Irish Mathematical Society
Cumann Matamaitice na hÉireann

Bulletin

Number 92 Winter 2023

ISSN 0791-5578

Irish Mathematical Society Bulletin

The aim of the *Bulletin* is to inform Society members, and the mathematical community at large, about the activities of the Society and about items of general mathematical interest. It appears twice each year. The *Bulletin* is published online free of charge.

The *Bulletin* seeks articles written in an expository style and likely to be of interest to the members of the Society and the wider mathematical community. We encourage informative surveys, biographical and historical articles, short research articles, classroom notes, book reviews and letters. All areas of mathematics will be considered, pure and applied, old and new.

Correspondence concerning books for review in the *Bulletin* should be directed to

`mailto://reviews.ims@gmail.com`

All other correspondence concerning the *Bulletin* should be sent, in the first instance, by e-mail to the Editor at

`mailto://ims.bulletin@gmail.com`

and only if not possible in electronic form to the address

The Editor
Irish Mathematical Society Bulletin
Department of Mathematics and Statistics
Maynooth University
Co. Kildare W23 HW31

Submission instructions for authors, back issues of the *Bulletin,* and further information about the Irish Mathematical Society are available on the IMS website

`http://www.irishmathsoc.org/`

The Irish Mathematical Society is a registered charity
(RCN: 20020279).

Irish Mathematical Society Bulletin 92, Winter 2023

CONTENTS

EDITORIAL

In your editor's youth, it was customary to expose primary schoolcildren to fine verse, including that of Thomas Gray. Reflecting on the lives of the rural poor, he wrote:

Full many a gem of purest ray serene
The dark unfathomed caves of ocean bear:
Full many a flower is born to blush unseen,
And waste its sweetness on the desert air.

His point was that outstanding achievement in public life, in art and literature, or in any worthy field of endeavour, requires both talent and opportunity. The simple fact is that much talent goes to waste for lack of opportunity. From the point of view of our discipline, it is to be regretted whenever people who might have elucidated its major problems are prevented by personal circumstances from giving the problems their undivided attention. For this reason, we disapprove of war, famine, pestilence, and any prejudice embedded in societal structures that might hinder the exposure of some genius to the question about the location of the zeta zeros. Accordingly, I applaud the spread of universal education, and positive action to overcome the barriers related to characteristics such as class, gender, and disability. In this, I am not adopting a controversial position. It's just commonsense.

We received a letter from the Association for Mathematical Research (AMR), a fairly new organisation devoted to supporting mathematical research and scholarship. On enquiry, it emerged that the AMR has been the subject of controversy, so rather than publishing their letter I have chosen to refer members to the article[1] about the matter by Rachel Crowell in the Scientific American for January 2022. The essence of the controversy seems to involve the question whether it is appropriate for any such organisation to have a policy of saying nothing at all about the injustices that suppress some talent.

Other correspondence included a letter informing us about the publication of a text: *Introduction à l'Etude des Probabilités Expérimentales. Un livre de probabilités pour les Ingénieurs* by Bernard Beauzamy. It is published by his Société de Calcul Mathématicque. Members who are familiar with Beauzamy's achievements and original views on the proper conduct of mathematical research and applications may wish to take note[2].

The Problem Page is a perennially-popular feature of the Bulletin, and is ably curated by Ian Short. Members will note a bit of an innovation this time, in that two of the new problems are supplemented by mention of related open questions. It is an interesting fact that the frontier of our understanding is never very far away from the things we actually know: the 'inner diameter' of mathematics is small. Many of the most entertaining puzzles are by-products of attempts to do something serious.

During 2023, Thomas Ungar stepped down from the Editorial Board, and Colm Mulcahy joined. I would like to thank Thomas for his unfailing help and expertise. Colm will be looking after obituaries.

---

[1] `https://www.scientificamerican.com/article/new-math-research-group-reflects-a-schism-in-the-field/`

[2] ISBN : 979-10-95773-02-3. Pour éditer un bon de commande : `http://www.scmsa.eu/livres/SCM_IEPE_order.htm` .

For a limited time, beginning as soon as possible after the online publication of this Bulletin, a printed (grayscale, not full-colour) and bound copy may be ordered online on a print-on-demand basis at a minimal price[3].

Editor, Bulletin IMS, Department of Mathematics and Statistics, Maynooth University, Co. Kildare W23 HW31, Ireland.
E-mail address: ims.bulletin@gmail.com

[3]Go to www.lulu.com and search for *Irish Mathematical Society Bulletin*.

LINKS FOR POSTGRADUATE STUDY

The following are the links provided by Irish Schools for prospective research students in Mathematics:

DCU: mailto://maths@dcu.ie
TUD: mailto://chris.hills@tudublin.ie
ATU: mailto://leo.creedon@atu.ie
MTU:
http://mathematics.mtu.ie/datascience
UG: mailto://james.cruickshank@universityofgalway.ie
MU: mailto://mathsstatspg@mu.ie
QUB:
http://web.am.qub.ac.uk/wp/msrc/msrc-home-page/postgrad_opportunities/
TCD: http://www.maths.tcd.ie/postgraduate/
UCC: https://www.ucc.ie/en/matsci/study-maths/postgraduate/#d.en.1274864
UCD: mailto://nuria.garcia@ucd.ie
UL: mailto://sarah.mitchell@ul.ie

The remaining schools with Ph.D. programmes in Mathematics are invited to send their preferred link to the editor.

E-mail address: ims.bulletin@gmail.com

Letters to the Editor

Martin Gardiner
From Graeme Cohen

Dear Professor O'Farrell,

In Issue Number 85 (Summer 2020, 3-–15), you published an article of mine, *Martin Gardiner: the first Irish–Australian mathematician*, in which I recounted much of what I knew then regarding Gardiner's life and work in Australia in the second half of the nineteenth century. He was a surveyor by trade, but, though he had only two years' formal training in surveying, civil engineering and other studies (in Queen's College, Galway), he proved to be a perceptive and acknowledged geometer, exceeding the output of Australia's other mathematicians of the time.

I was aware, when I wrote the article, of a number of descendants that Gardiner had in Australia, and, as I had done before, I continued afterwards to seek out information on his life. Numerous sources were available: his publications in local Royal Society Journals and Proceedings; his letters to newspapers as a constant commentator on the actions of government; and birth, death and marriage certificates. Important aspects of his life however remained opaque to me, so I ignored those (or, rather, invented alternative situations) and went ahead to write my first novel, i*The Possibly True Story of Martin Gardiner.*

Within a few weeks of the article in your Bulletin appearing online, I was contacted by John Gardiner, a great grandson of Martin, living in Sydney, with further news of Martin's life. Although I had written of Martin spending time in Queensland, quoting newspaper references concerning his surveying work there in the 1870s and 1880s, for example, John's news floored me. I knew of Martin's wife Bridget whom he brought from Ireland to Melbourne in 1856, and his three children with her; and I knew of his wife Emma whom he married in Sydney after Bridget's death, and his three children with her; but I knew nothing of his subsequent liaison with Caroline in Brisbane, and his three children with her. That is the news John gave me. To John Gardiner, his great grandfather was known as George. That is the name that Caroline knew him by. Yet there was enough family history to convince me that Martin and George were the same person. So, if nothing else, my novel needed more work. What I had previously written became Part I, necessarily modified. Martin Gardiner's life with Caroline became Part II.

May I recommend that your readers obtain a copy of *The Possibly True Story of Martin Gardiner*? The work is so much the better because of the publication of his really true story a few years before, what I knew of it at the time, in your Bulletin.
Yours sincerely

Graeme Cohen

Received 28-7-2023
g.cohen@bigpond.net.au

NOTICES FROM THE SOCIETY

Officers and Committee Members 2023

President	Dr Leo Creedon	ATU
Vice-President	Dr Rachel Quinlan	UG
Secretary	Dr Derek Kitson	MIC
Treasurer	Dr Cónall Kelly	UCC

Dr T. Carroll, Dr R. Flatley, Dr R. Gaburro, Dr D. Mackey, Prof. M. Mathieu, Prof. A. O'Shea, Dr R. Ryan, Assoc. Prof. H. Šmigoc, Dr N. Snigireva.

Officers and Committee Members 2024

President	Dr Leo Creedon	ATU
Vice-President	Dr Rachel Quinlan	UG
Secretary	Dr Derek Kitson	MIC
Treasurer	Dr Cónall Kelly	UCC

Dr C. Boyd, Dr T. Carroll, Dr R. Flatley, Dr R. Gaburro, Dr T. Huettemann, Prof. A. O'Shea, Assoc. Prof. H. Šmigoc, Dr N. Snigireva.

Local Representatives

Belfast	QUB	Prof M. Mathieu
Carlow	SETU	Dr D. Ó Sé
Cork	MTU	Dr J. P. McCarthy
	UCC	Dr S. Wills
Dublin	DIAS	Prof T. Dorlas
	TUD, City	Dr D. Mackey
	TUD, Tallaght	Dr C. Stack
	DCU	Prof B. Nolan
	TCD	Prof K. Soodhalter
	UCD	Dr R. Levene
Dundalk	DKIT	Mr Seamus Bellew
Galway	UG	Dr J. Cruickshank
Limerick	MIC	Dr B. Kreussler
	UL	Mr G. Lessells
Maynooth	MU	Prof S. Buckley
Sligo	ATU	Dr L. Creedon
Tralee	MTU	Prof B. Guilfoyle
Waterford	SETU	Dr P. Kirwan

Applying for I.M.S. Membership

(1) The Irish Mathematical Society has reciprocity agreements with the American Mathematical Society, the Deutsche Mathematiker Vereinigung, the Irish Mathematics Teachers Association, the London Mathematical Society, the Moscow Mathematical Society, the New Zealand Mathematical Society and the Real Sociedad Matemática Española.

(2) The current subscription fees are given below:

Institutional member	€250
Ordinary member	€40
Student member	€20
DMV, IMTA, NZMS, MMS or RSME reciprocity member	€20
AMS reciprocity member	$25
LMS reciprocity member (paying in Euro)	€20
LMS reciprocity member (paying in Sterling)	£20

(3) The subscription fees listed above should be paid in euro by means of electronic transfer, a cheque drawn on a bank in the Irish Republic, or an international money-order.

The subscription fee for ordinary membership can also be paid in a currency other than euro using a cheque drawn on a foreign bank according to the following schedule:

If paid in United States currency then the subscription fee is US$ 40.
If paid in sterling then the subscription is £30.
If paid in any other currency then the subscription fee is the amount in that currency equivalent to US$ 40.00.

The amounts given in the table above have been set for the current year to allow for bank charges and possible changes in exchange rates.

(4) Any member with a bank account in the Irish Republic may pay his or her subscription by a bank standing order using the form supplied by the Society.

(5) Any ordinary member who has reached the age of 65 years and has been a fully paid up member for the previous five years may pay at the student membership rate of subscription.

(6) Subscriptions normally fall due on 1 February each year.

(7) Cheques should be made payable to the Irish Mathematical Society.

(8) Any application for membership must be presented to the Committee of the I.M.S. before it can be accepted. This Committee meets three times each year.

(9) Please send the completed application form, available at
`http://www.irishmathsoc.org/links/apply.pdf`
with one year's subscription to:

Dr Cónall Kelly
School of Mathematical Sciences
Western Gateway Building, Western Road
University College Cork
Cork, T12 XF62
Ireland

Deceased Members

It is with regret that we report the deaths of members:

Seán Tobin, of UG, who died on 11 July 2023.

David Flannery, of MTU, who died on 8 August 2023.

E-mail address: subscriptions.ims@gmail.com

PRESIDENT'S REPORT 2023

Committee changes: From January 2024, the Officers of the Society remain unchanged. Thanks to Martin Mathieu and Ray Ryan for their many years of service to the Society, including decades of service as committee members, Officers and Editors of the Bulletin. Christopher Boyd (UCD) and Thomas Huettemann (QUB) are new committee members of the IMS, having been elected at the AGM on September 1, 2023.

IMS Bulletin: The Bulletin is freely available online from the Society's homepage. The Society buys a complimentary printed copy for Institutional members and the copyright libraries.

IMS meetings: The Society's annual 'September Meeting', was held at the University of Limerick on August 31 and September 1, 2023. The meeting was very well organised by Romina Gaburro (UL, co-chair), Derek Kitson (MIC, co-chair), Ronan Flatley (MIC), Eugene Gath (UL), Natalia Kopteva (UL), Kevin Moroney (UL), and Clifford Nolan (UL). The next meeting of the IMS will take place in Queen's University Belfast on August 29 and 30, 2024. The report on the 2023 IMS Annual Meeting and the Minutes of the 2023 IMS AGM are available elsewhere in the Bulletin.

Given that 2023 was my first year as President of the IMS, I have thought a great deal about the IMS, its nature, its role and its future. To me one of the main functions of the IMS is to bring us together, to build a mathematical community which is open and welcoming, especially to young mathematicians and to mathematicians who are new to Ireland. Building a welcoming community will attract people to study and work in mathematics and will benefit all of us and benefit Ireland, north and south. We have many new members, so allow me to quote from the aims of the IMS from its constitution:

> "The Society is incorporated for the purpose of promoting and extending the knowledge of mathematics and its applications. Activities proper to the Society shall include the following:
> (a) holding meetings of the members of the Society and visitors introduced by them,
> (b) publishing and distributing the Bulletin of the Society,
> (c) organizing and supporting conferences, lectures, and discussions on subjects of special and general interest to mathematicians,
> (d) discovering and making known the views of the members of the Society on mathematical matters of public interest,
> (e) co-operating with other organizations to achieve the purpose of the Society."

The IMS is a registered charity and is run by volunteers. The IMS Committee now meets in person three times a year and the four IMS Officers meet online more frequently. Thankfully, IMS membership numbers have grown rapidly since then, but as of August 2023, there were 320 members. I thank our Treasurer Conall Kelly for providing the data used to summarise membership by membership type and gender in the following two figures. This analysis also showed that the vast majority of members are resident in the Republic of Ireland, with others resident in Northern Ireland, Great Britain, USA, and elsewhere.

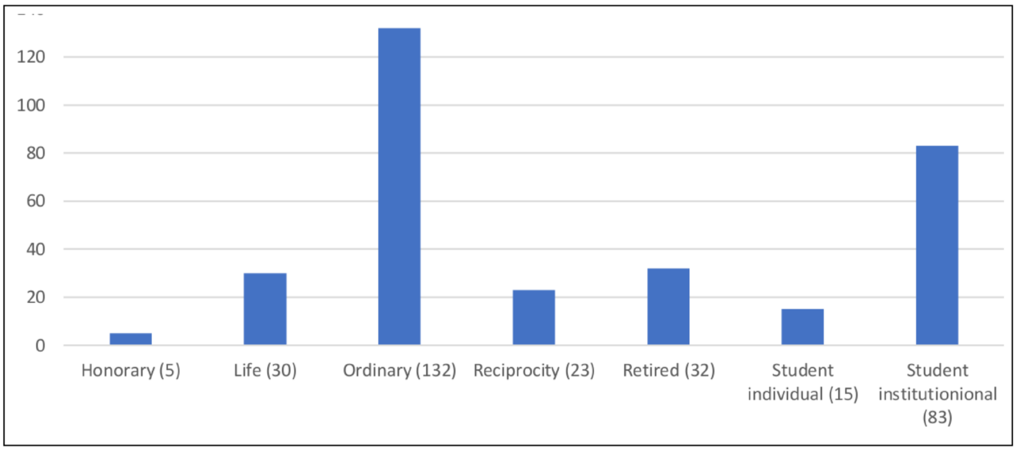


FIGURE 1. IMS Individual Membership (320) August 2023

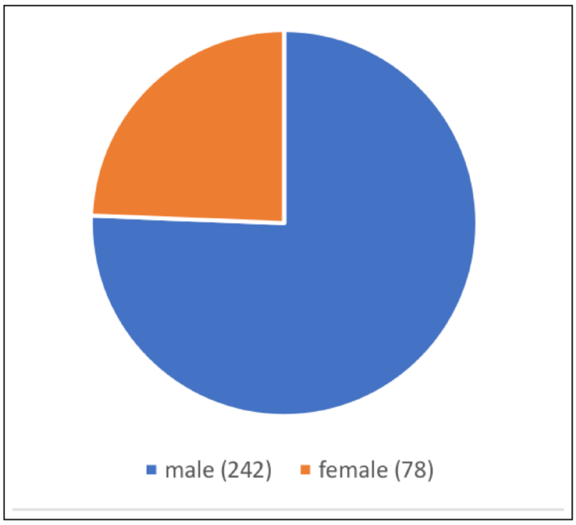


FIGURE 2. IMS Membership Gender (approximate figures) August 2023

Organisations relevant to mathematics in Ireland (in addition to the universities and schools) include the Irish Mathematics Trust, the Irish Mathematics Teachers' Association, the Royal Irish Academy, and two government departments: the Department of Education and the Department of Further and Higher Education, Research, Innovation and Science. Relevant overseas and international organisations include mathematical societies in other countries (including many with whom the IMS has reciprocity arrangements for membership), the International Mathematical Union (IMU), the International Commission on Mathematical Instruction (ICMI), and the European Mathematical Society.

