply note here that $\mu(n,m)$ and $\mu^*(n,m)$ are no longer asymptotic to $2n/3$ for large n, when $m \geq 4$. For instance, we prove in [1] that $\mu(n,4)$ is asymptotic to $4n/7$, and that $\liminf_{n\to\infty} \mu(n)/n = 1/2$.

After some preliminaries in the next section, we prove general formulae for $\nu(n,m)$ and $\nu^*(n,m)$ in Section 3. We then prove Theorem 1.1 in Section 4 and Theorem 1.2 in Section 5.

We wish to thank David Malone for pointing out the connection between our results and SAT. We are grateful to the referee for some comments that improved the exposition.

2. Notation and terminology

2.1. **Graphs.** The notation and terminology introduced in this section will be used throughout the article. We begin by recasting our problem. First note that we can replace the twin notions of buttons and bulbs with the single notion of vertices: when a vertex is pressed, the on/off state of that vertex and some other vertices is switched. The vertex set $S := S(n) := \{1, \ldots, n\}$ is associated with a directed graph G: we draw an edge from vertex i to each vertex whose on/off status is altered by pressing vertex i. Figure 2 shows a representation of the directed graph corresponding to the wiring in Figure 1. Notice that to avoid clutter we do not draw the loop from each vertex to

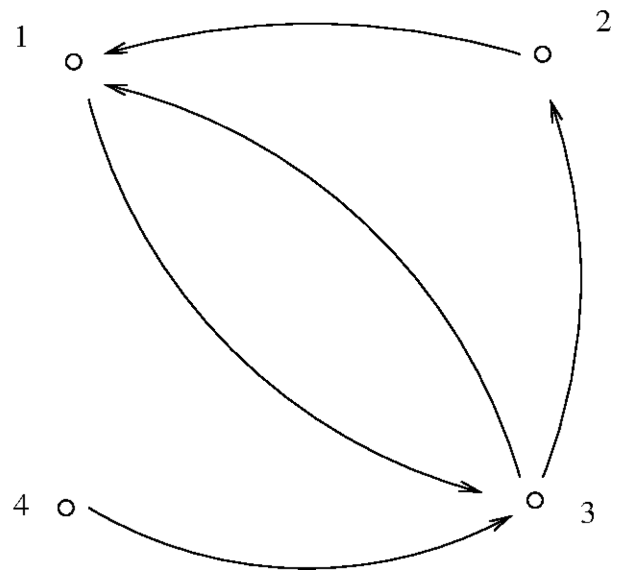

Figure 2. Graph for the wiring in Figure 1

itself, which is always present since a given button always switches the corresponding bulb.

2.2. **Edge function.** Associated with a given directed graph G is the *edge function* $F : S \to 2^S$, where $j \in F(i)$ if there is an edge from i to j, and the *backward edge function* $F^{-1} : S \to 2^S$, where $j \in F^{-1}(i)$ if there is an edge from j to i. In the case where G represents a button-bulb wiring W, $F(i)$ corresponds to the set of bulbs whose on/off status changes when Button i is pressed while $F^{-1}(i)$ corresponds to the set of buttons that, when pressed, change the on/off status of Bulb i. These functions specify the target of each outgoing edge and the source of each incoming edge, i.e. the head of each outgoing arrow and the feathers of each incoming arrow. We extend the definitions of F and F^{-1} to 2^S in the usual way: $F(T)$ and $F^{-1}(T)$ are the unions of $F(i)$ or $F^{-1}(i)$, respectively, over all $i \in T \subset S$. We say that $T \subset S$ is *forward invariant* if $F(T) \subset T$, or *backward invariant* if $F^{-1}(T) \subset T$. We denote by G_T the subgraph of G consisting of the vertices in T and all edges between them.

2.3. **Matrix reformulation.** If we examine the effect of a finite sequence of vertex presses $i_1, \dots, i_k$, on a fixed vertex i_0, it is clear that the final on/off state of vertex i_0 depends only on its initial state and the parity of the number of indices j, $1 \le j \le k$, for which $i_0 \in F(i_j)$. In particular, the order of the vertices in our finite sequence is irrelevant to the final state of i_0. Since this is true for each vertex, we readily deduce the following:

- The order of a finite sequence of vertex presses is irrelevant to the final on/off states of all vertices.
- We may as well assume that each vertex is pressed at most once, since pressing it twice produces the same effect as not pressing it at all.

Thus, instead of talking about a *finite sequence* of vertex presses, we can talk about a *set* of vertex presses and represent this set as an n-dimensional column vector $x \in \mathbb{F}_2^n$ (where $\mathbb{F}_2 = \{0, 1\}$ denotes the field with two elements), with $x_i = 1$ if and only if vertex i is pressed once and $x_i = 0$ if it is not pressed at all. Similarly, we represent the initial on/off state of the vertices by a column vector $c \in \mathbb{F}_2^n$, with $c_i = 1$ if and only if vertex i is initially lit. Lastly, we represent the wiring W as an element in $\mathcal{M}(n, n; \mathbb{F}_2)$, the space of $n \times n$ matrices over $\mathbb{F}_2$. To be specific, $W = (w_{i,j})$, where $w_{i,j} = 1$ if and only if vertex j affects the on/off status of vertex i; we note that $w_{i,i} = 1$ for all $i \in S$. The non-zero entries in the i-th row of W lists those vertices that switch vertex i on or off. The non-zero entries in the j-th column list those vertices that are switched on or off by vertex j. The matrix W is, in fact, the transpose of the adjacency matrix for the directed graph G. With these conventions, the vector $v = Wx + c \in \mathbb{F}_2^n$ is such that $v_i = 1$ if and only if vertex i is lit, assuming we have initial configuration c, wiring W, and vertex presses given by x.

2.4. **Degree.** The *degree of vertex* i, $\deg(i)$, is the number of 1's in the ith column of W (or, equivalently, the cardinality of $F(i)$. In graph-theoretic terms, this degree is the *out-degree* of the vertex). We define the *degree of* W, $\deg(W)$, to be $\max\{\deg(i) : i \in S\}$.

For $u \in \mathbb{F}_2^n$, we define $|u|$ to be the *Hamming norm* or Hamming distance from u to the origin, i.e. the number of 1 entries in u. Then $\deg(i)$ for a wiring W is the norm of the i-th column of the matrix W. Also, $|Wx + c|$ is the number of lit vertices, assuming we have initial configuration c, wiring W, and vertex presses given by x. Thus the function $M(W, c)$ defined in the Introduction can now be described as

$$M(W, c) = \max\{\, |Wx + c| : x \in \mathbb{F}_2^n \,\}.$$

For $n, m \ge 1$, we define $A(n, m)$ to be the set of matrices $W \in \mathcal{M}(n, n; \mathbb{F}_2)$ that have 1's all along the diagonal and satisfy $\deg(W) \le m$. If also $n \ge m$, we define $A^*(n, m)$ to be the set of matrices in $A(n, m)$ for which $\deg(i) = m$, for all $i \in S$. These classes of matrices are the classes of admissible wirings for the functions defined in the Introduction:

$$\begin{aligned}
\mu(n,m) &= \min\{M(W,0) : W \in A(n,m)\}\,,\\
\mu^*(n,m) &= \min\{M(W,0) : W \in A^*(n,m)\}\,,\\
\nu(n,m) &= \min\{M(W,c) : W \in A(n,m),\ c \in \mathbb{F}_2^n\}\,,\\
\nu^*(n,m) &= \min\{M(W,c) : W \in A^*(n,m),\ c \in \mathbb{F}_2^n\}\,,
\end{aligned}$$

The largest class of admissible wirings on n vertices that interests us is $A(n) := A(n,n)$. This gives rise to the numbers $\mu(n) := \mu(n,n)$ and $\nu(n) := \nu(n,n)$, as defined in the Introduction. It is convenient to define $\mu(0,m) = 0$ for all $m \in \mathbb{N}$.

2.5. **Connection to coding.** Although the Hamming distance is a central part of the problems under consideration, these problems are on the surface quite different from those in coding theory, since we are looking for wirings that minimize the maximum distance from the origin of Mx, $x \in \mathbb{F}_2^n$, whereas in coding theory we are looking for codes that maximize the minimum distance between codewords. However, it is shown in [1] that Sylvester-Hadamard matrices, which are known to give rise to Hadamard codes that possess a certain optimality property, also give rise to certain optimal wirings.

2.6. **Augmented complete graphs.** In graph theory, a *complete directed graph on r vertices* (also called a K_r) has an edge from each vertex to each other vertex. A wiring of r bulbs for which each button switches all the bulbs corresponds to a graph which has a K_r augmented by a loop at each vertex. We call such a graph an *augmented complete graph*, or a $\hat{K}_r$. Given the graph G of a wiring, we say that a subgraph H of G is an *augmented complete subgraph on r vertices*, or *a $\hat{K}_r$ in G,* if there is an edge from every vertex of H to every vertex of H. If H is such a subgraph, we call the set of its vertices a *$\hat{K}_r$ set in G.*

For $t \in \{0,1\}$, we denote by $t_{p\times q}$ the $p \times q$ matrix all of whose entries equal t, and let $t_p = t_{p\times p}$. The matrix 1_p should not be confused with the $p \times p$ identity matrix I_p.

2.7. **Relationship to Satisfiability.** The problems under consideration in this article are closely related to MAX-XOR-SAT problems in Computer Science. These problems are in the general area of propositional satisfiability (*SAT*). To be specific, we want to assign values to Booloean variables so as to maximize the number of clauses that are true, where each clause is composed of a set of variables connected by XORs. Since XOR in Boolean logic corresponds to addition mod 2, this problem can be written in our notation as follows: given a matrix $W \in \mathcal{M}(N,n;\mathbb{F}_2)$, we wish to choose a *variables vector* $x = (x_1,\ldots,x_n) \in \mathbb{F}_2^n$ so as to maximize the Hamming norm $|Wx|$; the N entries in $Wx \in \mathbb{F}_2^N$ are the *clauses*. Thus the goal is to compute $M(W,0)$.

XOR-SAT and MAX-XOR-SAT have been studied extensively; see for instance [2], [3], [4], [5]. Algorithms for solving such problems are useful in cryptanalysis [6], [7].

The relationship between MAX-XOR-SAT and our wiring problem is plain to see, so let us instead mention the differences:

- MAX-XOR-SAT is concerned with finding $M(W,0)$ for a fixed but arbitrary W, rather than seeking the minimum of $M(W,0)$ over a class of admissible wirings W. The main problems in MAX-XOR-SAT revolve around the efficiency of the computation of $M(W,0)$ for large n rather than the computation of a minimum for all n.
- In MAX-XOR-SAT, there is no requirement that $N = n$, and so no matching of clauses with variables (or bulbs with buttons in our terminology) and no requirement that $w_{ii} = 1$.

- In MAX-XOR-SAT and other SAT problems, the typical simplifying assumption is that there are either exactly, or at most, m variables in each clause. Thus in SAT we typically bound the Hamming norms of the rows of W, while in our wiring problem we bound the Hamming norms of the columns of W.

In spite of the differences, we would hope that the lower bounds in $M(W,0)$ given by our results might be of some interest to MAX-XOR-SAT researchers.

3. Formulae for ν and ν^*

3.1. **Trivial bounds.** Loosely speaking, larger sets of numbers have smaller minima. More precisely, if $E \subset F \subset \mathbb{N}$, then $\min F \le \min E$. Thus given $n \ge m$, the following inequalities are immediate:

$$\nu(n,m) \le \nu^*(n,m) \le \mu^*(n,m) \tag{3.1.1}$$

$$\nu(n,m) \le \mu(n,m) \le \mu^*(n,m) \tag{3.1.2}$$

3.2. **A lower bound for** $M(W,c)$**.**

Lemma 3.1. *Let $n \in \mathbb{N}$. For all $W \in A(n)$ and $c \in \mathbb{F}_2^n$, the mean value of $|Mx+c|$ over all $x \in \mathbb{F}_2^n$ is $n/2$. In particular, $M(W,c) \ge n/2$ and $M(W,c) > n/2$ if the cardinality of $\{i \in [1,n] \cap \mathbb{N} : c_i = 1\}$ is not $n/2$.*

Proof. Fix W and c. Let $S_i = \{x \in \mathbb{F}_2^n : x_i = 0\}$ and $T_i = \mathbb{F}_2^n \setminus S_i$. Both S_i and T_i have cardinality 2^{n-1}. Then, $f : S_i \to T_i$ is a bijection, where $f(x)$ differs from x in the i-th position and only in the i-th position. Since pressing vertex i toggles its own on/off status, $(Wx+c)_i = 1$ if and only if $(Wf(x)+c)_i = 0$. Let k be the number of sets of vertex presses x in S_i for which $(Wx+c)_i = 1$. Then exactly k sets of vertex presses x in T_i lead to $(Wx+c)_i = 0$ and so $2^{n-1}-k$ lead to $(Wx+c)_i = 1$. In total, therefore, there are 2^{n-1} sets of vertex presses x in $\mathbb{F}_2^n$ for which $(Wx+c)_i = 1$. The mean value of $(Wx+c)_i$ is therefore $\frac{1}{2}$ for each i. The mean value of $|Wx+c|$ is then $n/2$ since this mean value is given by

$$\frac{1}{2^n}\sum_{x\in\mathbb{F}_2^n}|Wx+c| = \frac{1}{2^n}\sum_{x\in\mathbb{F}_2^n}\sum_{i=1}^{n}(Wx+c)_i = \sum_{i=1}^{n}\frac{1}{2^n}\sum_{x\in\mathbb{F}_2^n}(Wx+c)_i = \frac{n}{2}.$$

The last statement in the lemma follows easily. ☐

3.3. The above lemma is a key tool in proving the following result which gives the general formula for $\nu(n,m)$ and $\nu^*(n,m)$. In this result, we ignore the case $m=1$ since trivially $\nu(n,1) = \nu^*(n,1) = n$.

Theorem 3.2. *Let $n, m \in \mathbb{N}$, $m > 1$.*

(a) $\nu(n) = \nu(n,m) = \left\lceil \frac{n}{2} \right\rceil$.

(b) *If $n \ge m$, then*

$$\nu^*(n,m) = \begin{cases} \nu(n,m)+1, & \text{if } n \text{ is even and } m \text{ odd},\\ \nu(n,m), & \text{otherwise}.\end{cases}$$

In particular, $\nu^*(n,2) = \nu^*(n) = \nu(n)$ for all $n > 1$.

Proof. We will prove each identity by showing that the right-hand side is both a lower and an upper bound for the left-hand side.

By Lemma 3.1, $M(W,c) \ge \left\lceil \frac{n}{2} \right\rceil$ for all $W \in A(n)$ and $c \in \mathbb{F}_2^n$. This global lower bound yields the desired lower bound for $\nu(n)$ and *a fortiori* for $\nu(n,m)$ and for $\nu^*(n,m)$ except in the case where n is even and m is odd. We postpone the proof of the lower bound in this case, until we have completed the proof of (a).

To prove the reverse inequalities, the upper bounds, we take as our initial configuration the *even indicator vector* $e \in \mathbb{F}_2^n$ defined by $e_i = 1$ when i is even, and $e_i = 0$ when n is odd. We split the set of integers between 1 and n into pairs $\{2k-1, 2k\}$, $1 \le k \le n/2$, with n being unpaired if n is odd; corresponding to the pairs of integers, we have *pairs of rows* in the wiring matrix W and *pairs of vertices.* For each proof of sharpness, we will define $W = (w_{i,j})$ such that $M(W, e)$ equals the desired lower bound. Pressing vertex j has no effect on the pair of vertices $2k-1$ and $2k$ if $w_{2k-1,j} = w_{2k,j} = 0$, and it toggles both of them if $w_{2k-1,j} = w_{2k,j} = 1$. Since initially one vertex in each pair is lit, this remains true regardless of what vertices we press if the corresponding pair of rows are equal to each other (as will be the case for most pairs of rows). Thus, in calculating $M(W, e)$, we can ignore all pairs of equal rows, for which the corresponding vertex presses leaves the number of lit vertices unchanged, and we only have to consider the unpaired vertex, if present.

To finish the proof of (a), it suffices to show that $\nu(n, 2) \le \left\lceil \frac{n}{2} \right\rceil$. Define the $n \times n$ block diagonal matrix

$$W = \begin{cases} \mathrm{diag}(1_2, \ldots, 1_2), & n \text{ even}, \\ \mathrm{diag}(1_2, \ldots, 1_2, 1_1), & n \text{ odd}, \end{cases} \tag{3.3.1}$$

In case $n = 9$, this matrix corresponds to the wiring of nine buttons and bulbs represented by Figure 3. In this figure, the boxes labelled by the number 2 represent augmented complete directed graphs on two vertices, and the small circle represents a single vertex (and its loop). We shall always indicate an augmented complete $\hat{K}_v$ subgraph by a box labelled v.

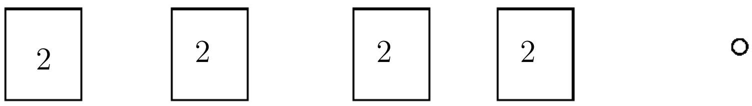

FIGURE 3. $n = 9$

Then $W \in A(n, 2)$ and $M(W, e) = \left\lceil \frac{n}{2} \right\rceil$. To see this, note that rows $2k-1$ and $2k$ of W are equal to each other for each $1 \le k \le n/2$. Thus when n is even, $|Wx + e|$ is independent of x, while it toggles between the two values r and $r-1$ when $n = 2r-1$ is odd, due to the change in the state of vertex n each time that vertex is pressed.

Now we prove the lower bound for (b) in the exceptional case. Fix $c \in \mathbb{F}_2^n$ and $W \in A^*(n, m)$ for some odd $m > 1$ and $n \ge m$. Each vertex press must change the parity of the number of lit vertices and, since the mean value of $|Wx + c|$ is $n/2$, it follows that $|Wx + c| > n/2$ for some $x \in \mathbb{F}_2^n$. Since $\nu(n, m) = n/2$ if n is even, we deduce that $\nu^*(n, m) \ge \nu(n, m) + 1$ if n is even and m odd.

It remains to prove that the desired formula in (b) for $\nu^*(n, m)$ is also an upper bound for $\nu^*(n, m)$ when $n \ge m > 1$. Suppose first that $n - m$ is even. First, define the block diagonal matrix $W' \in A(n, m)$ by the formula $W' = \mathrm{diag}(1_2, \ldots, 1_2, 1_m)$, where there are $(n-m)/2$ copies of 1_2. We modify $W' = (w'_{i,j})$ to get a matrix $W = (w_{i,j}) \in A^*(n, m)$ by adding $m - 2$ 1's to the end of each of the first $n - m$ columns, i.e. let

$$w_{i,j} = \begin{cases} 1, & i > n - m + 2 \text{ and } j \le n - m, \\ w'_{i,j}, & \text{otherwise} \end{cases}$$

In case $n = 9$ and $m = 3$, the matrix W corresponds to a wiring of the kind indicated in Figure 4. In this diagram, the boxes indicate augmented complete subgraphs having

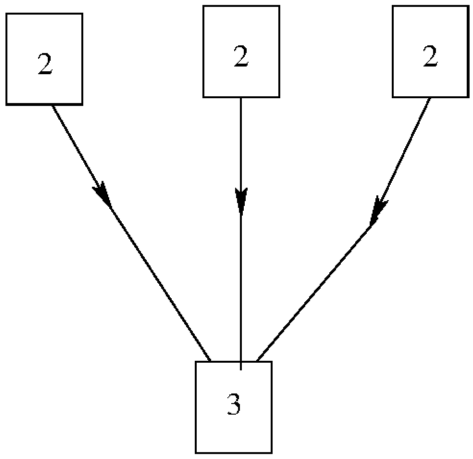

FIGURE 4. $n = 9$, $m = 3$

two or three vertices, as indicated. A single arrow coming from a $\hat{K}_2$ box indicates an edge from *each* of the two vertices in the box and going to *the same* vertex in the $\hat{K}_3$. The target vertex may be the same or different for the three $\hat{K}_2$'s, but the vertices in a given $\hat{K}_2$ share the same target. In general, in our diagrams, we will use the convention that **all the buttons corresponding to vertices in a given $\hat{K}_r$ box produce exactly the same effect.** Notice that nonisomorphic graphs may correspond to the same "box diagram", in view of the fact that a box diagram is not specific about the targets of some arrows.

All paired rows of W are equal, so if n and m are both even, then $|Wx+e| = n/2$ for all $x \in \mathbb{F}_2^n$, whereas if n and m are both odd, the value of $|Wx+e|$ is either $(n+1)/2$ or $(n-1)/2$, depending on the parity of $|x_i|$. In either case, we have found a matrix $W \in A^*(n,m)$ with $M(W,e) = \nu(n,m)$, and so $\nu^*(n,m) = \nu(n,m)$.

Suppose next that n is odd and m even, with $n > m+1$. We first define the block diagonal matrix $W' \in A(n,m)$ by the formula $W' = \operatorname{diag}(1_m, 1_2, \ldots, 1_2, W_3)$, where there are $(n-m-3)/2$ copies of 1_2 and

(3.3.2)
$$W_3 = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

and then define $W = (w_{i,j})$ by the equation

(3.3.3)
$$w_{i,j} = \begin{cases} 1, & 3 \le i \le m \text{ and } j > m, \\ w'_{i,j}. & \text{otherwise} \end{cases}$$

The corresponding wiring is indicated schematically in Figure 5.

The circled subgraph corresponds to the matrix W_3. The double arrows coming from each $\hat{K}_2$ each represent four edges in the graph, i.e. two pairs of edges, where each pair has a distinct target and the $\hat{K}_2$ set is the set of sources for the pair.

The first $n-3$ rows can be split into duplicate pairs as before, so the associated pairs of vertices will always be of opposite on/off status and the number of them that is lit is always $(n-3)/2$.

Initially, two of the last three vertices are lit. Since m is even, the parity of the number of lit vertices is preserved, and so no more than two of the last three vertices can be lit. Thus, $M(W,e) = (n+1)/2$ in this case, as required.

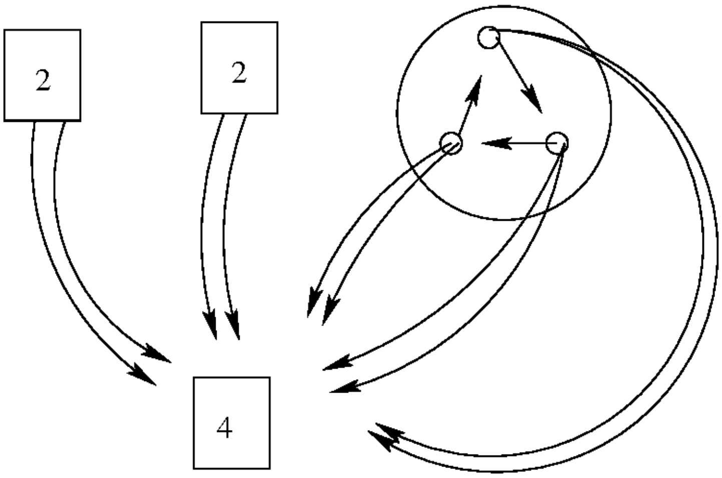

FIGURE 5. $n = 11$, $m = 4$

The case where m is odd and $n > m + 1$ is even, is similar. We first define $W' \in A(n, m)$ by the formula $W' = \text{diag}(1_m, W_3, 1_2, \ldots, 1_2)$, and then define $W = (w_{i,j})$ from W' by (3.3.3). The corresponding wiring is indicated schematically in Figure 6. (In this figure, following our convention, we indicate the multiple edges emanating from a $\hat{K}_2$ and going to the same target node by a single edge.)