The IMS has established a new 'Irish Committee for Equality, Diversity and Inclusion in Mathematics' (ICEDIM) which is chaired by Romina Gaburro and has commenced its work. On July 1, 2023, the IMS appointed Professor Ann O'Shea as the Ireland Country Representative on the International Commission on Mathematical Instruction ICMI. Ann replaced Maurice O'Reilly who had served in this role for several years. Ann is chair of the Irish Committee for Maths Education (ICME). See `https://www.mathunion.org/icmi/organization/overview-icmi` for details. Having been nominated by the IMS for membership of the European Mathematical Society's EMS Young Academy

(EMYA), Róisín Neururer (UCD) was subsequently elected as chair of EMYA. EMYA will be organising some activities at the 9th European Congress of Mathematics in Seville in July 2024 and there will be a call for additional members of EMYA with a deadline of 31 July 2024.

The EMS and IMU issue calls for nominations for several prizes - see `https://euromathsoc.org/` and `https://www.mathunion.org/` for details. Members are welcome to nominate individuals or to contact committee members of the IMS to suggest institutional nominations. IMS Treasurer Conall Kelly has joined the EMS Committee for Developing Countries. Among its activities the committee supports travel opportunities for early career researchers, emerging research centres of excellence, and a book donation programme. There will be future vacancies on EMS committees next year.

Academics from the University of Luxembourg contacted the IMS to find someone to help with translation of mathematical terms. Some work has started on this and is available at `https://math.uni.lu/dictionary/` The IMS is now a signatory of the DORA Declaration on Research Assessment. Work on promotion of the Society is ongoing, including branded items and a poster, with one eye on the 50th anniversary of the founding of the IMS which was on April 14, 1976.

I must convey my gratitude to Michael Mackey for maintaining the IMS website; to Tony O'Farrell and the editorial team at the Bulletin; to the IMS committee, especially the Treasurer Conall Kelly and the Secretary Derek Kitson; to the chair Ann O'Shea and the members of the Irish Committee for Maths Education; and the chair Romina Gaburro and the members of the Irish Committee for Equality, Diversity and Inclusion in Mathematics.

Leo Creedon
December 2023
E-mail address: `president@irishmathsoc.org and leo.creedon@atu.ie`

Draft minutes of the Irish Mathematical Society Annual General Meeting held on 1st September 2023 at University of Limerick

Present: J. Butler, T. Carroll, L. Creedon, J. Cruickshank, S. Dendrinos, A. Fatah, R. Flatley, R. Gaburro, E. Gath, J. Grannell, P. Greaney, R. Hill, N. Hoffmann, T. Huettemann, C. Kelly, D. Kitson, A. Krishnan, G. Lessells, D. Mackey, M. Mackey, M. Manolaki, M. Mathieu, P. Mellon, F. Murphy, C. O'Brien, P. Ó Catháin, S. O'Rourke, A. O'Shea, G. Pfeiffer, H. Šmigoc, N. Snigireva, K. Wendland, D. Wraith.

Apologies: C. Boyd, J.P. McCarthy, A. O'Farrell, M. O'Reilly, R. Quinlan, R. Ryan.

1 Agenda / Conflicts of interest

The agenda was accepted and no conflicts of interest were declared.

2 Minutes

The minutes of the AGM held on 2nd September 2022 at TU Dublin, Grangegorman were accepted.

3 Matters Arising

None.

4 Correspondence

- The IMU Secretary General has sent notification that the 20th IMU General Assembly will be held in New York City, USA, on 20–21 July 2026 and ICM 2026 will take place over 23–30 July 2026 in Philadelphia, USA.
- The IMU Secretary General circulated a full membership application (Group I) from associate member Mongolia which was subsequently voted on and approved by the IMU membership.
- Prof. Antonella Perucca and Dr. Olha Nesterenko of the University of Luxembourg contacted the Society regarding a project to create a Visual Mathematics Dictionary. The project involves "Preparing free and open-source didactical material aimed at learning mathematical terms in a different language". Assistance was requested in preparing Gaeilge materials. L. Creedon had circulated the request on MathDep and a volunteer had come forward.

5 President's Report

L. Creedon highlighted developments during the year including the creation of ICEDIM, chaired by R. Gaburro, and presented a breakdown of the IMS membership with statistics on membership type, gender and geographic location. A full report will be submitted to the Bulletin. Following nomination by the Society, R. Neururer was elected to the EMS Young Academy (EMYA) and subsequently appointed as Chair of EMYA, and C. Kelly was elected to the EMS Committee for Developing Countries. M. O'Reilly has stepped down as Ireland's representative to the International Commission on Mathematical Instruction (ICMI) and was thanked for his service. A. O'Shea has been appointed as the new ICMI representative for Ireland. The Committee introduced a new meeting in May in addition to regular meetings in August and December and monthly meetings between the executive officers. M. Mackey and A. O'Farrell were thanked for their continued service in maintaining the IMS webpage and as Editor of the Bulletin respectively. M. Mathieu is in the final year of a six year term on the Committee and was thanked for his service to the Society having previously served terms as President and Vice President, and as Editor of the Bulletin for 11 years. Next year's Annual Meeting is to be hosted by M. Mathieu and colleagues at Queen's University Belfast.

6 New members

30 new membership applications were approved since the last AGM. The new members

are: Fintan Hegarty; Fergal Murphy; Katrin Wendland; David Quinn; Róisín Hill; Abdul Fatah; David Cormican; Stephen Coughlan; Negin Nazari; Kevin Burke; Mehakpreet Singh; Ashok Das; Michaela Ottaviani; David McMahon; Doireann O'Kiely; Clifford Nolan; Sean Kelly; Alan Hegarty; Oisin Flynn-Connolly; Spyridon Dendrinos; Arundhathi Krishnan; Kevin Moroney; Cian O'Brien; Carl Sullivan; Páraic Treacy; Hazel Murray; Patrick Emmet Farrell; Paul Greaney; Laura Cooke and Sowmiya Krishnaraj.

7 Treasurer's Report
Accounts for 2022 were presented. For 2023, the figures for Subscriptions and Funding of Conferences are expected to increase. No shortfall is expected. A Donations button is now available on the IMS website.

8 Conference Support Fund
The following workshops were supported this year:

- Workshop on Key Lemmas in Analysis and Dynamics (UCD): January 2023
- SIAM UK and Ireland Sectional Meeting (TCD): April 2023
- Workshop on Nonlinear Waves (UCC): April 2023
- Groups in Galway (U Galway): May 2023
- 6th Conference on Irish History of Mathematics (Maynooth): August 2023
- 9th Conference on Research in Mathematics Education (DCU): October 2023.

Conference organisers were reminded to submit a report to the Bulletin. Members were reminded that Fáilte Ireland offer financial support for meetings with over 100 participants. The next call for the Conference Support Fund will be issued in November with a deadline in December.

9 Bulletin
C. Mulcahy has joined the editorial board with special responsibility for obituaries. Members are encouraged to notify C. Mulcahy or A. O'Farrell of deaths of IMS members. Hardcopies of the Bulletin can be purchased via a link on the IMS webpage. Organisers of meetings supported by the Society are encouraged to include abstracts when submitting reports to the Bulletin. Members are encouraged to submit research or expository articles to the Bulletin.

10 Membership fees
C. Kelly outlined proposed increases to memberships fees and the rationale for the increases. The proposed increases are: €30 -> €40 for individual membership, €15 -> €20 for student membership and members over the age of 65 with 5 years membership, €300 -> €400 for lifetime membership, €200 -> €250 for institutional membership. Following discussion the proposed increases were voted on and approved unanimously.

11 Report from Irish Committee for Mathematics Education (ICME)
A. O'Shea outlined ICME activities during the year. A full report will be published on the IMS website. The ICME has continued its work on textbook quality at post-primary level. Resources for the new applied maths syllabus are under development and an online Maths Education seminar series is set to continue. Activities of the International Commission on Mathematical Instruction (ICMI) were highlighted, including upcoming conferences, publications and the Klein Project mathematical resources. The 2024 International Congress on Mathematical Education will take place in Sydney. M. Kerin has stepped down from the ICME and was thanked for his service.

12 Report from Irish Committee for Equality, Diversity and Inclusion in Mathematics (ICEDIM)
R. Gaburro reported on ICEDIM activities during the year. An application involving four institutions has been made to the HEA gender equality enhancement fund to support new initiatives in Mathematics. Further funding applications are planned.

Potential future activities include a summer school, a Women in Maths day and an online seminar series. New members of ICEDIM are being sought and interested Society members are encouraged to contact L. Creedon.

13 Elections

The current terms of the following committee members come to an end this year: Ronan Flatley; Cónall Kelly; Derek Kitson; Martin Mathieu; Ray Ryan; Helena Šmigoc; Nina Snigireva. M. Mathieu has reached the end of a six year term as committee member, and is consequently not eligible for re-election as committee member. The remaining committee members are eligible for re-election. R. Ryan is stepping down from the committee and was thanked for his service.

The following nominations had been received and election to these positions was approved by the meeting:

Candidate	**Role**	**Nominated by**	**Seconded by**
Chris Boyd	Member	Ray Ryan	Nina Snigireva
Ronan Flatley	Member	Leo Creedon	Cónall Kelly
Cónall Kelly	Treasurer	Leo Creedon	Martin Mathieu
Derek Kitson	Secretary	Leo Creedon	Cónall Kelly
Thomas Huettemann	Member	Martin Mathieu	Romina Gaburro
Helena Šmigoc	Member	Leo Creedon	Cónall Kelly
Nina Snigireva	Member	Leo Creedon	Cónall Kelly

14 AOB

None.

Derek Kitson (MIC)
derek.kitson@mic.ul.ie

IMS Annual Scientific Meeting 2023
University of Limerick and Mary Immaculate College
31st August - 1st September

This year's annual meeting of the Society took place on 31st August and 1st September at the University of Limerick. The meeting was organised collaboratively by the Department of Mathematics and Statistics and the Mathematics Application Consortium for Science and Industry (MACSI) at the University of Limerick (UL) and the Department of Mathematics and Computer Studies at Mary Immaculate College (MIC). The organising committee consisted of Romina Gaburro (co-chair), Eugene Gath, Natalia Kopteva, Kevin Moroney and Clifford Nolan at UL and Ronan Flatley and Derek Kitson (co-chair) at MIC. This was the 36th annual scientific meeting of the Society and the meeting attracted over 70 participants. The event opened with a welcoming address from Dr Marie Connolly, Director of Human Rights, Equality, Diversity and Inclusion at UL. The schedule of talks featured over a dozen experts from across Ireland and the United Kingdom (a complete book of abstracts follows this report). The invited speakers were:

- Norma Bargary, University of Limerick,
  Functional data analysis for sports analytics;
- Patrick Browne, Technological University of the Shannon,
  Segre's theorem on ovals in Desarguesian projective planes;
- John Butler, Technological University Dublin,
  Mathematical Modelling of Multisensory Neuronal Processing and Behavioural Responses;
- Julie Crowley, Munster Technological University,
  Exploring the relationship between Mathematics and Emotions;
- James Cruickshank, University of Galway,
  Rigidity of bar and joint frameworks: polyhedra and beyond;
- Patrick Farrell, University of Oxford,
  Computing multiple solutions of nonlinear partial differential equations;
- Ivan Graham, University of Bath,
  Convergence of iterative solvers for the Helmholtz equation at high frequency;
- Emma Greenbank, University of Limerick,
  Volcanism to Batteries; modelling fluid flow in porous media;
- Thomas Huettemann, Queen's University Belfast,
  Some remarks on the "fundamental theorem" in algebraic K-theory;
- Cónall Kelly, University College Cork,
  Adaptive numerical methods for stochastic jump differential equations;
- Bernd Kreussler, Mary Immaculate College,
  On twistor spaces from an algebraic geometry perspective;
- Myrto Manolaki, University College Dublin,
  Holomorphic functions with chaotic behaviour;
- Katrin Wendland, Trinity College Dublin,
  Some quartic K3 surfaces.

The programme included a poster session for postgraduate students and early career researchers. A €100 prize for best poster by a postgraduate student was sponsored by SIAM UKIE. This sponsorship was a new feature of the meeting and the winner was David McMahon (UL) for his poster titled *Microlocal Analysis of Multistatic Radar Imaging.* The poster presentations were:

- Milton Assunção, University of Limerick,
  Dissolution of drug particles subject to natural convection;
- Jason Curran, University of Limerick,
  Stability and Reconstructions for Anisotropic Diffuse Optical Tomography;
- Niall Donlon, University of Limerick,
  Stable Reconstruction of a special type of anisotropic conductivity;
- Abdul Fatah, Atlantic Technological University, Galway,
  Quantum Error Correction using Quantum Latin squares;
- Oisín Flynn-Connolly, Université Sorbonne Paris Nord,
  The geometry of iterated suspensions;
- Seán Kelly, University of Limerick,
  Pointwise-in-time error bounds for a fractional-derivative parabolic problem on quasi-graded meshes;
- David McMahon, University of Limerick,
  Microlocal Analysis of Multistatic Radar Imaging;
- Shraddha Naidu, University of Limerick,
  Inclusion Size detection with Deep Learning and electrical impedance tomography;
- Maged Shaban, Technological University Dublin,
  Influence of Ceramic Suspensions on Micro Stereolithography Printing Modeling and Simulation.

Another new feature of the meeting was to invite short talks and poster presentations from the most recent winners of the Royal Irish Academy Hamilton Prize in Mathematics. The contributors were:

- Tiernan Brosnan, University of Limerick,
  Real-Time High-Frequency Radar Imaging (Short talk);
- James Hayes, University of Galway,
  Contact Graphs (Short talk);
- Ryan McGowan, Trinity College Dublin,
  Classifying Domains in $\mathbb{C}^n$ *through the Study of Metrics (Poster)*;
- Kituru Ndee, Technological University Dublin,
  Fluid Dynamics: Slow Viscous Flows (Poster).

The Annual General Meeting took place on the Friday at which 30 new members were welcomed to the Society. The webpage for the meeting is archived at:
`https://www.ul.ie/scieng/schools-and-departments/department-mathematics-and-statistics/36th-annual-meeting-of-the`

The organisers would like to express their gratitude for financial support provided by the Irish Mathematical Society, Mary Immaculate College (Department of Mathematics and Computer Studies), the University of Limerick (Department of Mathematics and Statistics/MACSI, Faculty of Science and Engineering, UL President, CONFIRM, SSPC), IBEC and SIAM UKIE. The organisers would also like to thank all who assisted with the organisation of the meeting, particularly the Royal Irish Academy and Dana Mackey (TU Dublin). Finally, the organisers would like to thank all who contributed to the meeting; the invited speakers, those who presented posters, and all who attended for creating an informative and convivial meeting.

Report by Romina Gaburro (UL) & Derek Kitson (MIC)
Romina.Gaburro@ul.ie, Derek.Kitson@mic.ul.ie

FIGURE 1. Participants at the 2023 IMS Annual Meeting.
(Photo courtesy of Thomas Huettemann (QUB))

BOOK OF ABSTRACTS:

Functional data analysis for sports analytics

Norma Bargary
University of Limerick

Functional data analysis (FDA) is a statistical methodology that is suitable for modelling high-dimensional data collected over some continuum (typically time). FDA is particularly suitable for the modelling and analysis of data that are measured continuously, more recently via state-of-the-art sensor technologies. This talk will outline the role of FDA with applications to data collected in a variety of sports settings, and discuss the methodological challenges associated with the statistical modelling of these modern human movement datasets.

Segre's theorem on ovals in Desarguesian projective planes

Patrick Browne
Technological University of the Shannon

Segre's theorem on ovals in projective spaces is an ingenious result from the mid-twentieth century which requires surprisingly little background to prove. In this brief talk we give a self contained proof of Segre's theorem. This is accessible to most yet showcases some minor improvements to Segre's proof that allow for results in shorter time and simpler computations than the original.