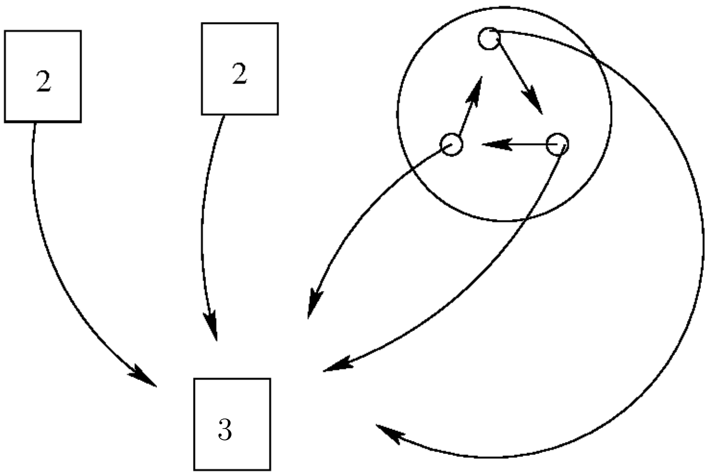

FIGURE 6. $n = 10$, $m = 3$

There are four unpaired rows, namely rows i, $m \leq i \leq m+3$. By an analysis similar to the previous case, at most three of these vertices can be lit (namely vertex m and at most two of the other three vertices), and half of the remaining $n - 4$ vertices are always lit. It follows that $M(W, e) = (n + 2)/2$, as required.

Finally, if $n = m + 1$, we define W to be the block diagonal matrix

$$W = \begin{pmatrix} 1_{(m-1)\times m} & 1_{(m-1)\times 1} \\ 1_{1\times m} & 0_{1\times 1} \\ 0_{1\times m} & 1_{1\times 1} \end{pmatrix}$$

See Figure 7.

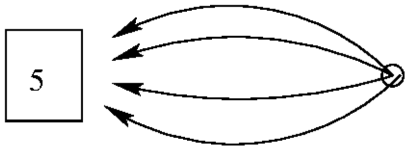

FIGURE 7. $n = 6$, $m = 5$

The first $m-2$ or $m-1$ rows are paired, depending on whether m is even or odd, respectively. Thus, $M(W,e) \le 1 + m/2$ if m is even, or $M(W,e) \le 2 + (m-1)/2$ if m is odd, as required. □

3.4. **Sublinearity.** Generalizing an idea used in the above proof, we see that if W and c have block forms

$$W = \begin{pmatrix} W_a & 0 \\ 0 & W_b \end{pmatrix} \qquad c = \begin{pmatrix} c_a \\ c_b \end{pmatrix},$$

then

$$M(W,c) = M(W_a, c_a) + M(W_b, c_b)\,. \tag{3.4.1}$$

This readily yields the following:

Corollary 3.3. *If λ is any one of the four functions μ, μ^*, ν, or ν^*, then it is sublinear in the first variable:*

$$\lambda(n_1 + n_2, m) \le \lambda(n_1, m) + \lambda(n_2, m)\,, \tag{3.4.2}$$

as long as this equation makes sense (i.e. we need $n_1, n_2 \ge m$ if $\lambda = \mu^*$ or $\lambda = \nu^*$).

4. THE CASE $m = 2$

4.1. **Proof of Theorem 1.1.**

Proof. Trivially $\mu(1,2) = 1$, and it is easy to check that $\mu(2,2) = 2$. Taking W_3 as in (3.3.2), we see that $M(W_3, 0) = 2$, and so $\mu(3,2) \le \mu^*(3,2) \le 2$. By combining (3.4.2) with these facts, we see that for $k \in \mathbb{Z}$, $k \ge 0$, and $i \in \{0,1,2\}$,

$$\mu(3k+i, 2) \le k\mu(3,2) + \mu(i,2) \le 2k + i\,.$$

Since $2k + i = \left\lceil \frac{2(3k+i)}{3} \right\rceil$, this gives the sharp upper bound for $\mu(n,2)$. The corresponding sharp upper bound for $\mu^*(n,2)$ follows similarly when $n \ge 1$ has the form $3k$ or $3k+2$, $k \ge 0$. If $n = 3k+1$, $k \ge 1$, only a small change is required to the μ-proof to get a proof of the sharp μ^* upper bound:

$$\mu^*(3k+1, 2) \le (k-1)\mu^*(3,2) + 2\mu^*(2,2) = 2k + 2\,.$$

It remains to show that we can reverse the above inequalities. We first examine the reverse inequalities for μ^*, so fix $W \in A^*(n,2)$. Writing $F : S \to 2^S$ for the edge function, where $S := S(n)$, we get a well-defined function $f : S \to S$ by writing $f(i) = j$ whenever there is an edge from i to $j \ne i$ in the associated graph G. For a dynamical system on any finite set, every point is either periodic or preperiodic. In our context, this just means that if we apply f repeatedly starting from any initial vertex $i \in S$, then we eventually get a repeat of an earlier value, and from then on the iterated values of f go in a cycle.

Note that the topological components of G do not "interfere" with each other: the vertices in any one component affect only the on/off status of vertices in this component, so maximizing the number of lit vertices can be done one component at a time (alternatively, this follows from (3.4.1) after reordering of the vertices).

A component of the graph G consists of a central circuit containing two or more vertices, with perhaps some directed trees, each of which leads to some vertex of the circuit, which we call the *root* of that tree. Starting from the outermost vertices of such a tree (those that are not in the range of f) and working our way down to the root, it is not hard to see that we can simultaneously light all vertices in each of these trees. Having done this, some of the vertices in the central circuit may not be lit up. We follow the vertices around the circuit in cyclic order, pressing each vertex that is unlit when we reach it until we have gone fully around the circuit. It is clear that at this stage at most one vertex in the circuit is unlit, and all the associated trees (excluding the roots) are still fully lit.

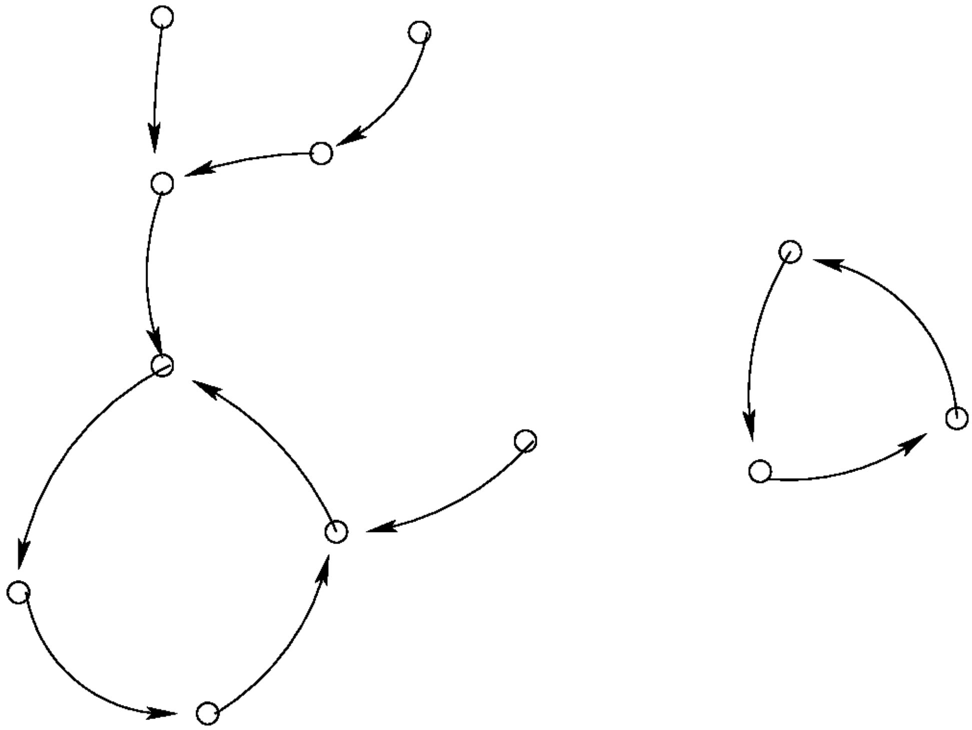

FIGURE 8. 'Dynamics' of $m = 2$

Note that any single vertex press either leaves the number of lit vertices in a given component unchanged, or changes that number by 2. Since initially all vertices are unlit, it follows that the number of lit vertices in a component is always even. It therefore follows that in a component of even cardinality all vertices can be lit, while in a component of odd cardinality all except one can be lit.

Thus, it follows that to minimize $M(W, 0)$ we need to maximize the number of components of odd cardinality (necessarily at least 3), and that the maximum proportion of lit vertices in any one component is at least $2/3$ (with equality only for components of cardinality 3). Thus $\mu^*(n, 2) \geq \lceil 2n/3 \rceil$, which gives the required lower bound except when $n = 3k + 1$, $k \in \mathbb{N}$. Since G has $n = 3k + 1$ vertices and all components have at least two vertices, it can have at most $k - 1$ components of odd cardinality, yielding the desired estimate $\mu^*(3k + 1, 2) \geq 3k + 1 - (k - 1) = 2k + 2$. Thus $\mu^*(n, 2)$ is given by the stated formula in all cases.

For μ, the above proof goes through with little change. We define $f(i)$ as before whenever Button i switches two bulbs, and $f(i) = i$ otherwise. The graph G can now have *prefixed components* where the central circuit contains only a single vertex, corresponding to a fixed point of f. However, it is clear from our earlier arguments that prefixed components can always be fully lit, so only the odd cardinality *non-prefixed components* (i.e. those without a fixed point) can contribute unlit bulbs. Thus, $\mu(n, 2) \geq \lceil 2n/3 \rceil$, as required. □

Although prefixed components do not contribute unlit vertices in the last paragraph of the above proof, singleton components (corresponding to a vertex with no inbound or outbound edge) are important since they allow us to get k, rather than just $k-1$, non-prefixed components of odd cardinality when $n = 3k + 1$. This accounts for the difference between $\mu(n, 2)$ and $\mu^*(n, 2)$ in this case.

It follows from the above proof that a wiring minimizes $M(W, 0)$ in either $A^*(n, 2)$ or $A(n, 2)$ if and only if its associated graph maximizes the number of non-prefixed components of odd cardinality among the allowed set of graphs. Such components have cardinality at least 3 so, for n a multiple of 3, this means that each component must have three vertices and correspond (up to permutation) to one or other of the matrices

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

For n of the form $3k + 1$ or $3k + 2$, it similarly follows easily from the extremality criterion that all components except at at most two are of cardinality 3 and have one of the two above forms. The possible exceptional components depend on the mod-3 nature of n, as well as whether we are looking at $A^*(n, 2)$ or $A(n, 2)$, but all are of cardinality 2, 4, 5, or 7. We leave to the reader the routine but tedious task of using the above extremality criterion to find all such sets of exceptional components.

5. Pivoting and the case $m = 3$

5.1. **Pivoting.** In preparation for the proof of Theorem 1.2, we introduce the concept of *pivoting*. Pivoting about a vertex i, $1 \leq i \leq n$, is a way of changing the given wiring W to a special wiring W^i such that $M(W^i, c) \leq M(W, c)$. Additionally, pivoting preserves the classes $A(n, m)$ and $A^*(n, m)$.