Mathematical Modelling of Multisensory Neuronal Processing and Behavioural Responses

John S. Butler
Technological University Dublin

Efficient navigation through the world heavily relies on the seamless integration of signals from multiple sensory modalities within the brain. Previous behavioural studies have suggested the existence of a winner-take-all sensory response mechanism as well as an optimal combination of sensory signals during multisensory processing. Conversely, evidence indicates that maladaptive multisensory processing could serve as an indicator of older adults' susceptibility to falls when compared to age-matched healthy controls.

While the work of Wong & Wang (2006) has been influential in modelling sensory decision-making, most research to date has focused on unisensory tasks. To address this gap, I will present an extension of their reduced two-variable model, designed to simulate both unisensory and multisensory neuronal processing and behavioural responses. This model is built upon ordinary differential equations, driven by biological data, enabling the investigation of audio-visual speeded reaction-time tasks and visual-vestibular decision making.

In this talk, I will illustrate how the extended model successfully re-creates and tests previously observed behavioural findings. Additionally, our model provides novel insights into the proportion of unisensory and multisensory neurons required for optimal multisensory integration. This work has been done in collaboration with Rebecca M. Brady (IRC funded PhD student).

Exploring the relationship between Mathematics and Emotions

Julie Crowley
Munster Technological University

Do you think it is reasonable to have the words Mathematics and emotions in the same sentence? Is Mathematics emotional? In this talk we investigate the relationship between Mathematics and emotions using a longitudinal case study. We also explore the possible relevance of this topic to a mathematics lecturer.

Rigidity of bar and joint frameworks: polyhedra and beyond

James Cruickshank
University of Galway

Bar and joint frameworks are objects that arise naturally in many mathematical contexts, from applications to mechanical and structural engineering, to protein folding, to commutative algebra, recreational mathematics and elsewhere. Their mathematical theory is a rich and currently very active area of research. In this talk I will survey some old and new results relating polyhedra and the rigidity theory of frameworks. The talk will be accessible to a general mathematical audience, by which I mean anyone who knows what a graph is and what a polyhedron is.

Computing multiple solutions of nonlinear partial differential equations

Patrick Farrell
Oxford University

Nonlinear problems may support multiple solutions, and these multiple solutions are typically very important for the application at hand. Examples include the buckling and snapping of structures, bistable devices in computer memory and displays, high- and low-confinement regimes in tokamak fusion reactors, multiple local minima of nonconvex optimisation problems, multiple Nash equilibria in games, and so on. Despite their mathematical interest and physical importance, the calculation of multiple solutions is not routinely carried out by practitioners, due to a lack of good algorithms. In this lecture I will present an elegant algorithm, based on Newton's method, for computing multiple solutions of nonlinear partial differential equations.

Convergence of iterative solvers for the Helmholtz equation at high frequency

Ivan Graham
University of Bath

Many interesting applications require the solution of the linear wave equation in heterogeneous media and/or complicated geometry - for example forward and inverse scattering problems in acoustics or electromagnetics. When the data oscillates on a restricted range of frequencies, application of Fourier transform can remove the time dependence, leading to the Helmholtz equation - an indefinite linear second order elliptic PDE. However (at high frequency), this equation is non-coercive (in standard settings) and has highly oscillatory solutions.

To compute solutions, fine discretizations (finite element methods) are required, resulting in (sparse) systems of linear equations with millions of unknowns and highly indefinite system matrices. Because of the system size and structure, application of modern direct methods (i.e., clever variants of Gaussian elimination) are problematic, so there is great interest in finding iterative methods which compute the solution via a sequence of 'local' approximations, (so-called 'domain decomposition' methods). Such methods are well-suited to implementation on modern parallel hardware and the search for fast convergent iterative methods for the discrete wave equation (guaranteed by theory) is thus a very active current research topic in numerical analysis/scientific computing.

In the low-frequency case, the PDE behaves like the Poisson equation, the linear systems are symmetric positive definite, and many 'optimal' methods (the most famous being 'multigrid') are available for fast iterative solution. However these methods generally fail at high-frequency.

In the talk I'll present some theory of the Helmholtz equation and its discretization, and a simple iterative procedure which forms the basis of several successful practical large-scale solvers. Then I'll present some recent theory which explains the convergence properties of this method.

The techniques of analysis involve combining the theory of the Helmholtz equation at high frequency with numerical analysis of finite element and domain decomposition methods.

The work is joint with Shihua Gong (Chinese University of Hong Kong Shenzhen) and Euan Spence (Bath). Some results are also joint with Martin Gander (Geneva) and David Lafontaine (Toulouse).

Volcanism to Batteries; modelling fluid flow in porous media

Emma Greenbank
University of Limerick

During this talk I will be discussing two different modelling projects arising from my PhD and Post-doctoral work. The first application I will be focusing on is from the field of volcanism. The eruptions that are of interest to this research are those that occur through crater lakes or shallow sea water. These eruptions are often some of the most dangerous in the world as they can cause tsunamis, lahars and base surges, but the phenomenon of interest for this research is that of the Surtseyan ejecta. Surtseyan ejecta are balls of lava containing an entrained material. They occur when a slurry of previously erupted material and water washes back into the volcanic vent. This slurry is incorporated into the magma and ejected, from the volcano, inside a ball of lava. Despite the formation of steam and anticipated subsequent high pressures inside these ejecta, many survive to land without exploding. The aim of this research was to explain the ejecta survival by describing the coupled evolution of pressure and temperature due to the flashing of liquid to vapour within a Surtseyan ejecta while it is in flight. Analysis of the model provides a criterion for fragmentation of the ejecta due to steam pressure build-up, and predicts that if diffusive steam flow through the porous ejecta is sufficiently rapid, the bomb will survive the flight intact. This criterion explicitly relates fragmentation to ejecta properties, and describes how a Surtseyan ejecta can survive in flight despite containing flashing liquid water, contributing to an ongoing discussion in volcanology about the origins of the inclusions found inside bombs. The remainder of the talk will be focusing on the modelling of a battery anode design introduced in 2019. This anode is composed of an array of copper nanowires, coated with Li-carrying copper silicide and surrounded by Li-alloying electrolyte. During the charging of the anode the bed of nanowires are deformed into a porous structure. This design has shown promise in alleviating the issues cause by silicon in lithium ion batteries. One major challenge of silicon is the extreme volumetric change of silicon during lithiation. The stresses resulting from this swelling can cause degradation and failure of the battery. In this research numerical solutions of the homogenised problem are used to predict the transport of lithium through the anode.

Some remarks on the "fundamental theorem" in algebraic K-theory

Thomas Huettemann
Queen's University Belfast

Two well-known K-theoretical results, the "fundamental theorem" and the computation of the K-theory of the projective line, are traditionally phrased in terms of Laurent polynomial rings. I will indicate how to adapt the statements and proofs to a much larger class of rings, containing, for example, skew Laurent polynomial rings and Leavitt paths algebras of (nice) graphs. In the first part of the talk I will introduce the classical lower K-groups and present some motivation for studying them, before formulating the general results (which are valid for "higher" K-theory as well).

Adaptive numerical methods for stochastic jump differential equations

Cónall Kelly

University College Cork

Stochastic differential equations (SDEs) are used to model the evolution of real-world phenomena subject to random noise and uncertainty. Consider, for example, asset prices or stochastic interest rates in finance, models of ecological systems with complex interaction between species or models of chemical reactions in biological cells. The random noise may act as a diffusion, for example reflecting market volatility, or as a jump process, for example when an ecosystem is subjected to random external shocks. For most nonlinear SDE models there is no closed-form solution and typically numerical methods are used by modellers. However, standard schemes based on solving to a final time using a uniform step size are not applicable for highly nonlinear systems and the methods that do exist are often inefficient.

In this talk we discuss the use of adaptive timestepping for SDEs driven by both a standard Brownian motion and a Poisson jump process. In the absence of jumps, we can ensure strong convergence of explicit schemes, under conditions where they fail to converge over a uniform mesh, by adjusting the timestep in response to the local behaviour of observed trajectories. We will motivate and characterise these strategies and show how they extend to include the jump case. Some implementation issues will be illustrated via a stochastic model of telomere shortening. This is joint work with Gabriel Lord (Radboud University, The Netherlands) and Fandi Sun (Heriott-Watt University, Edinburgh, UK).

On twistor spaces from an algebraic geometry perspective

Bernd Kreussler

Mary Immaculate College

The twistor spaces considered in this talk are 3-dimensional complex manifolds that can be described by polynomials in a concrete manner. I will report about joint work in progress with Jan Stevens in which we answer an interesting open question about deformations of such spaces. The history of the topic and the most important terms and ideas will be explained in a way that is accessible to the non-specialist.

Holomorphic functions with chaotic behaviour

Myrto Manolaki

University College Dublin

This talk is concerned with holomorphic functions which, under a certain countable process, can approximate every plausible function. It turns out that this behaviour, which seems quite pathological, is generic. After presenting several classical examples of this phenomenon, I will focus on some specific cases which have been recently investigated. As we will see, the boundary behaviour of the corresponding holomorphic functions is extremely chaotic.

Some quartic K3 surfaces

Katrin Wendland
Trinity College Dublin

We give an introduction to the geometry of certain complex surfaces, known as K3 surfaces, where we focus on a class of examples given by quartic equations. These special quartic K3 surfaces exhibit a number of beautiful geometric features which also open the door for applications in number theory and in conformal quantum field theory. We will highlight some of these applications in the talk.

Reports of Sponsored Meetings

Reports received of sponsored meetings held in 2023:

Groups in Galway 2023
May 18–19, 2023, University of Galway

The 2023 instalment of the series of meetings "Groups in Galway" took place at the University of Galway on May 18–19, 2023. This was the first in-person Groups in Galway after two online editions due to the pandemic (2020, 2021) and a special joint meeting of "Groups in Galway" and the "Irish Geometry Conference" (2022).

The meeting was organised by Angela Carnevale and Götz Pfeiffer, and was supported by the Irish Mathematical Society and by the Office of the Registrar and Deputy President of the University of Galway. There were 9 talks over three sessions, and over 30 participants.

Speakers and talks:

- Naomi Andrew (University of Oxford): *Automorphisms of groups and actions on trees*
- Javier Aramayona (ICMAT Madrid): *Asymptotically rigid mapping class groups*
- Ilaria Castellano (Bielefeld University): *Coxeter groups with more than two ends and groups acting on buildings*
- Leo Margolis (Universidad Autónoma de Madrid): *Modular Isomorphism Problem - progress, solution and open challenges*
- Padraig Ó Catháin (Dublin City University): *Monomial actions, group cohomology and complex Hadamard matrices*
- Colva Roney-Dougal (University of St Andrews): *Base size and relational complexity*
- Tobias Rossmann (University of Galway): *Orbits of unipotent groups: tame vs wild*
- Yuri Santos Rego (Otto-von-Guericke University Magdeburg): *Navigating the galaxy of Coxeter groups*
- Gerald Williams (University of Essex): *Generalized polygons and star graphs of cyclic presentations of groups*

The conference website `https://angelacarnevale.github.io/gig23/` contains abstracts of the talks and further information.

Report by Angela Carnavale, University of Galway
angela.carnavale@universityofgalway.ie

Nonlinear Dispersive Waves
April 24–25, 2023, University College Cork

A workshop took place at the School of Mathematical Sciences, UCC, from April 24^{th}–25^{th}. This workshop addressed some recent mathematical developments in the broad field of nonlinear dispersive waves, with a particular emphasis on waves arising in the ocean and atmosphere. It featured 12 international mathematicians as invited speakers, whose research backgrounds span the spectrum from pure to applied mathematics. The workshop was run on a hybrid-basis, achieving a global reach of over 60 participants from 17 countries (and 6 continents!), with thankfully many participants making it to Cork in

person. The speakers and their talks are listed below, while abstracts can be found at:
https://www.ucc.ie/en/media/academic/maths/pdfs/Workshop-Programme.pdf

Nonlinear Dispersive Waves Participants

- Didier Clamond (Université Côte d'Azur): *On the recovery of rotational gravity waves from the seabed pressure*
- Adrian Constantin (University of Vienna): *Frictional effects in wind-driven ocean currents*
- Olivia Constantin (University of Vienna): *A complex analytic approach to some problems in fluid flows*
- Joachim Escher (Leibniz University Hannover): *The Rayleigh–Taylor Condition for the Muskat Problem*
- Delia Ionescu-Kruse (Institute of Mathematics of the Romanian Academy): *On the short-wavelength stabilities of some geophysical flows*
- Rossen Ivanov (TU Dublin): *Modelling internal waves over variable bottoms*
- David Lannes (University of Bordeaux): *Wave structure interaction in the Boussinesq regime*
- Bogdan Matioc (University of Regensburg): *Stratified Periodic Water Waves with Singular Density Gradients*
- Emilian Parau (University of East Anglia): *A dissipative nonlinear Schrodinger model for wave propagation in the marginal ice zone*
- Jens Rademacher (University of Hamburg): *Rotating convection with kinetic energy backscatter*
- Raphael Stuhlmeier (University of Plymouth): *A discrete Hamiltonian perspective on the classical instabilities of deep-water waves*
- Samuel Walsh (University of Missouri): *Desingularization and global continuation for hollow vortices*

These scientific talks—and the resulting discussions and interactions—were greatly enjoyed by a diverse audience ranging from graduate students, upwards. The workshop was kindly funded by the School of Mathematical Sciences in UCC, the Irish Mathematical Society, and the Science Foundation Ireland, all of whose support is gratefully acknowledged. The proceedings of this workshop will be published in a forthcoming book volume by Birkhäuser.

Report by David Henry, University College Cork
d.henry@ucc.ie

Key Lemmas in Analysis and Dynamics
January 16–20, 2023, University College Dublin

The workshop "Key Lemmas in Analysis and Dynamics" took place at University College Dublin during 16th-20th January, 2023. The workshop, which was organised by Neil Dobbs and Myrto Manolaki from UCD, covered a variety of topics including Complex Analysis, Approximation Theory, Holomorphic Dynamics and Ergodic Theory. Its goal was to bring together early career and more established mathematicians to discuss challenging research problems and important mathematical tools used in these areas. There were 14 talks spread over 4 days and one day of excursion in Glendalough. Over 15 people attended the meeting. The organisers acknowledge financial support by the UCD School of Mathematics and Statistics and by the Irish Mathematical Society.

Participants in Key Lemmas Workshop

Speakers and titles of talks:

- **Neil Dobbs (UCD, Ireland):** *Hausdorff dimension and Julia sets (part I and II)*
- **Vasiliki Evdoridou (Open University, UK):** *Constructing oscillating wandering domains (part I and II)*
- **Gabriella Keszthelyi (Renyi Institute of Mathematics, Hungary):** *Dynamical properties of biparametric skew tent maps*
- **Alexey Korepanov (Loughborough University, UK):** *Mixing for the measure of maximal entropy for dispersing billiards (part I and II)*
- **Matteo Lotriglia (UCD, Ireland):** *On Class B Functions and the Area of their Julia Set*
- **Myrto Manolaki (UCD, Ireland):** *What can Potential Theory tell us about the boundary behaviour of holomorphic functions?*
- **Konstantinos Maronikolakis (UCD, Ireland):** *Properties of Abel universal functions*
- **David Marti-Pete (University of Liverpool, UK):** *Constructing entire functions with wandering continua (part I and II)*

- **Nina Snigireva (Atlantic Technological University, Ireland):** Noncontractivity in Fractal Geometry
- **Matteo Tabaro (Imperial College London, UK):** *Semi-Hyperbolicity Implies Existence of ACIPs for Real Multimodal Maps*

	Monday	Tuesday	Wednesday	Thursday	Friday
09:45-10.30 10:30-11.00	opening + coffee/tea	excursion (Maths in nature)	coffee/tea	Nina Snigireva coffee/tea	Matteo Lotriglia coffee/tea
11.00-11.45	Matteo Tabaro		Vasiliki Evdoridou	Vasiliki Evdoridou	Neil Dobbs
12.00-12.45	Myrto Manolaki		David Marti-Pete	David Marti-Pete	Alexey Korepanov
13.00-15.00	lunch		lunch	lunch	lunch
15.00-15.45	Gabriella Keszthelyi		Alexey Korepanov		
16.00-16.45	Neil Dobbs		Konstantinos Maronikolakis		
19.00-21.00				dinner	

For further information, please check the conference website:
`https://maths.ucd.ie/~ndobbs/KLAD2023/index.html`

Report by Neil Dobbs and Myrto Manolaki, University College Dublin
neil.dobbs@ucd.ie, myrto.manolakia@ucd.ie

Research in Mathematics Education in Ireland (MEI 9)
October 13–14, 2023, Maynooth University

The 9th conference on Research in Mathematics Education in Ireland (MEI 9) took place at the Institute of Education, St Patrick's Campus, DCU and focussed on the theme *Conceptualising Success in Mathematics Education.* It served to promote and facilitate discussion on mathematics teaching and learning across the continuum from early childhood education to third level education.