Fix a wiring $W = (w_{i,j})$ and initial configuration c, and let $F : S \to 2^S$ denote the edge function associated to W, where $S = S(n)$. Given $i \in S$, let $M_i = M(W^i, c)$ where the *pivoted wiring matrix* W^i is defined by the condition that its jth column equals the ith column of W if $j \in F(i)$, and equals the jth column of W otherwise. In other words, W^i rewires the system so that pressing the jth vertex has the same effect as pressing the ith vertex in the original system whenever $j \in F(i)$. On the other hand, it is easy to see that M_i is the maximum value of $|Wx + c|$ over all vectors x such that $x_j = 0$ whenever $j \in F(i) \setminus \{i\}$. In fact, any attainable set of lit bulbs for the wiring W^i and initial configuration c can be achieved without pressing any of the buttons in $F(i) \setminus \{i\}$. Hence, the same set of lit bulbs can be achieved with the original wiring W without pressing any of those buttons. In particular, $M_i \leq M(W, c)$. See Figure 9 for examples.

Pivoting about i, as defined above, is a process with several nice properties:

- it does not increase the value of M: $M(W^i, c) \leq M(W, c)$;
- it preserves membership of the classes $A(n, m)$ and $A^*(n, m)$;(In fact, if $j \in F(i)$, then $W^i_{j,j} = 1$, that is W^i still has 1's along the diagonal. This is the only property that actually requires checking in order to verify that the classes $A(n, m)$ and $A^*(n, m)$ are preserved.)
- if F^i is the edge function of W^i, then $F^i(i) = F(i)$ is a forward invariant augmented complete subgraph of the associated graph G^i.

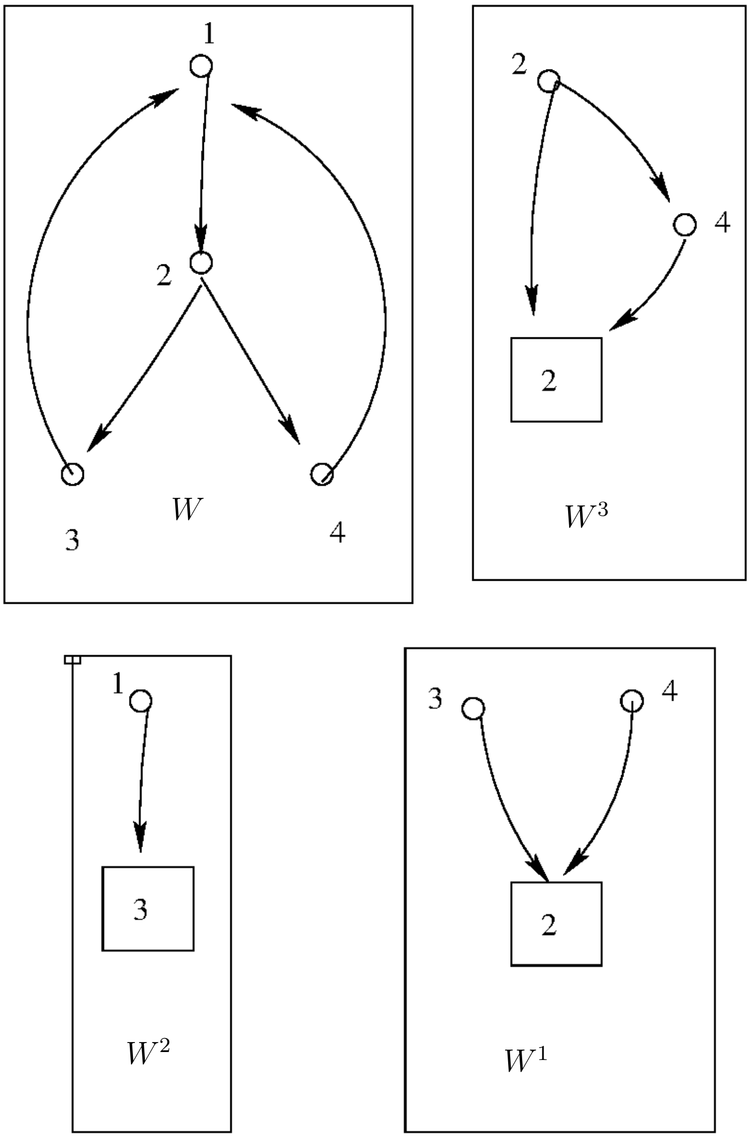

FIGURE 9. Pivoting

It is sometimes useful to pivot *partially* about i: given $T \subset S$, and $i \in S$, we define W' by replacing the jth column of W by its ith column whenever $j \in F(i) \setminus T$. Such *pivoting about* i *with respect to* T satisfies the same non-increasing property, preserves membership in $A(n, m)$ and $A^*(n, m)$, and $F(i) \setminus T$ is a (not necessarily forward invariant) augmented complete subgraph of the associated graph G'.

5.2. Pivoting is the key trick in the proof of the following lemma.

Lemma 5.1. *Let* $m \geq 2$ *and* $n \geq 1$. *Then either* $\mu(n+m, m) = \mu(n+m, m-1)$, *or*

$$\mu(n+m, m) \geq \mu(n, m) + \nu(m, m) = \mu(n, m) + \lceil m/2 \rceil \; .$$

Proof. Suppose $\mu(n+m, m) < \mu(n+m, m-1)$, and let $W \in A(n+m, m)$ be such that $M(W, 0) = \mu(n+m, m)$. Then, W has a vertex i of degree m. By minimality of W, pivoting about i gives $W^i \in A(n+m, m)$ with $M(W^i, 0) = \mu(n+m, m)$ (cf. Figure 10. The loop marked n just indicates an unspecified subgraph of order n.). For the wiring W^i, we first press a set of vertices in $S(n+m) \setminus F(i)$ so as to maximize the number

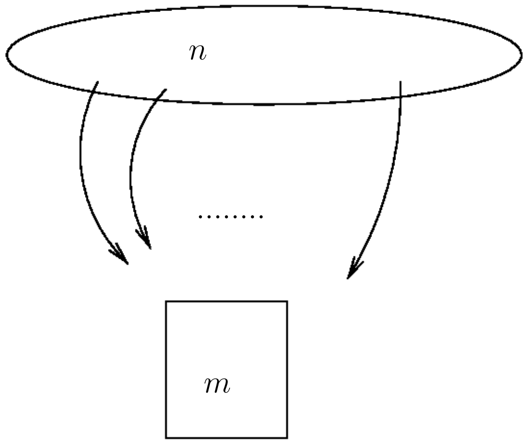

FIGURE 10. W^i

of lit vertices in $S(n+m) \setminus F(i)$, and then we press vertex i if fewer than half of the vertices in $F(i)$ are lit. By forward invariance of $F(i)$, the result follows. □

Proof of Theorem 1.2(a). Trivially, we have that $\mu(n,3) \le \mu(n,2)$, with equality if $n < 3$. It is also immediate that $\mu(3,3) = \mu(3,2) = 2$: any wiring that includes a vertex of degree 3 allows us to light all vertices by pressing the degree 3 vertex.

Suppose therefore that $\mu(n',3) = \mu(n',2)$ for all $n' < n$, where $n > 3$. Either this equation still holds when n' is replaced by n, or

$$\mu(n,2) = \mu(n-3,2) + 2 = \mu(n-3,3) + 2 = \mu(n-3,3) + \nu(3,3) \le \mu(n,3) \le \mu(n,2).$$

Here, the first equality follows from Theorem 1.1, the second from the inductive hypothesis, and the first inequality from Lemma 5.1. Since $\mu(n,2)$ is at both ends of this line, we must have $\mu(n,3) = \mu(n,2)$, and the inductive step is complete. □

5.3. For the proof of Theorem 1.2(b), we need another lemma.

Lemma 5.2. *Let $n, m, n' \in \mathbb{N}$, with $n \ge m$. Then*

$$\mu^*(n+n', m+1) \le \mu^*(n,m) + n' .$$

Proof. It suffices to prove the lemma subject to the restriction $n' \le n$, since this case, the trivial estimate $\mu^*(n,m) \le n$, and sublinearity (3.4.2) together imply the general case. Let us therefore assume that $n' \le n$.

Let $V = (v_{i,j}) \in A^*(n,m)$ be such that $M(V,0) = \mu^*(n,m)$. We now define a matrix $W = (w_{i,j}) \in A^*(n+n', m+1)$. First the upper left block of W is a copy of V, i.e. we let $w_{i,j} = v_{i,j}$ for all $1 \le i,j \le n$. Next, the $n' \times n$ block of W below V consists of copies of the $n' \times n'$ identity matrix; the last of these copies will be missing some columns unless n is a multiple of n'. Lastly, we define $w_{i,n+j} = w_{i,j}$ for all $1 \le j \le n'$. It is straightforward to verify that $W \in A^*(n+n', m+1)$; note that the assumption $n' \le n$ ensures that W has 1's along the diagonal. Refer to Figure 11 for a schematic. Note that vertex $6+i$ has the same targets as vertex i, but these edges going to vertices other than 7, 8 or 9 are not shown.

Since all columns after the nth column are repeats of earlier columns, it suffices to consider what happens when we press only combinations of the first n vertices. Such combinations light at most $\mu^*(n,m)$ of the first n vertices, so we are done. □

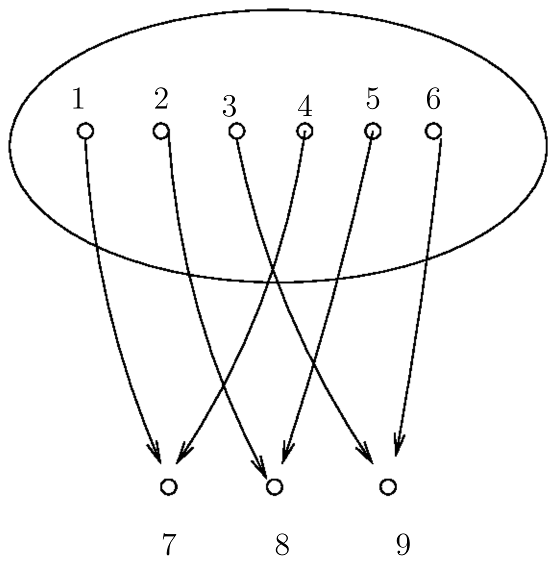

FIGURE 11. $n = 6$, $n' = 3$

5.4. **Proof of Theorem 1.2(b).**

Proof. Lemma 5.2 ensures that if $k, i \in \mathbb{N}$, then $\mu^*(3k+i,3) \le \mu^*(3k,2) + i = 2k+i$. This is the required sharp upper bound if $i = 1, 2$, since $2k+1 = \mu(3k+i,3)$ in this case. On the other hand, $\mu^*(3k+i,3) \ge \mu(3k+i,3) = 2k+i$, for all $k \in \mathbb{N}$ and $i = 1, 2$, and this gives the required converse for $i = 1, 2$.

It remains to handle the case where n is a multiple of 3. First, we show that the lower bound $\mu^*(3k,3) \ge \mu(3k,3) = 2k$ is sharp when $k = 2k'$ is even. Letting

$$W_6 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix} \in A^*(6,3)\,, \tag{5.4.1}$$

we claim that $M(W_6, 0) = 4$. Assuming this claim, (3.4.2) gives the desired sharpness: $\mu^*(6k',3) \le k'\mu^*(6,3) \le k'M(W_6,0) = 4k'$.