MEI Panellists

The keynote addresses were delivered by Prof Susanne Prediger (TU Dortmund) and Dr Niamh O'Meara (University of Limerick) whose talks were entitled

> *From task completion to learning progress: Shifting mathematics teachers' conceptualisations of success as a key challenge in professional growth*

and

> *What constitutes success in mathematics education in Ireland and what obstacles stand in the way of this success: A decade in review*

respectively. Over 90 national and international participants, including those from Germany, Norway and the United States, were in attendance. Teachers, academics, members of the teacher support services and policy makers participated in the conference.

The conference featured over 25 contributed talks, poster presentations and a panel discussion on *What counts as success in the assessment of mathematics?* The panel chair, Dr Joe Travers, is pictured with panellists Drs Gerry Sheil, Thérèse Dooley, Vasiliki Pitsia and Zita Lysaght below.

Also included in the conference were two symposia (one entitled *Mathematics Learning Support: where we have been, where we are now, where we are going* and the other *Conceptualising success for mathematics in modernity: Augmented reality, datascience and integrated STEM*).

Further details on the conference, including abstracts for the papers and posters presented, can be found on the conference website:
`https://sites.google.com/dcu.ie/meiconference/mei-9`.

The organisers would like to sincerely thank the Irish Mathematical Society for their support.

Report by Sinéad Breen, Dublin City University
sinead.breen@dcu.ie

Sixth Irish History of Mathematics Conference
August 30, 2023, Maynooth University

The 6^{th} Irish History of Mathematics Conference (IHoM6) was held in Renehan Hall at Maynooth University on Wednesday, August 30th. The organising committee was led by Ciarán Mac an Bhaird and also consisted of Mark McCartney (Ulster University) and Maurice OReilly (DCU).

The meeting was well attended, with more than 30 individuals in attendance at different times of the day. There was more international engagement than with previous IHoM conferences probably due to the other HoM events runnning at Maynooth that week. On Tuesday, Maynooth hosted the first workshop of the *History for Diversity in Mathematics Betwork* (https://mathshist4edi.wp.st-andrews.ac.uk/). On Wednesday evening, following IHoM6, and continuing on Thursday and Friday, we also had the *Consonances: Mathematics, Language, and the Moral Sense of Nature* Conference.

IHoM6 had nine talks across different aspects of the HoM. The list of all talks is as follows:

- Hadamard's Determinant Inequality - Padraig Ó Catháin
  Abstract A famous inequality due to Hadamard in 1893 establishes a bound for the determinant of a matrix in terms of an upper bound on the matrix entries. To the modern reader, the proof is curious as it uses techniques of nineteenth

century determinant theory, rather than results about inner product spaces. We will trace the historical development of this result and its generalisations, comparing nineteenth and twentieth century proof techniques, and concepts of proof.

- An enlightened archbishop: collecting mathematics in the Bolton Library - Olivia Lardner
Abstract - The core of what has become known in the 21st century as the Bolton Library – formerly Cashel Cathedral Library – was circa 75 years in the making, collected by two Irish men across the island of Ireland between 1669–1744, with activity in each of the four provinces. They would both go on to become Church of Ireland archbishops: William King (1650-1729) in Dublin and the eponymous Theophilus Bolton (-1744) in Cashel. Little research has been undertaken on the collecting activities of the latter, but a wealth of information exists on that of the former, due in the main to his own assiduous marking of and reflections on items acquired across 50 years of collecting activity. This talk will look at three mathematical volumes - two manuscripts and one early printed book - acquired by Archbishop King during this Age of Enlightenment.

- On Newton's series for sine and arcsine - Piotr Błaszczyk
Abstract We analyze Newton's Two Treatises of the Quadrature of Curves and Analysis by Equations of an Infinite Number of Terms from the perspective of mathematical techniques. On pages 336–338 of the 1745 edition, Newton derives series for sine and arcsine. To this end, he employs Euclidean proportion, Cartesian understanding of proportion in terms of the arithmetic of line segments, and Cartesian interpretation of the Pythagorean theorem, infinitesimals, formal power series, and binomial theorem – the technique exposed by every modern commentary. Moreover, Newton refers to the Euclidean concept of magnitudes of different kinds, which allows him to apply different units, namely infinitesimal unit line and the unit within – say – usual line segments. We focus on the technique of formal power series, which enables Newton to determine the derivative – to phrase it in modern terms – of the inverse function. Finally, we contrast Newton's derivation of the series for arcsine with modern calculus and show that the so-called arithmetization of calculus is not a complete success.

- Who was J. Walton, Berkeley's Dublin antagonist? - Roderick Gow
Abstract In 1734, an anonymous pamphlet entitled *The analyst: or a discourse addressed to an infidel mathematician* was published separately in Dublin and London. The work had an immediate impact and, although anonymous, it was rapidly surmised that its author was George Berkeley, who had been appointed to the bishopric of Cloyne a month or two earlier. Not the least part of its impact derived from the implied criticism of Newton's method of fluxions. It provoked a brief pamphlet war from supporters of Newton, with the occasional anonymous reply from Berkeley. Our interest is centred on two pamphlets written by a certain J. Walton, also published in Dublin and London, in 1735. One is *A vindication of Sir Isaac Newton's principles of fluxions*, the other *The catechism of the author of the minute philosopher fully answer'd.* The question we wish to raise here is: who was J. Walton? It is surprisingly difficult to give a conclusive answer, and other investigators have failed in the endeavour, not least even to identify the name signified by the initial J. Sufficient information has emerged to suggest that Walton was a wealthy man with scientific interests

and connections to leaders of Dublin society, but virtually unknown to historians.

- Potential Approaches to the Theory of Proportionality in Ancient Greek Geometry - David Wilkins
  Abstract This presentation seeks to explore the potential for geometric approaches to proportionality in the development of ancient Greek geometry prior to the establishment of the theory of proportionality attributed to Eudoxus. The aim is to show that, if appropriate geometric criteria are taken to represent proportionality, when applied in the context of straight line segments and parallelogrammic areas, then the majority of the propositions in Book 6 of Euclid's Elements of Geometry could be proved, consistent with the standards of proof typical of ancient Greek geometry, on the basis of the concepts, propositions and proof techniques exhibited in the first four books of Euclid's Elements. Such an approach should not introduce any logical dependence on the contents of Book 5 that present the theory of proportionality traditionally attributed to Eudoxus.

- Euclid's Elements in Irish: A 19th century tale - Ciarán Mac an Bhaird
  Abstract Special Collections in the University College Dublin (UCD) Library holds a manuscript which includes, amongst other non-mathematical material, approximately sixteen pages of Euclid's Elements written in old script (seancló) Irish. In this talk I will consider the contents of these pages which seem to have been written by the Irish language scholar John O'Donovan (Seán Ó Donnabháin) around the middle of the 19th century. We will look at Eoin MacNeill's commentary on O'Donovan's text in the Gaelic Journal (Irisleabhar na Gaedhilge) in the 1890's, which paid particular attention to the Irish words chosen by O'Donovan. We will also briefly outline the careers of the people involved, including O'Donovan, MacNeill, and James O'Laverty as we try to identify why these pages were written in the first place, and their curious route, via Belfast, to UCD.

- Considering conics: reading Apollonius in the collections of Marsh's Library - Sue Hemmens
  Abstract The seventeenth century saw a sustained fascination with the treatise on conic sections by the 'Great Geometer' Apollonius of Perga (c.240C–c190BCE). Characterised by some as the first significant advance in geometry since Euclid, Apollonius' writings were known to the Islamic world and subsequently rediscovered in Western Europe during the Renaissance where they formed the basis of many subsequent developments. Narcissus Marsh (1638–1713) is known to have been deeply interested in mathematics in general. He made extensive notes using his copy of La Hire's 1685 edition of Apollonius. Marsh also owned an important Arabic manuscript, now held in the Bodleian Library, which was used by Edmund Halley in preparation of his edition of the Conics, including a reconstruction of the 'lost book'. This paper will discuss the reception and reading of Apollonius as reflected in the collections of Marsh's Library.

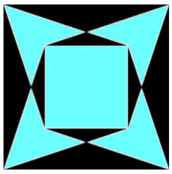
- The Lion, the Witch & the maths graduate: studying maths at Queen's College, Belfast in the 1880s - Mark McCartney
  Abstract The 1880s saw the dissolution of the Queen's University of Ireland, the formation of the Royal University of Ireland, and the admission of women as students to Queen's College Belfast. This talk will look at the mathematics curriculum and examinations around that time and aspects of the lives of Florence

Hamilton and Alice Everett.

- What's happening in the History of Mathematics Education? Perspectives from ICHME7 - Maurice OReilly
  Abstract In September 2022, the seventh International Conference on the History of Mathematics Education (ICHME7) took place over five days in Mainz, Germany, where 32 papers were presented. In an attempt to give an overview of areas of current interest in the field, I review a selection of the presentations under six headings. These are: pre-Enlightenment texts on mathematics, the emergence and development of engineering mathematics in military contexts in the 18th and 19th centuries, teaching mathematical analysis in the 19th century, school geometry in the 19th and early 20th centuries, international networks concerning mathematics teaching, and, last but by no means least, the advent and reception of the New Math from the 1960s. This review anticipates the publication of the ICHME7 proceedings at the end of August (see https://ichme7.uni-mainz.de/)

The organisers would like to thank the administrative and technical staff in the Department of Mathematics and Statistics for all their assistance, both Conference and Accommodation and Catering at Maynooth for their support, all speakers and participants, and finally the IMS for providing funding.

Report by Ciarán Mac an Bháird, Maynooth University
ciaran.macanbhaird@mu.ie



Irish Math. Soc. Bulletin
Number 92, Winter 2023, 27–32
ISSN 0791-5578

Weak Sequential Completeness of Uniform Algebras

J. F. FEINSTEIN AND ALEXANDER J. IZZO

Abstract. We give a simple, elementary proof that a uniform algebra is weakly sequentially complete if and only if it is finite-dimensional.

1. The Result

For X a compact Hausdorff space, we denote by $C(X)$ the algebra of all continuous complex-valued functions on X with the supremum norm $\|f\|_X = \sup\{|f(x)| : x \in X\}$. A *uniform algebra* on X is a closed subalgebra of $C(X)$ that contains the constant functions and separates the points of X. On every compact Hausdorff space X there is the trivial example of a uniform algebra, namely $C(X)$ itself. By the Stone-Weierstrass theorem, $C(X)$ is the only self-adjoint uniform algebra on the space X. However, there are many other (nonself-adjoint) uniform algebras. A typical example is the disc algebra which consists of the continuous complex-valued functions on the closed unit disc that are holomorphic on the open unit disc. The uniform algebras form a class of Banach algebras that is important both in the field of Banach algebras and in complex analysis, and uniform algebras also have applications to operator theory. In this paper we consider certain Banach space properties of uniform algebras, primarily weak sequential completeness and reflexivity. (These terms are defined in the next section.)

Every weakly sequentially complete uniform algebra is finite-dimensional. Although this fact is known to a few experts, the result is certainly not well known and seems not to be explicitly stated in the literature. In this paper we present a simple, elementary proof of the result. A different proof, using Arens regularity and bounded approximate identities, is given in the forthcoming book of Garth Dales and Ali Ülger [4, Section 3.6]. An anonymous referee has pointed out that using results in the literature a stronger statement can be obtained: every infinite-dimensional uniform algebra contains an isometric copy of the Banach space c of all convergent sequences of complex numbers. We give the referee's argument near the end of the paper.

Theorem 1.1. *Every weakly sequentially complete uniform algebra is finite-dimensional.*

Since every reflexive Banach space is weakly sequentially complete, we have the following as an immediate consequence.

Corollary 1.2. *For a uniform algebra A, the following are equivalent.*

(a) *A is weakly sequentially complete.*
(b) *A is reflexive.*
(c) *A is finite-dimensional.*

2020 *Mathematics Subject Classification.* Primary 46J10, 46B10.

Key words and phrases. uniform algebra, weakly sequentially complete, reflexive.

Received on 8-5-2023; revised 30-6-2023.

DOI:10.33232/BIMS.0092.27.32.

The authors' collaboration was partially supported by the Research in Teams program at the Banff International Research Station. The first author was also partially supported by a Simons collaboration grant and by NSF Grant DMS-1856010.

The fact that every reflexive uniform algebra is finite-dimensional does appear in the literature. However, the only explicit mention of this fact that we have found in the literature is at the very end of the paper [6] where the result is obtained as a consequence of the general theory developed in that paper concerning a representation due to Asimow [1] of a uniform algebra as a space of affine functions. A closely related result, which we will discuss at the end of our paper, appears in the paper [2] of Paul Beneker and Jan Wiegerinck: no separable infinite-dimensional uniform algebra is a dual space. The fact that every reflexive uniform algebra is finite-dimensional follows immediately since every infinite-dimensional uniform algebra contains a separable infinite-dimensional uniform algebra.

One can also consider what are sometimes called *nonunital uniform algebras*. These algebras are roughly the analogues on noncompact locally compact Hausdorff spaces of the uniform algebras on compact Hausdorff spaces. (The precise definition is given in the next section.) Every nonunital uniform algebra is, in fact, a maximal ideal in a uniform algebra, and hence is, in particular, a codimension 1 subspace of a uniform algebra. Since it is easily proven that the failure of weak sequential completeness is inherited by finite codimensional subspaces, it follows at once that the above results hold also for nonunital uniform algebras.

It should be noted that the above results do *not* extend to general semisimple commutative Banach algebras. For instance, for $1 \leq p \leq \infty$, the Banach space ℓ^p of pth-power summable sequences of complex numbers is a Banach algebra under coordinatewise multiplication and is of course well known to be reflexive for $1 < p < \infty$; for $p = 1$ the space is nonreflexive but is weakly sequentially complete [11, p. 140]. Also for G an infinite locally compact abelian group, the Banach space $L^1(G)$ is a Banach algebra with convolution as multiplication and is nonreflexive but is weakly sequentially complete [11, p. 140]. All of these Banach algebras are nonunital, with the exception of the algebras $L^1(G)$ for G a discrete group. However, adjoining an identity in the usual way where necessary, one obtains from them unital Banach algebras with the same properties with regard to reflexivity and weak sequential completeness.

In the next section, which can be skipped by those well versed in basic uniform algebra and Banach space concepts, we recall some definitions. The proof of Theorem 1.1 is then presented in Section 3. A proof of the stronger statement that every infinite-dimensional uniform algebra contains an isometric copy of the Banach space c is given in Section 4. In the concluding Section 5 we discuss the theorem of Beneker and Wiegerinck that no separable infinite-dimensional uniform algebra is a dual space.

2. Definitions

Recall from the introduction that a uniform algebra on a compact Hausdorff space X is an algebra of continuous complex-valued functions on X that contains the constant functions, separates the points of X, and is (uniformly) closed in the algebra $C(X)$ of all continuous complex-valued functions on X. For Y a noncompact, locally compact Hausdorff space, we denote by $C_0(Y)$ the algebra of continuous complex-valued functions on Y that vanish at infinity, equipped with the supremum norm. By a *nonunital uniform algebra* B on Y we mean a closed subalgebra of $C_0(Y)$ that strongly separates points in the sense that for every pair of distinct points x and y in Y there is a function f in B such that $f(x) \neq f(y)$ and $f(x) \neq 0$. If B is a nonunital uniform algebra on Y, then the linear span of B and the constant functions on Y forms a unital Banach algebra that can be identified with a uniform algebra A on the one-point compactification of Y, and under this identification B is the maximal ideal of A consisting of the functions in A that vanish at infinity.

Let A be a uniform algebra on a compact Hausdorff space X. A closed subset E of X is a *peak set* for A if there is a function $f \in A$ such that $f(x) = 1$ for all $x \in E$ and $|f(y)| < 1$ for all $y \in X \setminus E$. Such a function f is said to *peak on* E. A *generalized peak set* is an intersection of peak sets. A point p in X is a *peak point* if the singleton set $\{p\}$ is a peak set, and p is a *generalized peak point* if $\{p\}$ is a generalized peak set. A closed subset E of X is an *interpolation set* for A if $A|E = C(E)$, where $A|E$ denotes the algebra of restrictions of functions in A to E. The set E is a *peak interpolation set* for A if E is both a peak set and an interpolation set for A. For Λ a bounded linear functional on A, we say that a complex regular Borel measure μ on X represents Λ if $\Lambda(f) = \int f \, d\mu$ for every $f \in A$.

A Banach space A is *reflexive* if the canonical embedding of A into its double dual A^{**} is a bijection. The Banach space A is *weakly sequentially complete* if every weakly Cauchy sequence in A is weakly convergent in A. More explicitly the condition is this: for each sequence (x_n) in A such that (Λx_n) converges for every Λ in the dual space A^*, there exists an element x in A such that $\Lambda x_n \to \Lambda x$ for every Λ in A^*.

3. The Proof

Our proof of Theorem 1.1 hinges on the following lemma.

Lemma 3.1. *Every infinite-dimensional uniform algebra has a peak set that is not open.*

The proof of the above lemma uses two preliminary lemmas.

Lemma 3.2. *Let A be a uniform algebra on a compact Hausdorff space X, and let P be an open peak set for A. Then the characteristic function of P lies in A.*

Proof. Choose a function f that peaks on P. Then the sequence (f^n) of powers of f converges uniformly to the characteristic function χ_P of P, and hence χ_P lies in A. □

Lemma 3.3. *Every infinite compact Hausdorff space X contains a closed G_δ-set that is not open.*

Proof. Let $(x_n)_n = 1^\infty$ be an infinite sequence of distinct points of X. For each $n = 1, 2, 3, \ldots$ choose by Urysohn's lemma a continuous function $f_n : X \to [0,1]$ such that $f_n(x_k) = 0$ for $k < n$ and $f_n(x_n) = 1$. Let $F : X \to [0,1]^\omega$ be given by $F(x) = \big(f_n(x)\big)_{n=1}^\infty$. Then $F(x_m) \neq F(x_n)$ for all $m \neq n$. Thus the collection $\{F^{-1}(t) : t \in [0,1]^\omega\}$ is infinite. Each of the sets $F^{-1}(t)$ is a closed G_δ-set because F is continuous and $[0,1]^\omega$ is metrizable. Since these sets form an infinite collection of disjoint sets that cover X, they cannot all be open, by the compactness of X. □

Proof of Lemma 3.1. Let A be an infinite-dimensional uniform algebra on a compact Hausdorff space X.

In case $A = C(X)$, the result follows immediately from Lemma 3.3, since in that case it follows from Urysohn's lemma that the peak sets of A are exactly the closed G_δ-sets in X (see for instance [7, Section 33, exercise 4]).

Now consider the case when A is a proper subalgebra of $C(X)$. In that case, by the Bishop antisymmetric decomposition [3, Theorem 2.7.5] there is a maximal set of antisymmetry E for A that has more than one point. Since every maximal set of antisymmetry is a generalized peak set, and every generalized peak set contains a generalized peak point (see the proof of [3, Corollary 2.4.6]), E contains a generalized peak point p. Choose a peak set P for A such that $p \in P$ but $P \not\supseteq E$. The set P is not open in X, for if it were then the characteristic function of P would be in A by Lemma 3.2, which would contradict that E is a set of antisymmetry for A. □

Proof of Theorem 1.1. Let A be an infinite-dimensional uniform algebra on a compact Hausdorff space X. By Lemma 3.1, there exists a peak set P for A that is not open. Choose a function $f \in A$ that peaks on P.

For a bounded linear functional Λ on A, and a complex regular Borel measure μ on X that represents Λ, we have by the Lebesgue dominated convergence theorem that

$$\Lambda(f^n) = \int f^n \, d\mu \to \mu(P) \quad \text{as} \quad n \to \infty. \tag{1}$$

Thus the sequence $(f^n)_{n=1}^\infty$ in A is weakly Cauchy. Furthermore (1) shows that, regarded as a sequence in the double dual A^{**}, the sequence $(f^n)_{n=1}^\infty$ is weak*-convergent to a functional $\Phi \in A^{**}$ that satisfies the equation $\Phi(\Lambda) = \mu(P)$ for every functional $\Lambda \in A^*$ and every regular Borel measure μ that represents Λ.

For $x \in X$, denote the point mass at x by δ_x. Denote the characteristic function of the set P by χ_P. Then

$$\Phi(\delta_x) = \chi_P(x) \tag{2}$$

while for any function $h \in A$ we have

$$\int h \, d\delta_x = h(x). \tag{3}$$

Since P is not open in X, the characteristic function χ_P is not continuous and hence is not in A. Consequently, equations (2) and (3) show that the functional $\Phi \in A^{**}$ is not induced by an element of A. We conclude that the weakly Cauchy sequence $(f^n)_{n=1}^\infty$ is not weakly-convergent in A. □

4. Every infinite-dimensional uniform algebra contains c

In this section we give a proof of the following theorem along the lines suggested by a referee.

Theorem 4.1. *Every infinite-dimensional uniform algebra contains an isometric copy of the Banach space c.*

As mentioned in the introduction, this theorem strengthens Theorem 1.1. To see this, first note that the Banach space c_0 of sequences of complex numbers converging to zero is not weakly sequentially complete because in c_0 the sequence $(1, 0, 0, 0, \ldots)$, $(1, 1, 0, 0, \ldots)$, $(1, 1, 1, 0, \ldots)$ is weakly Cauchy but not weakly convergent. Since every norm-closed subspace of a weakly sequentially complete Banach space is itself weakly sequentially complete, it follows immediately that a weakly sequentially complete Banach space can not contain a copy of c_0, and hence can not contain a copy of c.

The proof of Theorem 4.1 uses the following two results. The first of these is due to Alain Bernard while the second is due to Aleksander Pełczyński. Proofs of these results can be found in [10, pp. 217–219, 241–242].

Theorem 4.2. *If A is a uniform algebra on an infinite, compact metrizable space, then there exists an infinite peak interpolation set for A.*

Theorem 4.3. *Let A be a uniform algebra on a compact Hausdorff space X, and let K be a peak interpolation set for A. Then there exists a linear isometry $L : C(K) \to A$ such that $(Lf)|K = f$ for all $f \in C(K)$.*

We also use the following result which is surely known but whose proof we include for the reader's convenience.

Lemma 4.4. *Let S be an infinite, compact metrizable space. Then $C(S)$ contains an isometric copy of the Banach space c.*

Proof. Choose a sequence of distinct points s_n in S converging to a point $s \in S$. Set $E = \{s_n : n = 1, 2, \ldots\} \cup \{s\}$. Clearly $C(E)$ is isometric to c, and by Theorem 4.3, for example, there is an isometric copy of $C(E)$ in $C(S)$. □

Proof of Theorem 4.1. Let A be an infinite-dimensional uniform algebra on a compact Hausdorff space X. By replacing A by a suitable closed subalgebra, we may assume that A is separable and hence that X is metrizable. Then by theorem 4.2, there exists an infinite subset K of X such that K is a peak interpolation set for A. By Theorem 4.3, there is an isometric copy of $C(K)$ in A, and hence by Lemma 4.4, A contains an isometric copy of c. □

5. No separable infinite-dimensional uniform algebra is a dual space

By a theorem of Bessaga and Pełczyński [5, Theorem 10, p. 48], if the dual space of a Banach space contains an isomorphic copy of the Banach space c_0, then it contains an isomorphic copy of ℓ^∞. (Two Banach spaces are *isomorphic* if there is a linear homeomorphism between them. Isomorphic Banach spaces need not be isometric.) Thus the following result of Beneker and Wiegerinck [2] follows immediately from Theorem 4.1.

Theorem 5.1. *No separable infinite-dimensional uniform algebra is a dual space.*

Note, however, that there are nonseparable uniform algebras that are dual spaces. For instance, the uniform algebra $C(\beta\mathbb{N})$ of all continuous complex-valued functions on the Stone-Čech compactification of the positive integers $\mathbb{N}$ can be identified with ℓ^∞ and thus is isometrically isomorphic to the dual of ℓ^1.

Beneker and Wiegerinck obtained Theorem 5.1 as a corollary of the main theorem of their paper [2] which concerns strongly exposed points. A point f in the closed unit ball B of a Banach space A is said to be *strongly exposed* if there exists $\Lambda \in A^*$ with the properties $\Lambda(f) = \|\Lambda\| = 1$ and for every sequence $(g_n)_{n=1}^\infty$ in A such that $\lim_{n\to\infty} \Lambda(g_n) = \lim_{n\to\infty} \|g_n\| = 1$, we have $\lim_{n\to\infty} g_n = f$ in A. Beneker and Wiegerinck's main result [2] states that the unit ball of an infinite-dimensional uniform algebra has no strongly exposed points. As noted by Beneker and Wiegerinck, Theorem 5.1 follows immediately since the the unit ball of a separable dual space is the closed convex hull of its strongly exposed points [9]. A completely elementary proof of a result stronger than the main theorem of [2] was later proven by Olav Nygaard and Dirk Werner. Denoting the real part of a complex number z by $\operatorname{Re} z$, a *slice* of B is a set of the form

$$S(\Lambda, \varepsilon) = \{g \in B : \operatorname{Re} \Lambda(g) \geq \sup \operatorname{Re} \Lambda(B) - \varepsilon\},$$

for $\Lambda \in A^*$ and $\varepsilon > 0$. Nygaard and Werner [8] showed that every slice of the closed unit ball of an infinte-dimensional uniform algebra has diameter 2. There are thus several routes to proving Theorem 5.1.

Acknowledgements: The question of whether every reflexive uniform algebra is finite-dimensional was posed to us by Matthias Neufang, and this was the inspiration for the work presented here. We thank Neufang for his question. We thank the anonymous referee who brought to our attention Theorem 4.1. We also thank Olav Nygaard for bringing the papers [2] and [8] to our attention.

References

[1] L. Asimow: *Decomposable compact convex sets and peak sets for function spaces*, Proc. Amer. Math. Soc. (1) **25** (1970), 75–79.

[2] P. Beneker and J. Wiegerinck: *Strongly exposed points in uniform algebras*, Proc. Amer. Math. Soc. **127** (1999), 1567–1570.

[3] A. Browder: *Introduction to Function Algebras*, Benjamin, New York, 1969.

[4] H. G. Dales and A. Ülger: *Banach Function Algebras, Arens Regularity, and BSE Norms*, Springer (in press).
[5] J. Diestel: *Sequences and Series in Banach Spaces*, Springer, New York, 1984.
[6] A. J. Ellis: *Central decompositions and the essential set for the space $A(K)$*, Proc. London Math. Soc. (3) **26** (1973), 564–576.
[7] J. R. Munkres: *Topology*, 2nd ed., Prentice-Hall, Upper Saddle River, New Jersey, 2000.
[8] O. Nygaard and D. Werner: *Slices in the unit ball of a uniform algebra*, Arch. Math. (Basel) **76** (2001), no. 6, 441–444.
[9] R. R. Phelps: *Dentability and extreme points in Banach spaces*, J. Functional Anal. **17** (1974), 78–90.
[10] E. L. Stout: *The Theory of Uniform Algebras*, Bogden & Quigley, New York, 1971.
[11] P. Wojtaszczyk: *Banach Spaces for Analysts*, Cambridge Studies in Advanced Mathematics, 25, Cambridge University Press, 1991.

Joel Feinstein received his Ph.D. from the University of Leeds under the guidance of H. Garth Dales in 1989. His research deals primarily with functional analysis, especially uniform algebras and more general Banach function algebras. Outside of mathematics, he enjoys playing violin in the Nottingham Symphony Orchestra alongside his wife, Uta, and spending time with his family.

Alexander Izzo received his Ph.D. from the University of California, Berkeley under the guidance of Donald Sarason also in 1989. His research lies primarily at the interface of complex analysis and functional analysis with an emphasis on uniform algebras. Outside of mathematics, he enjoys classical music and spending time in nature hiking and backpacking.

(Feinstein) School of Mathematical Sciences, University of Nottingham, University Park, Nottingham NG7 2RD, UK

(Izzo) Department of Mathematics and Statistics, Bowling Green State University, Bowling Green, OH 43403, USA

E-mail address, Feinstein: `Joel.Feinstein@nottingham.ac.uk`
E-mail address, Izzo: `aizzo@bgsu.edu`

Irish Math. Soc. Bulletin
Number 92, Winter 2023, 33–47
ISSN 0791-5578

An Interview with Professor John J. H. Miller

MAURICE OREILLY

Abstract. This interview with Professor John Miller gives an overview of his mathematical experiences from his childhood to his current research on wave energy. He provides some detail about his undergraduate days in TCD, his graduate studies in MIT, his extensive international collaborations, and his research in numerical analysis and its applications. Throughout, the reader will appreciate his passion for research and his strong engagement with researchers in Ireland, America, Europe and Asia.

Introduction

On 6th March 2023, I recorded an informal interview with John Miller about his mathematical life, at his home. After transcribing the audio recording (with the help of Otter software), John reviewed my edited version of the transcript, and I prepared the final version for publication in the Bulletin of the IMS.

Early experience of mathematics

MOR: *Good morning, John. It's a great privilege to talk to you about your decades of work with mathematics.*

JJHM: Thank you very much for suggesting it, Maurice.

MOR: *Would you like to say a little bit about how your interest in mathematics began, your earliest memories of mathematics?*

2020 *Mathematics Subject Classification.* Primary 01A70.
Key words and phrases. biography, numerical analysis, singular perturbations, wave energy, funding mathematics.
Received on 12-12-2023.
DOI:10.33232/BIMS.0092.33.47.

JJHM: Well, I remember my mother telling me that a teacher in Park House School, my kindergarten, remarked that "numbers mean something to that fellow". That's my earliest memory.

MOR: *Right, so your teachers had an important influence on you?*

JJHM: Oh, definitely, yes. My next school was Avoca School, Blackrock, now Newpark Comprehensive School. In those days, it was a small private school, run by Mr Parker, who was an engineer from Trinity. He liked mathematics, so I had a good teacher for the four years that I was there, before going to a boarding school called Kingswood School in England at the age of 14.

MOR: *What was the mathematics like in the boarding school?*

JJHM: It was good, but I studied it there only for the first two years. This was due to the English school system, which required pupils to specialize in three subjects after O-levels. It seemed then that modern languages would be a good choice, especially if I was to enter the family business, which was focussed on office machinery. The business had been founded by my grandfather, and my father built it into quite a well-known company. He was the sole agent in Ireland for IBM, and he was later invited to be the Managing Director of IBM (Ireland) Ltd, when they opened their first office here. I want to stress that there was never any pressure on me from either of my parents to choose this as a career. It was always made clear that I was free to choose whatever I wished. But I found, as time passed, that I had no particular ability in languages, and I didn't particularly like studying them. I took mathematics and applied mathematics in O-levels. But then I stopped all mathematics during the formative years from 16 to 18. It wasn't accidental that they turned out to be rather unsatisfying years for me. In the subsequent and final two years at Kingswood, the subjects I was studying did not inspire me. I had no serious goals, because the O-level results I'd got — I think I had eight subjects — were enough in those days for entry into Trinity College. I was just hanging around waiting to be 18 years old, which was considered appropriate to enter university.

Undergraduate years in Trinity

MOR: *And then you had a couple of years of Modern Languages.*

JJHM: Yes, to be precise, a year and a half of Mod Lang. And, just from a chance remark by a family friend, I suddenly realized that maybe I should be learning mathematics and science. The key thing then was that I was advised to get grinds from Mr Victor Graham, who was both a teacher of mathematics in the High School and a lecturer in mathematics to the engineers in Trinity College. He taught mathematics to many generations of schoolboys and engineering undergraduates — a brilliant teacher! It was a recommendation through one of the administrators in Trinity College, who said, "Oh, if you want to do mathematics, there's just one person to go to, and that's Victor Graham". And that was absolutely correct.

MOR: *And that really changed the trajectory of your study.*

JJHM: And my life! And, of course, as you know, INCA [the Institute for Numerical Computation and Analysis] created a perpetual trophy in his honour.

MOR: *Okay, while we're talking about that, do you want to say a little bit about that trophy?*

JJHM: That happened much later, when INCA funded the Victor Graham Trophy.

MOR: *Which was to be awarded to an Applied Mathematics schoolteacher, every year?*

JJHM: Yes, and you have done great work in promoting that.