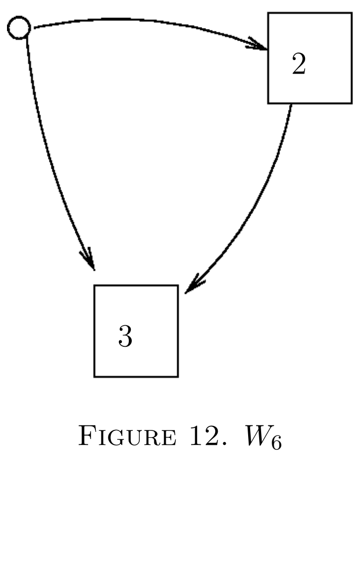

FIGURE 12. W_6

To establish the claim, it suffices to consider sets of vertex presses involving only vertices 1, 2, and 4. With this restriction, we proceed to list all eight possible values of x, and deduce that $M(W_6, 0) = 4$:

x^t	$(W_6x)^t$	$\lvert W_6x\rvert$
(0,0,0,0,0,0)	(0,0,0,0,0,0)	0
(1,0,0,0,0,0)	(1,1,0,0,1,0)	3
(0,1,0,0,0,0)	(0,1,1,1,0,0)	3
(1,1,0,0,0,0)	(1,0,1,1,1,0)	4
(0,0,0,1,0,0)	(0,0,0,1,1,1)	3
(1,0,0,1,0,0)	(1,1,0,1,0,1)	4
(0,1,0,1,0,0)	(0,1,1,0,1,1)	4
(1,1,0,1,0,0)	(1,0,1,0,0,1)	3

(Here x^t denotes the row-vector transpose of the column vector x.)

It remains to handle the case where $n = 6k' - 3$ for some $k' \in \mathbb{N}$. It is trivial that $\mu^*(3,3) = 3$. Next note that Lemma 5.2 ensures that for $k \geq 2$, $\mu^*(3k,3) \leq \mu^*(3k-3,2) + 3 = 2k+1$, so we need to show that this is sharp if $k > 1$ is odd.

Supposing $\mu^*(n,3) \leq 2k$ for some fixed $n = 3k$, $k \in \mathbb{N}$, $k > 1$, we will prove that k must be even. Let $W = (w_{i,j}) \in A^*(n,3)$ be such that $M(W,0) \leq 2k$, let $S = S(n)$, and let $F : S \to 2^S$ be the edge function associated to W.

We can assume that W is additionally chosen so that the associated graph G has a maximal number of (disjoint) $\hat{K}_3$'s among all matrices $W' \in A^*(n,3)$ for which $M(W',0) = 2n/3$. The maximum number of $\hat{K}_3$'s is always positive since we can get a $\hat{K}_3$ by pivoting about any one vertex; $\hat{K}_3$ sets are pairwise disjoint and forward invariant, since each vertex in a $\hat{K}_3$ uses up its two allowed outbound edges within the same $\hat{K}_3$.

We define A to be the union of all the $\hat{K}_3$ sets. If $i \in S \setminus A$, then $F(i) \cap A$ must be nonempty, since otherwise pivoting about i would create an extra $\hat{K}_3$. Thus, each $i \in S \setminus A$ has at most one edge from it to another vertex in $S \setminus A$. Suppose there is such a vertex i with $F(i) \setminus \{i\}$ not a subset of A. Then, we can pivot about i relative to A to get a $\hat{K}_2$, and the only edges coming from this $\hat{K}_2$ are single edges from both of its vertices to the same element in A. We repeat such pivoting of vertices relative to A to create more such $\hat{K}_2$s until this is no longer possible. From now on, W will denote this modified wiring matrix. We denote by B the union of the $\hat{K}_2$ vertices and write $C = S \setminus (A \cup B)$, and we refer to each vertex in C as a $\hat{K}_1$ (which it is, trivially).

We already know that there is an edge from each vertex in C to some vertex in A. If there is only a single edge from some $i \in C$ to $A \cup B$, then there must be an edge from i to some $j \in C$. Pivoting about i relative to $A \cup B$ (or equivalently, relative to A), we create a new $\hat{K}_2$, contradicting the fact that this cannot be done. Thus, there are two edges from each $i \in C$ to $A \cup B$. See Figure 13.

We have shown that there are edges from C to $A \cup B$, and from B to A, but that both A and $A \cup B$ are forward invariant. Also, there are no links between elements in C, or between elements in distinct $\hat{K}_2$'s or in distinct $\hat{K}_3$'s. There are $3s$ elements in A, $2t$ elements in B, and u in C, for some integers s, t, u, and we have $3s + 2t + u = n$.

The forward invariance of both A and $A \cup B$ suggests two algorithms for lighting many of the vertices. The first is to begin by pressing all these vertices in C to light all these vertices. After this first step, we can ensure that at least one vertex in each $\hat{K}_2$ is lit by pressing a vertex in any $\hat{K}_2$ without a lit vertex. Finally, we ensure that at least two vertices are lit in each $\hat{K}_3$ by pressing a vertex in any $\hat{K}_3$ in which fewer

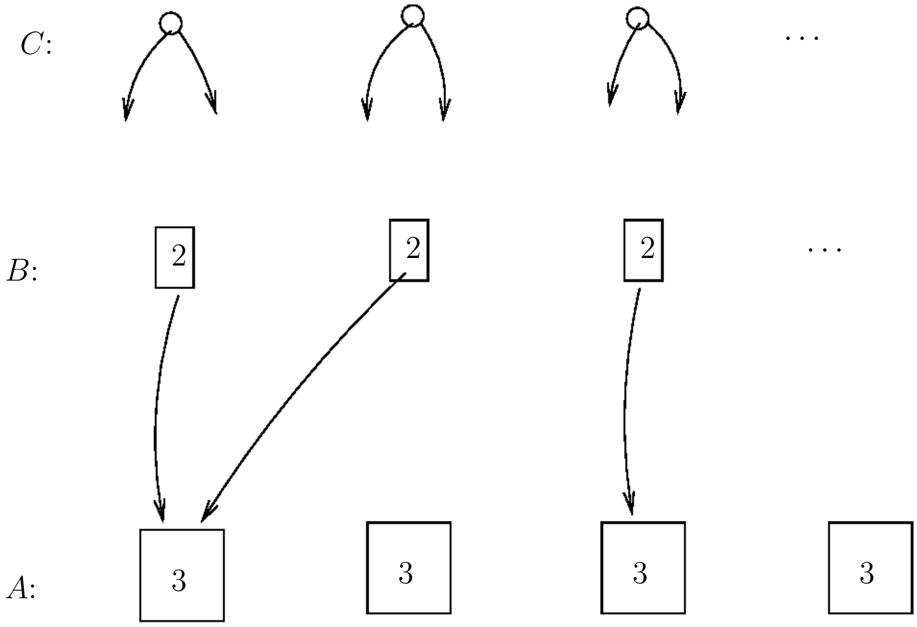

Figure 13.

than two vertices are lit. Having done this, we have at least $2s+t+u$ lit vertices, so $2s+t+u \le \mu^*(n,3)$. Thus, $6s+3t+3u \le 3\mu^*(n,3) \le 2n$. When we compare this with the equation $6s+4t+2u=2n$, we deduce that $t \ge u$.

An alternative algorithm for lighting the vertices is to first press one vertex in each $\hat{K}_2$, thus lighting all $\hat{K}_2$ vertices. As a second step, press a vertex in any $\hat{K}_3$ in which fewer than 2 vertices are lit. Having done this, at least two vertices in each $\hat{K}_3$ are lit as well as both vertices in each $\hat{K}_2$. Consequently, $2s+2t \le \mu^*(n,3) \le 2n/3$. Thus, $6s+6t \le 2n$, while $6s+4t+2u=2n$. It follows that $u \ge t$, and so $u=t$.

Note that the first lighting algorithm gives at least $2s+2t=2n/3$ lit vertices, and it actually gives more than this number unless after the first step exactly one vertex in each $\hat{K}_2$ is lit. Since any larger number contradicts $\mu^*(n,3)=2n/3$, there must be an edge from C to each $\hat{K}_2$. But, since the numbers of $\hat{K}_1$'s and of $\hat{K}_2$'s are equal, and there is at most one edge from each $\hat{K}_1$ to B (since at least one edge from each $\hat{K}_1$ goes to A), it follows that from each $\hat{K}_1$ there is an edge to a $\hat{K}_2$, and no other vertex in C is linked to the same $\hat{K}_2$, i.e. we can pair off each $\hat{K}_1$ with the unique $\hat{K}_2$ to which it is linked in the graph. See Figure 14. We refer to the subgraph of G given by the union of a $\hat{K}_1$ and a $\hat{K}_2$ plus the edge between them as a $C_{1,2}$; the set of its three vertices is a $C_{1,2}$ set.

The second lighting algorithm will give more than $2s+2t=2n/3$ lit vertices unless the first step ends with one or two lit vertices in each $\hat{K}_3$. Thus, there is an edge from at least one $\hat{K}_2$ to each $\hat{K}_3$. Since any one $\hat{K}_2$ is linked to only a single $\hat{K}_3$, it follows that $t \ge s$.

We now define the *active vertices* to be all $\hat{K}_1$ vertices, together with one vertex from each $\hat{K}_2$, and the *active edges* are all the edges coming from active vertices. When considering the effect of pressing sets of vertices in $B \cup C$, we can restrict ourselves to considering only sets of active vertices, hence the terminology.

To light more than two thirds of the vertices, it suffices to first light two vertices in every $C_{1,2}$ set in such a way that there is at least one $\hat{K}_3$ that is either fully lit or

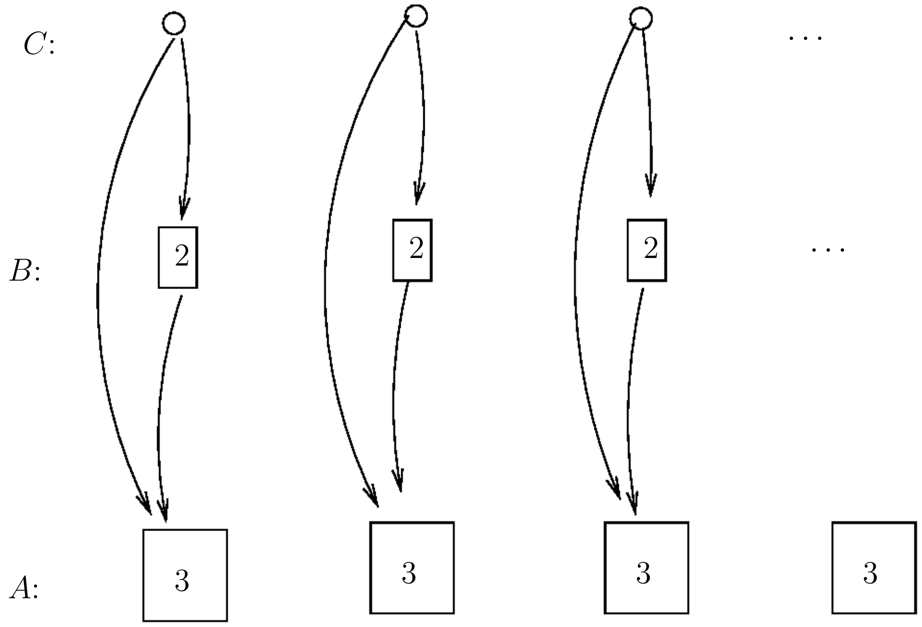

FIGURE 14.

fully unlit, since we can subsequently light two thirds of all vertices in all other $\hat{K}_3$ sets, together with all vertices in the fully unlit or fully lit $\hat{K}_3$, by pressing only $\hat{K}_3$ vertices. Since each $\hat{K}_3$ is forward invariant, we are done.

But, given a $C_{1,2}$ set with all vertices unlit, pressing one or both of its active vertices leaves exactly two of its vertices lit. This gives us three ways of lighting two thirds of the vertices in that $C_{1,2}$ set, and this flexibility will be crucial to proving that n must be a multiple of 6. In particular, it means that for any given $\hat{K}_3$, there must be an associated $C_{1,2}$ both of whose active vertices have edges leading to that $\hat{K}_3$, since if this were not so, we could light two vertices in each $C_{1,2}$ without ever pressing a vertex linked to that $\hat{K}_3$. Furthermore, even if a $C_{1,2}$ is doubly linked to a $\hat{K}_3$, but the two active edges between them connect to the same vertex, then by pressing both active vertices, the on/off status of all vertices in the $\hat{K}_3$ remains unchanged. Let us therefore say that a $C_{1,2}$ set with two active links to distinct vertices in a $\hat{K}_3$ is *well linked* to that $\hat{K}_3$ set. We say that they are *badly linked* if they are linked but not well linked.

It follows that S can be decomposed into a collection of $C_{1,2}$ sets, each of which is paired off with a distinct $\hat{K}_3$ set to which it is well linked, plus $t - s$ extra $C_{1,2}$ sets that have not been paired off with any $\hat{K}_3$, but are linked (well or badly) to some of the $\hat{K}_3$'s. We claim that if $t > s$ then the residual $C_{1,2}$ sets always allow us to arrange that at least one $\hat{K}_3$ is fully lit or fully unlit after we light two vertices in every $C_{1,2}$. It follows from this claim that n cannot be an odd multiple of 3, since then we would have $t - s > 0$, and we could light more than two thirds of the vertices.

Suppose therefore that $t > s$, and so there exists some particular $\hat{K}_3$ with vertex set $D = \{a, b, c\}$, say, that has more than one $C_{1,2}$ linked to it, at least one of which is well linked. We wish to show that we can press one or both of the active vertices in each of the $C_{1,2}$'s linked to D while keeping D *in sync* (meaning that all three of its vertices are in the same on/off state).

Now D is initially in sync, and we can handle any two well-linked $C_{1,2}$'s while keeping D in sync. To see this, note that if the two pairs of active links go to the same pair of vertices in D, then we press all four active vertices in both $C_{1,2}$'s. If on the other hand, they do not go to the same pair of vertices then without loss of generality, one $C_{1,2}$ is linked to a and b and the other to b and c. By pressing three of the four active vertices, we can toggle the on/off status of all three vertices in D.

Since we can handle well-linked $C_{1,2}$'s two at a time, and we can handle badly linked ones one at a time, while keeping D in sync, we can reduce to the situation of having to handle only two or three $C_{1,2}$'s, with at least one of them well linked. We have already handled the case of two well-linked $C_{1,2}$'s, so assume that there are two $C_{1,2}$'s and exactly one is well linked, to a and b, say, while the other is badly linked, with either one or two links to a single vertex $v \in D$. By symmetry, we reduce to either of two subcases: if $v = a$, then we press one active vertex in both $C_{1,2}$'s that is connected to a, while if $v = c$, then we press three vertices so as to toggle the on/off status of all of D.

There remains the case of three linked $C_{1,2}$'s. If two are well linked and one badly linked, then we just handle the two well-linked ones together as above, and separately handle the badly linked one. Finally, all three may be well linked. If all three $C_{1,2}$'s link to the same pair of vertices, a and b, say, then we press both active vertices in one of them and one in the other two, to ensure that both a and b are toggled twice (and so unchanged). If two $C_{1,2}$'s link to the same pair of vertices, a and b, say, and the third links to b and c, say, then we can press one vertex in each $C_{1,2}$ to ensure that all three vertices in D are toggled once. Finally, if no two $C_{1,2}$'s leads to the same pair of vertices, then one leads to a, b, another to b, c, and a third to c, a. We can press all six of the active vertices so as to toggle each of a, b, c twice. This finishes the proof of the theorem. □

5.5. **Remark.** Note that even when n is a multiple of 6, the above argument gives us some extra information: after suitable pivoting, any wiring $W \in A^*(n, 3)$ with $M(W, 0) = 2n/3$ must reduce to a collection of $C_{1,2}$'s each of which is well linked to a distinct $\hat{K}_3$. Each associated subgraph with six vertices is a component of the full graph and is unique (up to relabeling of the vertices). Moreover, it is the graph of the wiring W_6 in (5.4.1) so, after suitable pivoting, any wiring $W \in A^*(n, 3)$ with $M(W, 0) = 2n/3$ reduces to $n/6$ copies of W_6.

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A simpler proof of Lima's dilogarithm identity

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Abstract. From a closed-form expression for a hyperbolic integral, I derived in 2012 a non-trivial two-term dilogarithm identity for $\mathrm{Li}_2\left(\sqrt{2}-1\right)+\mathrm{Li}_2\left(1-1/\sqrt{2}\right)$. In recent works published in this Bulletin, Campbell (2021) has applied a series transform obtained via Fourier-Legendre theory to find a new proof for that identity, whereas Stewart (2022), working independently, used known functional relations for the dilogarithm function to develop three other proofs, which has renewed interest in this subject. In this short note, I show how Hill's five-term relation can be applied to suitable algebraic points in order to get a simpler proof of that identity.

1. Introduction

The dilogarithm function is a classical function introduced by Leibnitz in 1696, defined as $\mathrm{Li}_2(z) := \sum_{n=1}^{\infty} z^n/n^2$, which converges for all complex z with $|z| \le 1$. This function can be extended to all $z \in \mathbb{C}\backslash(1,\infty)$ through the integral representation

$$\mathrm{Li}_2(z) := -\int_0^z \frac{\ln(1-t)}{t}\, dt\,. \tag{1}$$

Although this integral cannot be expressed as a finite combination of elementary functions, as follows from a theorem by Liouville (1837) [10], closed-forms are currently known for only a few special values, namely $\mathrm{Li}_2(0)=0$, $\mathrm{Li}_2(1/2)=\pi^2/12-\ln^2 2/2$, $\mathrm{Li}_2(-1)=-\pi^2/12$, $\mathrm{Li}_2(1)=\pi^2/6$, $\mathrm{Li}_2(\pm i)=-\pi^2/48\pm i\,G$, $\mathrm{Li}_2(1\pm i)=\pi^2/16\pm i\,(G+\pi\ln 2/4)$, $\mathrm{Li}_2(1/2\pm i/2)=5\,\pi^2/96-\ln^2 2/8\pm i\,(G-\pi\ln 2/8)$, $\mathrm{Li}_2(-\phi)=-\pi^2/10-\ln^2\phi$, $\mathrm{Li}_2(-1/\phi)=-\pi^2/15+\frac{1}{2}\ln^2\phi$, $\mathrm{Li}_2(1/\phi)=\pi^2/10-\ln^2\phi$, $\mathrm{Li}_2(1/\phi^2)=\pi^2/15-\ln^2\phi$, where $G:=\sum_{n=0}^{\infty}(-1)^n/(2n+1)^2$ is Catalan's constant and $\phi:=\left(1+\sqrt{5}\right)/2$ is the golden ratio. In fact, closed-form expressions remain scarce even for two-term linear combinations with rational coefficients of this function at algebraic points (some examples are given in Refs. [4] and [8, Chaps. 1 and 2], and references therein). Interestingly, in 2012, on investigating a hyperbolic version of the trigonometric change of variables introduced by Beukers, Calabi and Kolk to show that $\sum_{n=1}^{\infty} 1/n^2=\pi^2/6$ (the so-called *Basel problem*) [1], I found that (see Theorem 3 of Ref. [9])

$$\int_{\alpha/2}^{\infty} \ln(\tanh z)\, dz = \frac{\alpha^2}{4}-\frac{\pi^2}{16}\,,$$

where $\alpha := \ln(\sqrt{2}+1)$. This allowed me to derive the following two-term dilogarithm identity (see Theorem 4 of Ref. [9]):

$$\mathrm{Li}_2\left(\sqrt{2}-1\right)+\mathrm{Li}_2\left(1-\frac{1}{\sqrt{2}}\right)=\frac{\pi^2}{8}-\frac{\alpha^2}{2}-\frac{\ln^2 2}{8}\,. \tag{2}$$

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Since then, this result has attracted the interest of some mathematicians, among them Campbell, who in 2021 used a series transform obtained via Fourier-Legendre theory (see Refs. [3] and [6, Sec. 7.14]) to present an independent proof in this Bulletin [2], a work which he complemented one year later with a careful investigation of previous equivalent results, which has led to a result of 1915 by Ramanujan, as seen in Eqs. (3) and (4) of Ref. [4]. Also in 2022, Stewart has found three other distinct proofs for Eq. (2) by exploring some known functional relations for the dilogarithm function [11].

However, all these approaches involve complex mathematical steps or are somewhat lengthy. In this short note, I apply Hill's five-term relation to get a simpler proof of Eq. (2).

2. Main Result

According to a conjecture of 1995 by Kirillov [7], it should be possible to derive all two-term dilogarithm identities from Hill's five-term relation (1830) [5][1]

$$L(x\,y) = L(x) + L(y) - L\left(\frac{x\,(1-y)}{1-x\,y}\right) - L\left(\frac{y\,(1-x)}{1-x\,y}\right)\,, \tag{3}$$

where x and y are two complex numbers such that $|x| < 1$ and $0 < y < 1$, or $|y| < 1$ and $0 < x < 1$, or $x < 1$ and $0 < y < 1$, or $y < 1$ and $0 < x < 1$. Note that, for simplicity, it is stated in terms of the normalized Rogers' dilogarithm $L(z) := \frac{6}{\pi^2}\left[\mathrm{Li}_2(z) + \frac{1}{2}\ln z\,\ln(1-z)\right]$, as usual. On taking Kirillov's conjecture as a motivation, after many attempts I have succeeded in finding a suitable pair of algebraic arguments x and y for which Hill's five-term relation reduces to the identity in Eq. (2).

Proof of Eq. (2). On taking $x = 2-\sqrt{2}$ and $y = 1/\sqrt{2}$, for which $x\,y = \sqrt{2}-1$, in Hill's five-term relation, our Eq. (3), one finds

$$L\left(\left(2-\sqrt{2}\right)\frac{1}{\sqrt{2}}\right) = L\left(2-\sqrt{2}\right) + L\left(\frac{1}{\sqrt{2}}\right) - L\left(\frac{3-2\sqrt{2}}{2-\sqrt{2}}\right) - L\left(\frac{1/\sqrt{2}-\sqrt{2}+1}{2-\sqrt{2}}\right), \tag{4}$$

which promptly simplifies to

$$L\left(\sqrt{2}-1\right) = L\left(2-\sqrt{2}\right) + L\left(\frac{1}{\sqrt{2}}\right) - L\left(1-\frac{1}{\sqrt{2}}\right) - L\left(\frac{1}{2}\right)\,. \tag{5}$$

Now, one applies Euler's reflection formula (1768) $L(z) = 1 - L(1-z)$ to both $L\left(2-\sqrt{2}\,\right)$ and $L\left(1/\sqrt{2}\,\right)$. Since Euler's reflection yields $L(1/2) = 1/2$, one finds

$$L\left(\sqrt{2}-1\right) = 1 - L\left(\sqrt{2}-1\right) + 1 - L\left(1-\frac{1}{\sqrt{2}}\right) - L\left(1-\frac{1}{\sqrt{2}}\right) - \frac{1}{2}\,, \tag{6}$$

which promptly reduces to

$$L\left(\sqrt{2}-1\right) + L\left(1-\frac{1}{\sqrt{2}}\right) = \frac{3}{4}\,, \tag{7}$$

which is just the Rogers equivalent of Eq. (2). □

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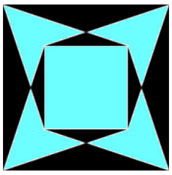

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Paul J. Nahin: The Mathematical Radio: Inside the Magic of AM, FM, and Single-Sideband, Princeton University Press, 2024. ISBN:978-0-69123-531-8, GBP 22.00, 376 pp.