MOR: *So, you're in Trinity now, and you're studying mathematics, at last. What was the study environment like in Trinity for mathematics at that time?*

JJHM: Well, I was tremendously happy and totally goal orientated. I simply worked as hard as I possibly could at two degree courses, one in mathematics, the other in natural sciences. Prior to this, in the final six months of my second year in College, I was catching up with the two years of school mathematics that I had missed. Mr Graham was teaching me calculus and all the preliminary stuff that was necessary to bring me up to first year Trinity level of mathematics. These were individual private tutorials. I think it was a couple of times a week. And I used to go up to his house, mainly in the evenings, and would have an hour's grind with him. He was a brilliant teacher, and I just absolutely loved the work. And I loved the first few years in Trinity doing science too.

MOR: *Who stands out as lecturers in Trinity, in those years?*

JJHM: Well, in the Freshman years, I can't remember any specific, outstanding lecturers. But, in the two Sophister years, there were two main lecturers. They were both outstanding, Mr Broderick and the Provost, AJ McConnell. I would still regard AJ McConnell as the best mathematics teacher that I ever encountered. His lectures for advanced undergraduates were simply brilliant, and they were held in lovely surroundings, too. We sat in the private library of the Provost's House, with French windows opened, in the summer, onto the lawn. There were fewer than ten of us, and we felt very privileged to be listening to lectures by the Provost. His pet cat was often in attendance too, curled up and asleep on an armchair.

MOR: *And were the lectures very interactive?*

JJHM: Not really. In general, Irish undergraduates then were quite timid. He was so brilliant; he was a master of his subject and we listened.

MOR: *And you essentially sat at his feet and took notes ...*

JJHM: ... and loved it.

MOR: *And they were all very clear notes afterwards?*

JJHM: Superb! He introduced us to relativity theory, quantum mechanics and so on; all sorts of modern fields like that at the time. And you got the impression that it was so simple [laughs]. And yet he was teaching us material from, really, the forefront of science.

MOR: *And what about Mr Broderick?*

JJHM: Mr Broderick was quite different. He was very thorough. He wasn't inspiring, but he was very thorough. And he was conscientious, because he took account of the interests of each of the students in the two Sophister years. Both McConnell and he were the backbone of the Sophister instruction in mathematics in those days. Broderick took account of what each student was interested in, especially if he was going to continue with his studies, which I think most of us did. He would teach us all a little bit of relevant material that each student was interested in studying.

MOR: *So, what subject matter did he teach that you were interested in?*

JJHM: He didn't tell us that at the time, he just did it, without comment. And, by the way, we all found out that he devoted one day a week to stay at home when he taught himself new topics. So that was the way he kept himself up to date, and presumably learned that new stuff to teach us. So, to answer your question, the relevant topics for me turned out to be stability theory, especially in numerical methods and differential equations, including norms, and a little bit of functional analysis.

MOR: *Yes. Going back to McConnell. Were there other areas that you associated with him?*

JJHM: Relativity and quantum mechanics, as I mentioned, but also fluid mechanics, aerodynamics, tensors — a broad range of topics in applied mathematics and mathematical physics.

MOR: *And what about books at that time? Did you use books much? Or did you rely mainly on the lecturers' notes?*

JJHM: We had to rely mainly on the notes we took ourselves at the lectures. There were no handouts from the lecturers. Books were a problem, because, well, I suppose you could buy them, but the more advanced we got, the harder it was to purchase books in Dublin. They had to be ordered from abroad. There was a good collection, of course, in the College Library, but the catalogue in the library was dreadful; it consisted of big volumes with each page having many little pieces of paper, one for each book, stuck onto it. I don't know if you ever saw those? And then you couldn't take any book out. So, it took you a long time to find the book in the catalogue, and then you had to wait maybe a day or more, before the book would appear, after you had filled in the request slip.

MOR: *And was the library up to date?*

JJHM: Yes, because it's a copyright library. So, it was a brilliant library. Well, I mean, up to date with, probably, a two-year delay.

MOR: *What about your peers? Did you tend to work together, or individually?*

JJHM: No, absolutely not. We didn't work together [laughs]. It was cut-throat competition!

MOR: *Thinking of things like Schol and all of that.*

JJHM: Exactly. Yes. And I was lucky, I studied and got Schol, and that meant a great deal to me, at the time.

America was the place be

MOR: *So, as you finished in Trinity then, how did the options for research emerge for you?*

JJHM: Well, again, Mr Graham, he was the guide. He told me what to do, just as he did for many of his other students. He was a very strict teacher, a very good teacher, very thorough, and he insisted that I write eight or ten letters of application to universities in America, because we had determined that I didn't want to go to Oxford or Cambridge, which were the standard choices for Trinity maths students in those days. I felt that they were too like Trinity, and, anyway, America was the place to be. It was an exciting time — the time when the Russians had sent up their first satellite, and the Americans were panicking that they were slipping behind in space research. It was 'all go' in America, and many young Irish students wanted to be part of that. Mr Graham made sure that we wrote those letters in the Christmas holidays of our final year.

MOR: *America was the place be!*

JJHM: Yes. I heard a 'grand old man' in Britain being asked where he would like to study, if he were young again. It was quite a casual radio interview. He replied that he would want to be a PhD student at MIT. So that was one of the influences that determined me to go to MIT rather than another university. I received offers from three or four top universities, and I chose MIT, because I knew it was excellent, and I imagined it would be a totally different experience from a 'posh' university.

MOR: *And was it clear early on that numerical analysis was the way you wanted to go, or were there different options open to you?*

JJHM: Well, in retrospect, I didn't understand what numerical analysis was. There was a lot of confusion in those days, because people didn't really make a clear distinction between computer science and mathematics. So, it was all the one to me. I don't think the term computer science was in use then. So, numerical analysis seemed to encompass everything interesting about computers, and the programming of computers was not regarded as a serious academic subject. I remember mathematicians being quite disparaging about computers, and saying that the only intellectually interesting thing about computers was to use them to solve mathematical problems. There was no concept of the internet or computer languages being intellectually interesting. It was a snobbish approach, and a rather degrading attitude to the people who had designed computers. But I didn't understand any of that at the time.

MOR: *You wanted to be involved with these wonderful machines, in some way?*

JJHM: Yes. I suppose that was due partly to the fact that the IBM company magazine, *Think*, was left lying around our house, maybe deliberately, in those years. It was a beautifully produced magazine, full of discussions about the future prospects for computers in business. They were not used in business at that time, but they were coming. One of the most advanced machines for business at that time was the IBM electric typewriter. There were no computers or word processors.

MOR: *It's interesting, just going back to your father now, since you mentioned him in this context, that earlier on he thought your future was in languages rather than in some technical aspects.*

JJHM: Possibly because he had no specific technical training himself. He was always more interested in the overall system rather than the nitty gritty technical details.

MOR: *Or maybe he felt that languages would be more rewarding?*

JJHM: I remember that he told me once that he would have liked to be able to talk with his business associates in their own languages. He thought that would be useful to me, if I were to take over the family business. I was thinking of that too. But once I got hooked on mathematics through Mr Graham's influence, probably I was thinking more and more about an academic life. I never gave much thought to my future. I never planned it. I thought everything would just happen. I always felt things would work out well [laughs]. I don't know why.

MOR: *Let's say a little bit more about your time in MIT. How did your experience there evolve over the four years?*

JJHM: It was great flying to MIT in one of Aer Lingus's beautiful new Boeing 707s. In those days, people were proud that Aer Lingus had Boeings, and that there was a direct flight from Dublin to Boston, on an Aer Lingus Boeing, and I was proud to arrive at MIT as an Irish student. Of course, I was very proud of being Irish, and it was at the height of the Kennedy era. This was a time when academics across America were optimistic that many problems could be solved. People in general were very positive, and there was an air of excitement. The one disappointment was the discovery that I had to do two more years of courses and exams in MIT. I wanted to do research. I was ready at that stage to do research, having been an undergraduate for six years, which was far too long. But it was delayed by two years because MIT's new numerical analyst, Gilbert Strang, was not due to arrive till two years later. So, it fitted in quite nicely, although I could have started research had he been there. I spent a lot of that two years, going to lectures on different topics. I really went to lots of introductory lectures on a variety of subjects.

MOR: *Which ones did you find particularly interesting?*

JJHM: I went to lectures on astrophysics, mathematical logic and meteorology, for example. Of course, there were a lot of famous scientists there. I took courses in mathematics and physics, which would count towards what I had to do for the PhD. For example, plasma physics applied to tokamak devices to generate electricity. That was still an unsolved problem, even though it was said then that it would take just twenty years for it to become commercial. Nowadays, the period to commercial application is still claimed to be twenty years.

MOR: *Who do you remember from those famous scientists?*

JJHM: I remember Charles Townes, who later received the Nobel prize for the invention of the laser. He offered me a position as a student on his team. I also saw, from a distance, Norbert Wiener and one of the Browder brothers. An interesting encounter was with John McCarthy, author of the programming language, LISP. At that stage he was not well known publicly, but he has now become extremely famous as one of the founders of artificial intelligence. That meeting turned out to be a remarkable coincidence. A few years later, at a party in Boston, I met and, later, married an intelligent and beautiful young Kerry nurse called Mary O'Reilly. Many years passed before we learned that she and John McCarthy were first cousins.

Early years in research and international collaborations

MOR: *Then Strang arrived. What thesis topic did he suggest?*

JJHM: He suggested that a recent theorem on power-bounded matrices could be strengthened. I managed to do this in two years. I considered that a great topic and I thoroughly enjoyed working on it, day and night, under Gil's expert and kind guidance. It is interesting to note that Gil is just a few years older than me and, as we speak, he has still not retired. I was lucky to encounter him for the first two years of his stellar career at MIT. We have remained in contact ever since. After finishing my PhD studies, I became an Associate Professor in the first year of operation of the University of Massachusetts at its new campus in Boston. I remained there for five years, but with several visiting positions in Chalmers University, Gothenburg, the Istituto di Elaborazione dell'Informazione, Pisa, and Trinity College Dublin. During my stay in Pisa, I clarified and expanded my thesis work, which was published in monograph form by the IEI [2]. This is now available on ResearchGate. In 1970, I accepted a two-year position in the Economics Department, TCD, with the promise that it would be followed immediately by a permanent position in the Mathematics Department. Fellowship followed a few years later, and the number of postgraduates I was supervising began to increase. I organised three international conferences on numerical analysis, under the auspices of the Royal Irish Academy, and then I took a two-year visiting position as a professor in the University of Nijmegen with an offer that this would be made permanent after two years.

MOR: *Your involvement in further conferences was connected with events in the Netherlands?*

JJHM: Yes. I was approached by Simon Polak, a senior employee of Philips in Eindhoven. Out of the blue, he came and talked me into organizing a conference on the numerical analysis of semiconductor devices. He said that Philips, through him, would help me to set up the conference. He would put me up for a couple of nights in a hotel at Philips' expense, where I could use the telephone without limits. He expected me to spend a day or two, in my hotel room, telephoning people worldwide to find out about numerical analysis for semiconductor devices. This was such an extraordinary and imaginative offer, that I certainly couldn't resist it. It was so exciting! So, I chatted with total strangers in many countries. I formed a network using a bootstrapping technique. I'd ask the person I was talking to "Well, who else do you know?" and I'd get additional names and telephone numbers. It all just grew and grew. I think I'd covered the main people in those 48 hours of work. So here was me, sitting in Eindhoven, talking for maybe an hour to an American on the telephone. I even remember talking to a famous American engineer, who told me at the end of the conversation that he was in the shower for the entire conversation [laughs]. They thought this was an exciting project. I quickly got a picture of what was happening in an area of application of numerical analysis. This was the birth of the NASECODE [Numerical Analysis of Semiconductor Devices] conferences, most of which were held in Ireland. NASECODE I, held in Dublin, was the first conference in the world on this topic. The participants were delighted to be brought together, especially in such an unlikely place, in those days, as Ireland.

MOR: *People didn't like to divulge the latest discoveries in their research, but even more so because there was a lot at stake in the development of semiconductor devices.*

JJHM: Absolutely, yes. So, the topics that they gave talks on would, I'm sure, be vetted by their management. And especially because I had a very altruistic view of research in the Cold War era, I felt that science should be used to ameliorate the results of the Cold War, and, maybe stupidly, I really wanted to bring people together. I thought if people came together, that would help to maintain peace and so on. So, I insisted that the whole world was represented at these NASECODE conferences. I insisted that Russians, who were the only serious competitor to the US in those days, would be treated as equally as possible, and as equally as they were willing, because they were much more restricted than anybody else. I mean they were restricted by their government, whereas Americans were restricted by their companies. So, I did succeed in attracting scientists and engineers from many countries. Of course, you had to deal with the Academy of Sciences to get anywhere in the Soviet Union. And, through the Academy of Sciences, powerful people became involved. I was dealing with the top-level people in the Academy of Sciences, including the then President of the Academy of Sciences, the mathematician, Guri Ivanovich Marchuk, who had a strong interest in numerical analysis. They were all very good communists. But we collaborated as scientists, and, at any meetings, or anything to do with the running of the NASECODE conferences, politics were simply not mentioned. It seemed to me that everybody, on both sides, appreciated the opportunity to meet each other. These experiences led, on a St Patrick's Day in Moscow, to my having lunch in his official dacha with the President of the USSR Academy of Sciences, and, that same evening, attending the St Patrick's Day party in the American Embassy.

MOR: *And so, as far as Marchuk is concerned, what are your memories of your first contact with him? And how did it come about?*

JJHM: This was on the occasion of a seminar he gave at Chalmers. The next meeting was during a visit to the Institute of Theoretical Physics in Trieste. We had both been invited there by Professor Jacques-Louis Lions to give some lectures at a summer school in the Institute. Incidentally, it was there that I was approached by a Thai student, who asked if he could become a PhD student of mine. Since he had his own funding, I agreed immediately.

MOR: *Lions was one of the founders of IRIA* [*l'Institut de recherche en informatique et en automatique*]*, which became INRIA later, I think.*

JJHM: Yes. Lions was an excellent mathematician, and a very powerful man in French science. He had many students in numerical analysis. He published numerous papers and wrote many books. He was a member of the Académie Française. He was a close colleague of Marchuk; they exchanged many visits.

MOR: *So those encounters with Marchuk and Lions really opened doors in big ways to mathematicians in the Soviet Union.*

JJHM: Marchuk was certainly politically powerful. I know that, at one stage, he was one of the Vice-Presidents of the Russian Socialist Republic. This may have been due to his position in the Academy of Sciences. He was also an author of many papers and books. It is widely believed, that he was responsible for the mathematical computations required in the building of the Soviet nuclear bombs.

Research in singular perturbations

MOR: *Were you already working on singular perturbations at that time, or did that come a little later?*

JJHM: I think Lions mentioned singular perturbations in a lecture that I attended. I would have been visiting France on an exchange programme between the Irish and the French governments. In those days, these exchange agreements, luckily for me, didn't seem to be very popular amongst Irish mathematicians, and I grabbed every one of them that I possibly could.

MOR: *From the Irish point of view, were they administered by the Royal Irish Academy?*

JJHM: Yes, and by an equivalent French organization. Lions was involved on the French side. It was on these visits that I got to know Lions quite well, and I went to a lot of his inspiring lectures in his university and INRIA. I also met people like Philippe Ciarlet, Pierre-Arnaud Raviart, Roland Glowinski and many other well-known French mathematicians. Lions had a very extensive group of his own students and colleagues. He was a friendly and helpful man. At one of his lectures, he spoke about singular perturbations, and mentioned the Russian mathematician, A. M. Il'in, who was doing good work on this topic. Apparently, there were two Il'ins, and the singular perturbation one was hard to find. That sounded great to me — a good subject that Lions is interested in. I tracked down Il'in, the wrong one first, and then I found the right one.

MOR: *How did you communicate with them — by post? This was before the internet was available. Did you phone them, like the others?*

JJHM: It was telephoning and snail-mail. And, notably, in the early days, during one of the conferences I organized, we had a postal strike in Ireland and, I think, a bank strike too. I must say, it was interesting organizing a conference with no postal services [laughs]. It all had to be done by suitcases of letters taken on the mailboat to post offices in Holyhead, and also by phone and telex, where possible. Yes, fax, also, was very important. That was the really advanced way of communicating in those days.

MOR: *So, your early contacts amongst Soviet mathematicians were Marchuk and Il'in?*

JJHM: Yes, mainly Marchuk. I didn't meet Il'in for a long time. But I met his students. He was not an organizer. He was good academic.

MOR: *Who was the first person you actually met in Il'in's circle of singular perturbation people?*

JJHM: I think that was Grigorii Ivanovich Shishkin. This happened in Novosibirsk at the BAIL conference that I organised there. This was one of a series of BAIL [Boundary and Interior Layers] conferences. At that conference an important relationship was established between one of my Irish graduate students and Shishkin, which led to many visits to Dublin by the latter, and the publication of numerous joint papers and books on numerical methods for singular perturbation problems.