REVIEWED BY EDWARD L. BACH

The book title is "The Mathematical Radio: Inside the Magic of AM, FM, and Single-Sideband". This suggests that the author, Paul J. Nahin, wishes to discuss mathematical aspects of radio technology, and that he regards this technology as something magical. In order to enjoy the book, readers need to expend some effort to immerse themselves in the mathematical and technological detail, while keeping in mind that the mathematics is beautiful and the technology enables them to have magical experiences. It is not a book that can be easily skimmed, and there are a lot of equations and circuit diagrams.

There seem to be two motivations for writing the book: the passion of an early and lifelong radio hobbyist, and the irritation of an applied mathematician, who has read "A Mathematician's Apology" and, instead of perhaps taking Hardy's opinions with a grain of salt, has taken them to heart. If the reader is not on the same wavelength as the author, the book can appear a bit laboured in places.

The book follows the history of radio and starts with the problem of how to generate and receive radio signals. It is recommended to read the appendix on Maxwell's equations first. There are further chapters on multiplier circuits, AM, sideband, and FM radio. The author points out the mathematical theories and results which impact or motivate the design of the components used in radio transmission and reception, and provides challenge problems at the end of each chapter, to allow the reader to work through some of the details or to elaborate on statements made in the chapter text.

A reader who has a good grounding in second-year university mathematics and physics should be able to understand the text and do the problems. Both text and problems provide a detailed illustration of how mathematical equations and techniques underpin a real world application.

It is not clear to me how this book would be used in teaching; I suppose it would be possible to include it as part of a one-term mathematical methods course. I believe the book is intended more for mathematicians to read for recreation or for enrichment.

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G. Cohen: The Possibly True Story of Martin Gardiner, Halstead Press, 2022.
ISBN:978-1-9-25043-69-3, AUD 34.95, 280+viii pp.

REVIEWED BY CLIFFORD GILMORE

This historical novel is inspired by the curious life of the Irish-Australian mathematician Martin Gardiner. The protagonist left an intriguing paper trail during his lifetime, that ranged from mathematical publications to judicial proceedings. However, very little is otherwise known about the man himself. In this work, the author Graeme Cohen takes the verifiable events from Gardiner's life and weaves the extraordinary tale of a driven researcher who became embroiled in political scandals, ill-fated business ventures and salacious liaisons.

Gardiner was born in Dublin around 1833 and studied at Queen's College Galway, before he dropped out of his engineering course after two years. He arrived with his wife and two small children in Melbourne in 1856, and it is in Australia where he left his mark. His mathematical passion was in the area of geometry, a topic on which he published over a dozen research articles. Although his publication record exceeded those of contemporary mathematicians in Australia, he struggled to secure the academic position that he felt he deserved.

Having to eke out a living as a surveyor or as an occasional mathematics teacher, his professional frustration was compounded by his propensity for workplace conflict, where he considered his superiors and colleagues his intellectual inferiors. This bitter cocktail led to a turbulent career, where the luckless protagonist never lasted very long in any position before the inevitably acrimonious parting of ways.

Cohen develops the plausible character of a conceited mathematician and malcontent, whose awkward personality frequently obstructed his own ambitions. However, this is not the simple story of a tortured genius battling against the world. Indeed, Gardiner's single-minded pursuit of mathematical research impacted on his personal life, where he revealed a less palatable dimension to his character.

In the genre of historical fiction, it is natural to wonder how much of a story is based on fact and how much is fiction. Fortunately for us, Gardiner teasingly left traces of his movements and activities through letters written to newspapers, advertisements placed offering services as a maths tutor, public challenges made to the mathematicians of Australia, and parliamentary and court proceedings. His legacy is highly unusual; thus it is unsurprising that Cohen became interested in the story of this eccentric character. So, in the spirit of the oft stated Irish adage of not letting the truth get in the way of a good story, the author takes these data points and masterfully performs an act of interpolation to link them via the engaging narrative of our antihero.

Set against the backdrop of the daily struggles of early colonial Australia, the author paints the picture of a brilliant researcher and a flawed character. Moreover, the technical aspects of Gardiner's research are kept to a minimum, so this novel can easily be enjoyed by readers that do not possess deep mathematical knowledge. Cohen has thus

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done an excellent job of candidly bringing Gardiner's story to life, so I fully recommend this as a highly entertaining and sometimes shocking read.

Editor's note: see also Graeme Cohen's *Letter to the Editor* in Number 92, Winter 2023, and his article *Martin Gardiner: the first Irish-Australian mathematician* in Number 85, Summer 2020.

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J. Cruickshank et al.: Irish Mathematical Olympiad Manual, Logic Press, 2023.
ISBN: 978-1447791355, EUR 12.99, 156 pp.

REVIEWED BY JESSICA SEARANCKE

One of the first things I noticed about the the Irish Mathematical Olympiad Manual is the way it begins: with general problem-solving advice, quickly followed by a summary of the basic knowledge any reader should already have from the Junior Certificate. I have to say it bluntly: the problem-solving advice is really good. Stuff that should be straightforward, and that makes a massive difference to your abilities, and yet so many students don't know it: Spread out your work because paper is cheap. Understand the target fully, and find proxy targets to help solve it. Lots of these tips I use already, some of them I don't - but regardless, I've never seen them written down in this way before. Lots of people view problem-solving as something you either can do, or you can't do, in a direct relationship with your IQ score. By putting problem-solving skills down to intelligence alone, people tend to ignore the core skills that make the difference between someone who's confused and gives up, and someone who's confused but figures it out. This book puts it differently, listing lots of the skills that make this difference.

My only criticism of this section of the book is its understandability. I could immediately understand and relate to the skills that I currently use, but it was less easy to fully grasp how to use the unfamiliar skills in a real problem. Inserting an example or two of how the techniques could be used, following this section (or each point individually), would make a real difference to how easily students can apply the advice given. The problems at the end of the chapter go some way toward this goal, but they lack worked solutions, or specific links to the tips. Worked solutions would demonstrate the advice, and would change the way students work through problems.

The next section of the book reviews the baseline knowledge that is needed before using the manual. The way in which it concisely lists rules and facts is ideal for this kind of book, since readers are likely to already be familiar with such information, but could do with a reminder, or a point for future reference. For me, such rules are far too easy to forget, but can also be quickly refreshed into the working memory. Similarly, the review of basic trigonometry clearly demonstrates the progression from this baseline knowledge to the far more complex use of these functions in later chapters (including De Moivre's theorem). Such progression illustrates the versatility of the manual for a range of abilities of reader, since it spans several years of schooling. Unfortunately page 6 appears to contain an error on its first line ('concyclic!points'). I expect this to change in future editions. *(Editor's Note: this is corrected in the direct-sales edition).*

Past this point, the book begins to review or teach (depending on your position) a range of different concepts. Having already studied maths at a high pre-university level, I am familiar with all of the initial concepts, I have encountered many of the intermediary concepts, and lots of the most complex concepts are completely new to me. The entire book is formatted in black and white text, which is spread out clearly and logically. This separates the manual from a revision guide: it is a workbook, to be

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worked through in order, if possible, to gradually develop further mathematical skills. I believe this book would be ideal, not only to prepare for a Mathematical Olympiad, but also to develop essential skills prior to studying maths or other related subjects at university. The skills developed in this book cannot be taught from a one-off YouTube video or web search, but can be gained over time by working through a manual such as this one. The inclusion of regular practice problems and examples supports this gradual skill development.

To criticise the main body of the book, I would like to first address the use of proofs to teach new topics. They are heavily used to introduce new ideas, from the sine and cosine rules, all the way to Ceva's theorem. For me, I completely endorse the use of proofs to do this, however I feel that these could sometimes be better explained. For example, many proofs might be useful in understanding a concept, but the application of this knowledge is more important. In some cases, the proof dominates the explanation of a concept. In other cases, it is unclear whether the authors intend the resulting fact to be learnt and used by rote, or whether they are simply introducing the reader to a way of relating different facts in a problem. Further worked problems would help solve this issue by demonstrating how the facts are intended to be used, and how they are most likely to come up in a real Olympiad question.

In order to better explain the issue I have just raised, I will give some examples: The beginning of Section 6.2 (Combinatorics and Binomial Coefficients) is very well explained - it begins by addressing the meaning of the factorial symbol, explains logically how numbers of combinations and permutations are worked out, and gives some examples to demonstrate this point. Similarly, Section 10.2 (Some Theorems about Triangles) addresses each theorem individually, accompanied by several example questions for each. However, also in Section 10.2, some of the exercises are heavily proof-dominated (such as for Ptolemy's theorem). The reader will have little understanding of how to apply their new knowledge, only how to prove it. Even if Olympiad questions do have a tendency to focus on proof, understanding can be significantly enhanced by candidates also knowing how to apply the concept.

On page 25, a complex proof is given involving the semiperimeter, however no diagram is included. As a result, it is unclear to a reader what 'semiperimeter' refers to (if this is explained elsewhere in the book, a page reference should be given). The meaning of a, b, c, A, B and C - all letters which are used in the proof - is also unclear. As well as this, on page 29, the relationship between an inscribed circle and the surrounding triangle is described. In my experience, this is extremely useful to recognise in Olympiad-style questions. However the formula given is one that I would be unlikely to memorise pre-exam. Do the authors simply wish to introduce the reader to this style of comparison, or do they actually recommend learning the formula? The answer to this question should be explicit. Also, if the former is true, I think there could be more effective ways of introducing this concept. My final point to address is the index, which is highly comprehensive. I would only like to question the inclusion of specific angles at the beginning of the index, for example $15°$ and $270°$, which seems unnecessary and of little use.

Overall, I believe the book is almost ideal for all those studying maths pre-university, including those preparing for Mathematical Olympiads. It is effective in developing problem-solving skills, including those involved with proof. My criticisms widely revolve around the necessity for further examples to demonstrate some of the concepts taught. I would highly recommend this book for able students wishing to push their studies in mathematics further in their final one or two years of school, and those developing essential mathematical skills prior to entering university.

Jessica Searancke is an A-Level Mathematics and Further Mathematics student, intending to study Engineering at university next year. She has participated in the UK Maths Trust Challenge several times, as well as the Mathematical Olympiad for Girls, achieving the Gold Award and a Distinction respectively. Outside of her mathematical studies, she enjoys fantasy and dystopian novels, poetry, running and investigating the risks posed by AI.

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PROBLEMS

J.P. MCCARTHY

Problems

We will ease into things with a problem from your erstwhile contributor, a version of which the previous editor felt was on the easy side, making it perhaps suitable for inviting undergraduates to engage with:

Problem 94.1. Let X, Y be independent and identically distributed random variables with values in a finite group G. Let $H < G$ be a subgroup such that $\mathbb{P}[X \in H] \in (1/2, 1)$. Prove that

$$\mathbb{P}[XY \in H] < \mathbb{P}[X \in H].$$

More meatier group theory, this time courtesy of Des MacHale of University College Cork:

Problem 94.2. If G is a group with centre Z and $|G/Z| = n!$, for some integer $n > 1$, show that G/Z is non-abelian.

The problem is stated for not-necessarily-finite groups, but solutions in the finite case are welcome. On the other hand, Des MacHale invites you to consider the following problem: for which numbers other than $n!$ does this result hold?

The following problem was provided by Anthony O'Farrell (Maynooth University) and Maria Roginskaya (Chalmers University of Technology):

Problem 94.3. A very large number of prizes are available for children at a big party thrown by a billionaire. The prizes are numbered $1, 2, 3, \ldots,$ and are to be shared between a boy and a girl. Each boy at the party is given a card with a number in 1, 2, 3, ..., different for each boy, and the same is done for each girl, but it is possible that a boy will have the same number as some girl. There are m boys and n girls. A number $d \geq 1$ is specified, and this determines the rule for the allocation of prizes, as follows. The prize labelled p is allocated to the first boy-girl partnership who present cards labelled a and b, where $a + b = p$, and where a differs from b by no more than d. Having claimed a prize with some girl, a boy is free to claim others with other girls, and similarly for girls. Thus, as the party progresses, the children will repeatedly pair up and claim prizes, until all the prizes that can possibly be claimed are taken. Show that the number of prizes that can be claimed is less than $13\sqrt{mnd}$.

The following hint is provided: let k be a nonnegative integer, and use induction on k to get the best inequality you can for the number of prizes under the additional assumption that $mn \leq 2^k \cdot d$.

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Solutions

Here are solutions to the problems from *Bulletin* Number 92.

The first problem was solved by the North Kildare Mathematics Problem Club, and the proposer, Des MacHale of University College Cork. We present the solution of Problem Club.

Problem 92.1. Show that the infinite cyclic group is not the full automorphism group of any group.

Solution 92.1. Suppose $\mathrm{Aut}(G)$ is cyclic. Then so is the subgroup of inner automorphisms, which is isomorphic to G/Z (where Z is the centre of G). Let kZ generate G/Z. For $g, h \in G$ choose $m, n \in \mathbb{N}$ and $z, w \in Z$ with $g = k^m z$ and $h = k^n w$. Then

$$gh = k^m z k^n w = k^{m+n} zw = k^{m+n} wz = hg.$$

Therefore $G = Z$.

Since G is abelian, $\tau : g \mapsto g^{-1}$ is an automorphism of G, and $\tau^2 = 1$. If we suppose, in addition, that $\mathrm{Aut}(G)$ is infinite or of finite odd order, then $\tau = 1$, i.e. each element of g has $g^2 = 1$. Thus G is a vector space over $\mathbb{Z}_2$. Each permutation of a basis of G over $\mathbb{Z}_2$ gives an automorphism of G. At dimension greater than two, these permutations give non-commuting elements in $\mathrm{Aut}(G)$. At dimension two $\mathrm{Aut}(\mathbb{Z}_2 \times \mathbb{Z}_2) = S_3$. It follows that G has dimension at most one. But then $\mathrm{Aut}(\mathbb{Z}_2)$ is trivial, not infinite cyclic. □

The Problem Club leaves an aside: the proposer also posed the question of determining which finite cyclic groups could be $\mathrm{Aut}(G)$ for some group G.

The infinite cyclic group has the cyclic group of order two as its automorphism group. We have already seen that $\mathbb{Z}_n$ is never $\mathrm{Aut}(G)$ if $n > 1$ and n is odd.

If G is cyclic of order m, then $\mathrm{Aut}(G)$ is isomorphic to the multiplicative group of the ring of integers modulo m, and the order of $\mathrm{Aut}(G)$ is $\phi(m)$. This group is cyclic if and only if $\phi(m)$ is a product of distinct primes, so if and only if m is a prime power p^k, and $(p-1)p^{k-1}$ is a product of distinct primes. Thus, whenever p is prime and $p-1$ is square-free we have two groups $\mathbb{Z}_p$ and $\mathbb{Z}_{p^2}$ with cyclic automorphism groups, of respective orders $p-1$ and p^2-p. The first few orders of $\mathrm{Aut}(G)$ that arise this way are $1, 2, 6, 10, 22, 30$, and $2, 6, 42, 110, 486, 930$ (resulting from the primes $2, 3, 7, 11, 23, 31$).

For an abelian product group $G \times H$, the automorphism group contains $\mathrm{Aut}(G) \times \mathrm{Aut}(H)$, and hence has at least two non-trivial involutions and is not cyclic, unless $\mathrm{Aut}(G)$ or $\mathrm{Aut}(H)$ is trivial. Thus the only finitely-generated abelian groups G with cyclic $\mathrm{Aut}(G)$ are the cyclic examples just described, and their products with groups having only the identity automorphism..

It remains to consider abelian G that are not finitely-generated.

Suppose $n > 0$, that $\mathbb{Z}_{2n}$ is isomorphic to $\mathrm{Aut}(G)$, and let σ generate $\mathrm{Aut}(G)$. We know that G is abelian, so $\tau : g \mapsto g^{-1}$ is an automorphism. There are two possibilities:

Case 1: $\tau = 1$. Then as before, G has dimension at most one over $\mathbb{Z}_2$, so $\mathrm{Aut}(G)$ is trivial, a contradiction.

Case 2: $\tau \neq 1$. Then $\sigma^n = \tau$. Replacing G by its quotient by the subgroup fixed by $\mathrm{Aut}(G)$, we may assume that each element of G except 1 is moved by some automorphism, and hence is moved by σ. So each nonzero element $g \in G$ moves in a cycle of order $\alpha(g)$ dividing $2n$, under the action of $\mathbb{Z}_{2n}$. Let

$$\beta = \mathrm{lcm}\{\alpha(g) : g \in G\}.$$

Then $\beta|2n$ and $\sigma^\beta = 1$, so $\beta = 2n$. We can choose a finite number of elements $g_1,\ldots,g_m$ such that

$$2n = \text{lcm}\{\alpha(g_1), \ldots, \alpha(g_m)\}.$$

Let $H = \langle g_1, \ldots, g_m\rangle$. Then H is finitely-generated, and invariant under σ, and $\sigma|H$ has order $2n$. If $\sigma|H$ generates $\text{Aut}(H)$, we have seen that $2n = p$ or $2n = p^2 - p$ for some prime p such that $p - 1$ is square-free.

But does every automorphism of H extend to an automorphism of G?

The second problem was solved by the North Kildare Mathematics Problem Club, and the proposer Andrei Zabolotskii of the Open University. We provide the solution of the Problem Club.

*Problem 92.2***.** Let A be a symmetric square matrix of even order over the ring of integers modulo 2. Suppose that all entries on the leading diagonal of A are 0. Let B be the square matrix obtained from A by replacing each 0 entry with 1 and replacing each 1 entry with 0. Prove that $\det A = \det B$.

Solution 92.2. First note that $x^2 = x$ for x in $\mathbb{Z}_2$. Also $+1 = -1$, so the sign of a permutation is 1 in $\mathbb{Z}_2$. Now let $A = [a_{ij}]$ be a symmetric $2n \times 2n$ matrix, entries in $\mathbb{Z}_2$, zero on the diagonal.

So $\det(A)$ is the sum

$$a_{1,\sigma(1)} \times \cdots \times a_{2n,\sigma(2n)}$$

where σ ranges over all permutations of $1, .., 2n$. As $a_{ij} = a_{ji}$, we can cancel such a term with that arising from σ^{-1}, when $\sigma \neq \sigma^{-1}$. Thus only permutations that are involutions can survive. Also we can remove terms from involutions which fix one or more points (as they involve a diagonal entry in A). Finally, each term a_{ij} will be matched by $a_{ji} = a_{ij}$. So their product can be recorded as a_{ij}.

Thus

$$\det(A) = \sum a_{i_1,i_2} \times \cdots \times a_{i_{2n-1},i_{2n}},$$

where $\{i_1, i_2\}, \{i_3, i_4\}, \ldots \{i_{2n-1}, i_{2n}\}$ ranges over all partitions of the set $\{1, 2, .., 2n\}$ into two-element subsets. There are $(2n)!/(2^n n!)$ such partitions, an odd number.

Let J be the all 1's matrix. We need to compare $\det(A)$ with $\det(A + J)$. Analysing as above, we now have to sum over all involutions of $1, .., 2n$ (counting the identity as an involution).

$$\det(A + J) = \sum(1 + a_{i1,i2}) \times \cdots \times (1 + a_{i_{2n-1},i_{2n}}),$$

plus all sums involving fewer products of the same type. When these products are all expanded, the coefficient of a given product $a_{i_1,i_2}a_{i_3,i_4} \cdots a_{i_{2r-1},i_{2r}}$ is (equal modulo 2 to) the number of involutions of $\{1, \ldots, 2n\}$ that fix $\{i_1, i_2, i_3, i_4, \ldots, i_{2r-1}, i_{2r}\}$. When $r < n$, the coefficient equals the number of involutions of a set of $2n - 2r$ elements, which is even, so zero modulo 2. (This applies even to the empty product, 1, so there is an even number of 1's). Hence the only terms that survive are those with $r = n$, and these sum to $\det(A)$. □

Readers were asked to consider the more challenging question of whether or not the characteristic polynomials of A and B are equal. The Problem Club provided a "leisurely version" of the above proof which was a wonderful interplay between orbits, involutions, and fixed points. The approach also spoke to the case of matrices with entries in a commutative ring R with identity, where key was the language of a matrix in $M_n(R)$ as a function $x : \mathcal{P}_1([n]) \sqcup \mathcal{P}_2([n]) \to R$. The technology in the leisurely version helped answer the challenging question in the positive: indeed the characteristic polynomials of A and B are equal.

The third problem was solved by the North Kildare Mathematics Problem Club; the proposer, Tran Quang Hung of the Vietnam National University at Hanoi, Vietnam; Kee-Wai Lau of Hong Kong, China; and your erstwhile contributor. Here is one of those solutions:

Problem 91.3. For $x > 0$, let $\mu(x)$ denote the ℓ_∞-norm of the sequence

$$u_n(x) = \frac{x^n}{n^n}, \quad n = 1, 2, \ldots .$$

Determine

$$\lim_{x\to\infty} \frac{\log \mu(x)}{x}.$$

Solution 92.3. For fixed $x > 1$, extend the sequence $\mu_n(x)$ to a function $f_x : [1, \infty) \to (0, \infty)$:

$$f_x(y) = \frac{x^y}{y^y}.$$

It is strictly positive as $f_x(y) = \exp\left(\log\left(\frac{x}{y}\right) y\right)$. Its derivative with respect to y is:

$$\frac{d}{dy}(f_x(y)) = f_x(y)\left(\log\left(\frac{x}{y}\right) - 1\right).$$

Note as $f_x(y)$ is strictly positive, this derivative is strictly positive on $[1, x/e)$, and strictly negative on $(x/e, \infty)$.

It follows that for fixed $x > 1$, $\mu(x)$ attains its maximum at $\lfloor x/e \rfloor$ or $\lceil x/e \rceil$. Therefore we know that for some $z_x \in (-1, 1)$ the maximum occurs at

$$\frac{x}{e} + z_x.$$

We calculate, using the fact that x can be chosen large enough to make each of the manipulations valid:

$$\mu(x) = \left(\frac{x}{\frac{x}{e} + z_x}\right)^{\frac{x}{e}+z_x}.$$

Then,

$$\begin{aligned}\log \mu(x) &= \left(\frac{x}{e} + z_x\right) \log\left(\frac{x}{\frac{x}{e} + z_x}\right) \\ &= \frac{1}{e}(x + ez_x) \log\left(e \cdot \frac{x}{x + ez_x}\right) \\ &= \frac{1}{e}(x + ez_x)\left[\log e + \log\left(\frac{x}{x + ez_x}\right)\right],\end{aligned}$$

and so

$$\frac{\log \mu(x)}{x} = \frac{1}{e}\left(1 + \frac{ez_x}{x}\right)\left[1 + \log\left(\frac{1}{1 + \frac{ez_x}{x}}\right)\right].$$

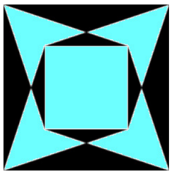
As a consequence,

$$\lim_{x\to\infty} \frac{\log \mu(x)}{x} = \frac{1}{e} \times 1 \times (1 + \log(1)) = \frac{1}{e}. \qquad \square$$

We invite readers to submit problems and solutions. Please email submissions to `imsproblems@gmail.com` in any format (but preferably LaTeX). Submissions for the summer Bulletin should arrive before the end of April, and submissions for the winter Bulletin should arrive by October. The solution to a problem is published two issues after the issue in which the problem first appeared. Please include solutions to any problems you submit, if you have them.

Finally, I would like to thank Ian Short for his many, many years of service to this problem page. With your help, we can continue Ian's great work.

Department of Mathematics, Munster Technological University



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