MOR: *That was the fourth conference, I think; the first three in Dublin, and then the fourth in Novosibirsk.*

JJHM: Yes. I'm delighted that the BAIL series is still going strong, independently of me. That conference led, of course, to our main working contact in Russia, namely Shishkin. He was a younger colleague of Il'in, who was the leader of the department. For the following 15 years or so, Irish numerical analysts had many happy years of collaboration with Shishkin. His long visits were almost entirely supported by the meagre funds we had available for this purpose. Shishkin was passionate about his approach to numerical methods for singular perturbation problems. He would undergo almost any kind of suffering in order to travel to promote his own methods. The Irish group involved at this time would probably all agree that he has never been properly recognized, at home or abroad, for the quality of his work. But I think most of us would agree, that this is probably due to his difficult personality.

MOR: *When one looks at citations on Google Scholar, the most cited work under your name is the book on fitted methods for singular perturbation problems, from 1996* [3]. *Taking that as an example of collaboration with Shishkin, what are your memories of writing it?*

JJHM: It was a satisfying experience, because we didn't have to worry at all about the quality of the mathematics. Shishkin was monitoring, line by line, as it was being written. Nothing appeared in that book that was not approved by Shishkin. He is an absolute master of the subject. There's no doubt about that. But he is a singular person.

He is deliberately not a teacher. He doesn't reveal anything beyond the immediate question. He has done far more work on every topic than he would ever reveal to a colleague. If you wanted some help to understand an argument, or you questioned a result of his, he wouldn't go out of his way to explain it to you. He would give you some guidelines about how to do it. But it was really your business to educate yourself about what he had done. In his own papers, he quotes results that require long algebraic manipulations, which you imagine he did not have to do because of some trick that he knew. Invariably that turned out not to be the case — he had done the same calculations twenty years ago. He knew that you were wrong, but all he would say was, "Go and try again", with no possibility of further help from him.

MOR: *It was quite a challenge to fill in the gaps then.*

JJHM: Absolutely, but I'm not sure that I ever found any significant mistake in anything that Shishkin has written. Unfortunately, much of what he has written is, in my opinion, almost incomprehensible. I doubt that much of the material will ever be checked by somebody else. Perhaps AI will be used to elucidate it at some stage!

MOR: *And so, his results that are not in the book on fitted methods for singular perturbation problems* [3] *or in the one on robust techniques for boundary layers* [1] *(or in other joint papers and books) may be neglected, because people won't make the required effort to understand them.*

JJHM: Exactly! Certainly, those of us who worked with him always felt that a main role of our work was to ensure that Shishkin's results were not lost. We felt that we were opening things up, and that lots of other people would follow us. But, for reasons I still don't fully understand, Shishkin's methods were not widely adopted, especially in the United States.

MOR: *And who were the other significant Irish collaborators?*

JJHM: Largely my own students and, in turn, their students, and then, of course, the highly successful group of Martin Stynes in Cork.

MOR: *And also, significant groups throughout Europe.*

JJHM: Yes. Nothing to do with me. Groups spearheaded originally by people such as Hans-Görg Roos in Dresden, and Lutz Tobiska in Magdeburg.

MOR: *You've also had very significant collaborations, I think, in Singapore and India, for example. Do you want to say something about those more recent ones?*

JJHM: Yes. I took early retirement in the year 2000, which was exactly thirty years after I had started in Trinity. That was some years before I had to retire, since I wanted to do so while I still had some energy for new ventures. I undertook a number of things, including two extended visits to Singapore, spread over a two-year period. My duties there were essentially research, and we were well treated. While in Singapore, I became aware of someone in India, who was working on singular perturbations. It turned out to be an Indian lady, Valarmathi Sigamani, known more concisely as Mathi. I got in touch with her and, one Christmas, we went to see her in Tiruchirappalli, otherwise known as Trichy. She made the important remark during our visit, that she was only interested in papers that contained theoretical analysis involving rigorous proofs of parameter uniform convergence. From that point on, I decided to support her in every way I could. She was very keen to help young female undergraduates to get a good education in mathematics. It is noteworthy that the college where she worked had been set up in the 19$^{\text{th}}$ century by a Lutheran pastor from Germany. I was told that they took in children literally off the street and, if they were intelligent, housed and educated them. Some of these students later became successful academics in different colleges. She built up a large group of devoted postgraduate students. My role was to provide some guidance on suitable research problems, and, sometimes, to help with the actual details of the proofs. I often had to suggest that a proof be clarified with more detail, as I did with my own students. We published many papers together, and she always insisted on adding my name to the list of authors. Most of the credit for obtaining the results is due to Mathi and her group.

MOR: *So that was a very fruitful collaboration and an unexpected one?*

JJHM: Yes, because there was so much energy coming out of the department she was running, so many bright young Indian people. I suppose I provided some useful general direction to them, but it was their own energy and their own abilities that counted. All they needed was a little guidance in the right direction.

MOR: *Well, John, we've spoken about your collaborations in so many places, but I think there are some others that come to mind which you might like to say a bit more about. I'm thinking of Bulgaria, Egypt, Thailand, Yugoslavia and China. You had significant contacts in those countries.*

JJHM: With regard to the first three, I had various visits at different stages of my career: in Bulgaria, in particular with my friend, Svetoslav Markov; in Egypt, where our family spent a memorable Christmas and New Year on the campus of the University of Assiut; in Thailand, on many occasions with Suwon Tangmanee. The first contact in Yugoslavia was with Zorica Uzelac, when I was asked to play a role on her PhD thesis committee. This was in the late 1970s, while we were living in the Netherlands. We drove to Ljubljana, took the train to Belgrade, and were then driven to Novi Sad. My role there was entirely nominal. The candidate became a personal friend of ours. Many years later, INCA funded Zora to translate Shishkin's doctoral thesis into English. She made a particularly good job of this onerous task, and her translation later played a big part in a book in English published by Shishkin. After the breakup of Yugoslavia, the Croatian mathematician, Mladen Rogina, invited me to several of his conferences in different venues. Sadly, he passed away before his time, at a relatively young age. I became interested in developing contacts with China, when colleagues in the US told me that Chinese PhD students were coming to the US, and that they were hard-working. An early contact was Guo Ben-yu, whom I invited to spend some time in Trinity. Subsequently, the BAIL V conference was organized in 1988, in Shanghai, with his help, and the BAIL VII conference was organised in 1994 by Zhuang Fenggan in Beijing. More recently, after a lavish celebration in Shanghai in honour of Guo Ben-yu, I visited Houde Han in Beijing to complete some joint work.

My students

MOR: *But even before that, you attracted quite a few Chinese students into the research group. How did that come about?*

JJHM: Most of my Chinese students were contacted through the BAIL conferences. I had plenty of money, due to the EU EVEREST contract, which involved illustrious partners such as Philips, GEC and Rutherford Appleton Laboratories. It was the largest European Commission contract that I participated in. It involved the numerical modelling of semiconductor devices in three dimensions. There was a lot of money available. That particular contract probably brought in a quarter of a million euro to Trinity. It was a huge amount of money in those days. To get the work done, I realised that there weren't enough suitable postgrad students in Ireland. Naturally, I would automatically support every Irish student, but I had to find more. This was just at the time when China was opening up, and my US colleagues were speaking highly of their Chinese students. On one occasion, while I was in Beijing on the way to the BAIL IV conference in Novosibirsk in 1986, I met some of these candidate students. It was arranged that I would meet them, so that I could recommend them for studentships in Trinity.

MOR: *You attracted a lot of graduate students. On my reckoning, you probably had, at the period, at least half of the graduate students in mathematics in the country. When you look back, how do you see the recruitment of graduate students to mathematics in Ireland in the 80s, which was when the Numerical Analysis Group or NAG arose? How do you see that in the context of graduate students staying in Ireland, and indeed attracting students from abroad to Ireland?*

JJHM: Well, of course, I'm delighted that that happened. I'm very proud of the fact that so many of my students did well in whatever they chose to do. For example, three of my former students became heads of university departments in Ireland and at least two others in other countries. So, I'm absolutely delighted that that worked out. As for the Chinese, I don't know whether it's sad or not, but the Chinese students neither stayed in Ireland nor went back to China. One of them has done very well in Canada, and two of them have done very well in Australia. Had it not been for visa problems, I think that at least one of them would have stayed in Ireland. Nothing was ever said, but it seemed to me that, deliberately, the Department of Justice in Ireland was not allowing Chinese students to stay here. How different things are now in Ireland!

MOR: *So that was Ireland's loss.*

JJHM: In my opinion, that was Ireland's loss.

Research funding

MOR: *One of the key things to consider is funding of research in Ireland, and you've got a lot of experience of that. What would be your overview of the funding from both Irish and, of course, European sources, and your experience of it all?*

JJHM: Well, my experience is very early experience, so I can say nothing about current funding — everything seems to have changed for the better. It's a completely different world now to the world then. There had been no serious funding in Ireland, for mathematics, when I was a young, or even middle-aged, academic in Trinity. There was simply nothing. Then, in the 1980s, European Union funding emerged. Of course, most academics in Ireland, of my age and generation, knew nothing about how to get or use that funding. I had been in the Netherlands for two years, as a full professor in Nijmegen. But, even there, there was no effort to teach new academics how to apply for this funding. The key problem was that you were asked to find half of the money yourself, and the European Commission would pay the other half. Finally, I learned how to do it. I think I was among the first to do so in Trinity. I probably would thrive in the present system, although it doesn't attract me nowadays. Unfortunately, Trinity didn't really understand how to make the best use of people who brought in those grants. The academics were also naïve. Now, a mathematician with a big grant would negotiate excellent terms for his students and himself, and if these were not forthcoming, he would be snapped up by another university. The concept of an academic making such a proposal and threatening to resign, would never even have occurred to any academic in those days. But it's the way things have gone, and it's a much healthier way, because you're making the best use of the available resources.

MOR: *Looking at sources of funding that you've dealt with over the years — initially, it was primarily from Europe, and then later, I think around 2005, there was the SFI funding — how do you see the introduction of significant funding from indigenous sources at that time?*

JJHM: Irish funding was transformed by Atlantic Philanthropies. Chuck Feeney had an enormous influence and showed the way, insisting that for every million dollars that he put into research in Ireland — and it was millions — the Irish government had to match it. So, that was brilliant. Then, I think, it was Mary Harney who set up SFI. That was important as well. It was run, initially, by experienced Americans from the NSF. I worked there for six months or so, after retiring. There were still some Americans there, so a lot was done in the American way, and, therefore, it was very fair and unbiased. This funding changed everything as far as Irish research goes. It's now an environment that, if I were a young man, I'm sure I would love to be part of. People who knew me, twenty years earlier, said that I was a little ahead of my time. I suppose, because of all the travelling and international experience, I was aware of things that people back home, in more senior positions, were not.

MOR: *Yes, but that critical mass wasn't achieved until later.*

JJHM: Yes. Hopefully, some of my activities contributed to some extent to creating that critical mass. Ireland is a brilliant place now for research, and a terrific environment for ambitious people to do their research.

Students and teaching

MOR: *When you look back over your career, how do you see your teaching and the readiness of students?*

JJHM: I must be frank about this. Up to a point, I quite enjoyed teaching, but teaching was never an important thing to me. I mean, I knew teaching was important, because I'd had such good teachers, and I knew who my good teachers were. The two outstanding teachers that I came across were, as I said, Provost McConnell and Victor Graham. They were both gifted teachers, especially Victor Graham, whose life was devoted to teaching. He was inspired to teach, and, I think, he loved teaching. He influenced so many people. As well as a natural instinct to make things simple, because of McConnell and Graham, I wanted to make my teaching simple. So, I hope my lectures were easy to understand, even though I sometimes lectured on complicated things. That's the nice thing about mathematics, once you really understand it, it's simple. If you don't think it's simple, I think you haven't fully understood it.

MOR: *So that's your perspective as a teacher. How did you find, overall, the preparedness of undergraduate Irish students?*

JJHM: I do want to stress, by the way, that I don't regard myself as being a particularly good teacher, and I wasn't motivated to become a brilliant teacher. But I knew I wasn't a bad teacher. Certainly, I could have given better lectures than I did. But what was your last question?

MOR: *How did you find the preparedness of students?*

JJHM: I was very impressed with the Irish students. I found that most of the Irish students, that I encountered, were very well trained and prepared. I admired their self-motivation, as well. They were great.

MOR: *These were the graduate students that you had?*

JJHM: Yes, the graduate students. But the undergraduates too were well prepared. The basic training in mathematics, I thought, in my day, was very good. Although I never taught first year students, I was impressed, always, by the students I met in the higher classes in Trinity.

My current research

MOR: *The last topic that I want to make sure to ask you about is your current research. Can you say something about that, John, and how it arose?*

JJHM: Well, when I retired early, I studied renewable energy, especially wind and wave. Then, luckily, through activities in INCA, I got to know Lawrence Crane much better than previously. He and I had been colleagues for about thirty years in Trinity. Strangely, we didn't really know each other well, in our Trinity days. He retired a little later than I did, even though he was a little older than I was. I retired in the year 2000. He retired a few years later. We then worked constructively for about twenty years together, getting to know each other very well, in the context of INCA. After trying several things, we finally became interested in renewable energy, which was a hot topic twenty years ago, and still is. After looking at wind, we realized that wind was well understood, and that a lot of the interesting research had been completed. Then, we found that there were many open research problems related to wave power. Lawrence and I both felt that two retired professors, like ourselves, with no money, would still be able to contribute. We made a decision not to worry about funding, but just to set up an organization that would survive, whether or not we got funding. So, we set up a company called Waveforce Energy Ltd, which still exists. It has never had any significant funding, and it pays no salaries. But we regard ourselves as extremely lucky to have found this research topic. We have learnt a lot about it, and we are passionate about the work. Sadly, Lawrence passed on two years ago. We had a very productive collaboration, and now the work is continuing with a colleague, in Limerick, who is a mechanical engineer. Recently, we have added an electrical engineer to the team. We have a digital model of a complete micropower device designed to produce electrical power from ocean waves. This device is ready to be built and tested. We hope to complete that in 2024. The device has a number of novel features. It is small, inexpensive, robust, efficient and simple. We hope it will produce micropower more cheaply than any previous micropower device. There are currently no successful large wave power devices.

Conclusion

MOR: *The public at large can find it difficult to see how mathematics works through numerical analysis, modelling, and cutting-edge technology. You've got an overview of all of that. What would you say to a youngster who hasn't seen this, maybe to spark their imagination, and to get them to see how mathematics can be really useful in the way that you've applied it over the years?*

JJHM: Well, I think I have been very lucky in life, and I cannot give any advice on how to find a project like the one Lawrence and I found to work on, in our old age. It was something that we were passionate about, and I still am. It meant that we were as busy as ever during the Covid pandemic, and we did not have to meet in order to work closely together — email and the telephone were sufficient to keep the work going. When you are young, the key thing is to concentrate on what you are good at and what you enjoy the most. If you can find a job that involves these things, then you are in luck. Later in life, you will probably realise that as long as you have sufficient money, it is your health, both physical and mental, that really matters. In short, ensure that you have an interest that inspires you, and the good health to pursue it.

MOR: *So, passion and inspiration really matter.*

JJHM: That's the right way of putting it.

MOR: *John, thanks very much. I really enjoyed talking with you.*

JJHM: Thank you very much, Maurice, for organizing it. Without your initiative and drive, it wouldn't have taken place.

References

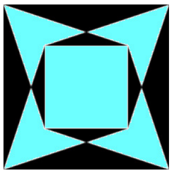
[1] P. Farrell, A. Hegarty, J.J.H. Miller, E. O'Riordan and G. I. Shishkin. *Robust computational techniques for boundary layers.* CRC Press, 2000.

[2] J.J.H. Miller. *On the L_2 Theory of Linear Hyperbolic Differential and Difference Equations with Constant Coefficients.* Istituto di Elaborazione dell'Informazione, Pisa, 1969.

[3] J.J.H. Miller, E. O'Riordan and G. I. Shishkin. *Fitted numerical methods for singular perturbation problems: error estimates in the maximum norm for linear problems in one and two dimensions.* World Scientific, 1996.

Maurice OReilly is recently retired Associate Professor in mathematics education at Dublin City University. Previously he was Head of Mathematics at St Patrick's College Drumcondra, Dublin. He completed his PhD in numerical analysis with Professor John Miller as advisor, spent 14 years in Dundalk Regional Technical College and three in the University of Dar es Salaam, Tanzania. His current interests are in mathematics education and the history of mathematics.

Maurice OReilly, CASTeL, Dublin City University.
E-mail address: maurice.oreilly@dcu.ie



Irish Math. Soc. Bulletin
Number 92, Winter 2023, 49–53
ISSN 0791-5578

The Sieve of Eratosthenes and a Partition of the Natural Numbers

PETER LYNCH

Abstract. The sieve of Eratosthenes is a method for finding all the prime numbers less than some maximum value M by repeatedly removing multiples of the smallest remaining prime until no composite numbers less than or equal to M remain. The sieve provides a means of partitioning the natural numbers. We examine this partition and derive an expression for the densities of the constituent "Eratosthenes sets". The densities must sum to unity, yielding an interesting result, equation (14), that may be new.

1. The Sieve of Eratosthenes

The *primorial*, P_K — often denoted $K\#$ — is defined to be the product of the first K primes:

$$P_K = \prod_{k=1}^{K} p_k \,.$$

The sequence of primorials is $\{2, 6, 30, 210, 2310, \dots\}$ and the terms of the sequence grow as K^K. It is convenient to set $M = P_K$. The algorithm of Eratosthenes goes as follows: starting from the set $I_M = \{1, 2, 3, \dots, M\}$,

- eliminate all multiples of 2 greater than 2;
- eliminate all remaining multiples of 3 greater than 3;
- eliminate all remaining multiples of 5 greater than 5;
- ... ...
- eliminate all remaining multiples of p_K greater than p_K.

All that remains is the set of the first m prime numbers, $\{2, 3, 5, \dots, p_m\}$, where p_m is the largest prime not exceeding $M = P_K$.

For all $k \in \mathbb{N}$, let D_k be the set of numbers in $\mathbb{N}$ that are divisible by p_k. For $k = 1, 2, \dots, K$, we define the set $D_{k,M} = D_k \cap I_M$ to be the set of numbers in I_M that are divisible by p_k. Thus, $D_{1,M}$ is the set of even numbers up to M, $D_{2,M}$ the multiples of 3 up to M, and so on.

The k-th "Eratosthenes set", E_k, is the set containing p_k together with all the numbers removed at stage k. Thus, E_1 is the set of all multiples of 2, that is, all the even numbers; E_2 is the set of odd multiples of 3; E_3 is the set of multiples of 5 not divisible by 2 or 3; E_4 is the set of multiples of 7 not divisible by 2, 3 or 5; and so on.

The Eratosthenes set E_k may be defined symbolically:

$$E_k = \{n \in \mathbb{N} : (p_k \mid n) \wedge (p_\ell \nmid n \text{ for } \ell < k)\} \,.$$

Some initial values of E_k are shown in Table 1.

2020 *Mathematics Subject Classification.* 11A41, 11B50.
Key words and phrases. Sieve, Partition, Prime numbers.
Received on 12-12-2023.
DOI:10.33232/BIMS.0092.49.53.
My thanks to Dr Kevin Hutchinson, UCD for comments on a draft of this paper.

Table 1. Arrangement of the natural numbers as multiples of the prime numbers in sequence. The k-th row contains the "Eratosthenes set" E_k.

2	4	6	8	10	12	14	16	18	20	22	24	26	28	30	...
3	9	15	21	27	33	39	45	51	57	63	69	81	87	93	...
5	25	35	55	65	85	95	115	125	145	155	175	185	205	215	...
7	49	77	91	119	133	161	203	217	259	287	301	329	343	371	...
11	121	143	187	209	253	319	341	407	451	473	517	583	649	671	...
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

We denote by $E_{k,M}$ the set $E_k \cap I_M$. It is the set containing all multiples of p_k up to M that are *not multiples of any smaller prime*. We see immediately that $E_{1,M} = D_{1,M}$, that $E_{2,M} = D_{2,M} \setminus D_{1,M} = D_{2,M} \cap D^{\complement}_{1,M}$ and, more generally, that

$$E_{k,M} = D_{k,M} \setminus (D_{1,M} \cup D_{2,M} \cup \cdots \cup D_{k-1,M}) = D_{k,M} \cap (D_{1,M} \cup D_{2,M} \cup \cdots \cup D_{k-1,M})^{\complement} .$$

Using De Morgan's law, we may write

$$E_{k,M} = D_{k,M} \cap (D^{\complement}_{1,M} \cap D^{\complement}_{2,M} \cap \cdots \cap D^{\complement}_{k-1,M}) . \tag{1}$$

Since all primes p_k for $k \leq K$ divide M, the sizes of the D-sets are known: $|D_{k,M}| = M/p_k$ and so $|D^{\complement}_{j,M}| = M - M/p_j = M(1 - 1/p_j)$.

The Inclusion-Exclusion Principle. The inclusion-exclusion principle provides a valuable means of calculating the sizes of unions of sets [1] We denote the cardinality of a finite set A by $|A|$. The size of the union of two finite sets is

$$|A \cup B| = |A| + |B| - |A \cap B| , \tag{2}$$

where the intersection term prevents double counting. For the union of three sets,

$$|A \cup B \cup C| = (|A| + |B| + |C|) - (|A \cap B| + |A \cap C| + |B \cap C|) + |A \cap B \cap C| . \tag{3}$$

This idea can be generalised using the inclusion-exclusion principle to give the magnitude of the union of n finite sets:

$$\left| \bigcup_{i=1}^{n} A_i \right| = \sum_{i=1}^{n} |A_i| - \sum_{1 \leqslant i < j \leqslant n} |A_i \cap A_j| + \sum_{1 \leqslant i < j < k \leqslant n} |A_i \cap A_j \cap A_k| - \cdots + (-1)^{n+1} |A_1 \cap \cdots \cap A_n| . \tag{4}$$

Thus, the size of the union of sets is expressed as a combination of sizes of intersections.

Density. We define the density of a set $A \subseteq I_M$ (relative to M) to be $\rho(A) = |A|/M$. Then $\rho(D_{k,M}) = 1/p_k$ and $\rho(D^{\complement}_{j,M}) = (1 - 1/p_j)$. We note that division of equations (2)–(4) by M converts the cardinalities to densities. Thus, for example, (2) becomes

$$\rho(A \cup B) = \rho(A) + \rho(B) - \rho(A \cap B) .$$

Clearly, density is additive for disjoint sets. Thus,

$$\rho(D_{k,M}) = \rho(D_{k,M} \cap (D_{\ell,M} \uplus D^{\complement}_{\ell,M})) = \rho(D_{k,M} \cap D_{\ell,M}) + \rho(D_{k,M} \cap D^{\complement}_{\ell,M})$$

and, as p_k and p_ℓ are coprime, $\rho(D_{k,M} \cap D_{\ell,M}) = 1/p_k p_\ell$ and $\rho(D_{k,M} \cap D^{\complement}_{\ell,M}) = (p_\ell - 1)/p_k p_\ell$, so that

$$\rho(D_{k,M} \cap D_{\ell,M}) = \rho(D_{k,M})\rho(D_{\ell,M}) \quad \text{and} \quad \rho(D_{k,M} \cap D^{\complement}_{\ell,M}) = \rho(D_{k,M})\rho(D^{\complement}_{\ell,M}) \tag{5}$$

Moreover,

$$\begin{aligned} \rho(D^{\complement}_{k,M} \cap D^{\complement}_{\ell,M}) &= \rho((D_{k,M} \cup D_{\ell,M})^{\complement}) = 1 - \rho(D_{k,M} \cup D_{\ell,M}) \\ &= 1 - [\rho(D_{k,M}) + \rho(D_{\ell,M}) - \rho(D_{k,M} \cap D_{\ell,M})] \\ &= 1 - \left(\frac{1}{p_k} + \frac{1}{p_\ell}\right) + \frac{1}{p_k p_\ell} \;=\; \frac{p_k - 1}{p_k}\frac{p_\ell - 1}{p_\ell} \\ &= \rho(D^{\complement}_{k,M})\rho(D^{\complement}_{\ell,M}) \,. \end{aligned} \tag{6}$$

By means of the inclusion-exclusion principle, we easily extend the product relationships (5) and (6) to show that the density of the set $E_{k,M}$ in (1) is the product of the densities of the component sets on the right side:

$$\rho(E_{k,M}) = \rho(D_{k,M})\rho(D^{\complement}_{1,M})\rho(D^{\complement}_{2,M}) \ldots \rho(D^{\complement}_{k-1,M}) \,. \tag{7}$$

Using explicit expressions for the terms on the right, the density of the set $E_{k,N}$ is

$$\rho(E_{k,M}) = \frac{1}{p_k}\frac{(p_1 - 1)}{p_1}\frac{(p_2 - 1)}{p_2} \;\ldots\; \frac{(p_{k-1} - 1)}{p_{k-1}} = \frac{1}{P_k}\prod_{j=1}^{k-1}(p_j - 1) \,, \tag{8}$$

where $P_k = p_1 p_2 \ldots p_k$. We observe that the numbers $\rho_{k,M} := \rho(E_{k,M})$ are generated by a recurrence relation

$$\rho_{k+1,M} = \left(\frac{p_k - 1}{p_{k+1}}\right)\rho_{k,M} \,, \tag{9}$$

with initial value $\rho_{1,M} = \frac{1}{2}$. This enables us to compute the sequence $\{\rho_{k,M}\}$. The first eight density values are given in Table 2.

TABLE 2. Density of the Eratosthenes sets E_k for $k \leq 8$.

k	1	2	3	4	5	6	7	8	...
p_k	2	3	5	7	11	13	17	19	...
P_k	2	6	30	210	2310	30,030	510,510	9,699,690	...
ρ_k	$\frac{1}{2}$	$\frac{1}{6}$	$\frac{1}{15}$	$\frac{4}{105}$	$\frac{8}{385}$	$\frac{16}{1001}$	$\frac{192}{17{,}017}$	$\frac{3072}{323{,}323}$	...

2. PASSAGE FROM I_N TO $\mathbb{N}$

For arbitrary $N \in \mathbb{N}$, let $I_N = \{1, 2, \ldots, N\}$ and let $D_{k,N}$ denote $D_k \cap I_N$, the set of all multiples of p_k not exceeding N. Then $|D_{k,N}| = [N/p_k]$ and $\rho(D_{k,N}) = [N/p_k]/N$. Since, for any real x, we have $x - 1 < [x] \leq x$, it follows that $(N/p_k) - 1 < [N/p_k] \leq N/p_k$, and thus $(1/p_k) - (1/N) < \rho(D_{k,N}) \leq 1/p_k$. Therefore, the limit of $\rho(D_{k,N})$ exists, so that

$$\rho(D_k) := \lim_{N\to\infty} \rho(D_{k,N}) = \frac{1}{p_k} \qquad \text{and also} \qquad \rho(D_k^c) = 1 - \rho(D_k) = 1 - \frac{1}{p_k} \,.$$

In this way, we can pass from I_N to $\mathbb{N}$, obtaining the densities of all the Eratosthenes sets in $\mathbb{N}$. In particular, the values of ρ_k in Table 2 are also the densities of the first eight (infinite) Eratosthenes sets relative to the natural numbers. Equations (7) and (8) remain valid in the limit $M \to \infty$, as does the recurrence relation for $\rho_k := \lim_{M\to\infty} \rho_{k,M}$. Thus,

$$\rho_{k+1} = \left(\frac{p_k - 1}{p_{k+1}}\right)\rho_k \,. \tag{10}$$

Convergence. We now show that the series $\sum \rho_n$ converges. The simple ratio test is inadequate, as $\lim \rho_{n+1}/\rho_n = 1$, telling us nothing. A more subtle and discriminating test is required.

In his classical text, *Introduction to the Theory of Infinite Series*, Bromwich [2, §12.1] describes an extension of the ratio test, originating with Ernst Kummer and refined by Ulisse Dini. To test a series $\sum a_n$ for convergence, we select a sequence $\{d_n\}$ such that the series $\sum d_n^{-1}$ is divergent. The criterion is as follows.

$$\text{Let}\quad t_n = d_n\left[\frac{a_n}{a_{n+1}}\right] - d_{n+1}\,. \quad \text{Then}\ \begin{cases}\text{if } \lim t_n > 0, & \sum a_n \text{ converges;}\\ \text{if } \lim t_n < 0, & \sum a_n \text{ diverges.}\end{cases} \tag{11}$$

If $\lim t_n = 0$, there is no conclusion and another choice of $\{d_n\}$ is required. The selection of the sequence $\{d_n\}$ depends on the series being tested.

This test can be used to show that the series $\sum \rho_n$ converges. From (9), the ratio of successive terms is $\rho_n/\rho_{n+1} = p_{n+1}/(p_n - 1)$. In his paper on infinite series, Euler [3] showed that the series $\sum 1/p_n$ diverges. Choosing $d_n = p_n$, we have

$$t_n = p_n\left[\frac{p_{n+1}}{p_n - 1}\right] - p_{n+1} = \left[\frac{p_{n+1}}{p_n - 1}\right] > 1\,,$$

which fulfils the convergence criterion $\lim t_n > 0$, so the series converges. We will show below that the sum to infinity is 1, but the convergence rate is quite slow. Writing $\sigma_N = \sum_{k=1}^{N} \rho_k$ we have $\sigma_{10} = 0.842$, $\sigma_{1,000} = 0.938$, and $\sigma_{100,000} = 0.960$.

Partitioning the Natural Numbers. Defining $E_0 = \{1\}$, we obtain a partition of the natural numbers $\mathbb{N}$:

$$\mathbb{N} = \biguplus_{n=0}^{\infty} E_n\,, \tag{12}$$

where the sets E_n may be listed explicitly:

$$\begin{array}{rcl}
E_0 & = & \langle 1\rangle \\
E_1 & = & \langle 2,4,6,8,10,12,\dots\rangle \\
E_2 & = & \langle 3,9,15,21,27,\dots\rangle \\
E_3 & = & \langle 5,25,35,55,65,85,\dots\rangle \\
 & & \dots\dots\dots\dots\dots\dots\dots\dots \\
E_K & = & \langle p_K, p_K^2, p_K p_{K+1},\dots\rangle \\
 & & \dots\dots\dots\dots\dots\dots\dots\dots
\end{array}$$

The disjoint union in (12) contains all the positive integers, each occurring just once, providing a partition of $\mathbb{N}$.

Totient Function Expression for ρ_k. Euler's totient function $\varphi(n)$ counts the natural numbers up to n that are coprime to n. In other words, $\varphi(n)$ is the number of integers k in the range $1 \le k \le n$ for which the greatest common divisor $\gcd(k,n)$ is equal to 1. Clearly, for prime numbers, $\varphi(p) = p - 1$. Gauss first proved that

$$\sum_{d|N} \varphi(d) = N$$

[4, Th. 63]. This states that the sum of the numbers $\varphi(d)$, extended over all the divisors d of any number N, is equal to N itself.

The number of values x coprime to $\prod_{j=1}^{k} m_j$ is, by definition, given by $\varphi(m_1 m_2 \dots m_k)$. But Euler's function is multiplicative for products of coprime numbers $\{m_1, m_2, \dots, m_k\}$:

$$\varphi(m_1 m_2 \dots m_k) = \prod_{i=1}^{k} \varphi(m_j)\,.$$

Thus, for $M = P_K$, we have

$$\varphi(P_K) = \varphi\left(\prod_{j=1}^{K} p_j\right) = \prod_{j=1}^{K} \varphi(p_j) = \prod_{j=1}^{K} (p_j - 1) = P_K \prod_{j=1}^{K}\left(1 - \frac{1}{p_j}\right)\,.$$

Now, using (8), we can write the density in terms of the totient function:

$$\rho_k = \frac{1}{P_k}\prod_{j=1}^{k-1}(p_j - 1) = \frac{1}{P_k}\prod_{j=1}^{k-1}\varphi(p_j) = \frac{\varphi(P_{k-1})}{P_k}\,. \tag{13}$$

For example, for $K = 4$ we have $p_K = 7$, $P_K = 210$ and $P_{K-1} = 30$ and, counting explicitly, $\varphi(30) = |\{1, 7, 11, 13, 17, 19, 23, 29\}| = 8$. Thus,

$$\rho_4 = \frac{\varphi(P_3)}{P_4} = \frac{8}{210} = \frac{4}{105}\,,$$

as already shown in Table 2.

An Interesting Result. Defining the cumulative density $\sigma_k = \sum_{j=1}^{k} \rho_k$ and noting that, as the sets E_k are mutually disjoint, σ_k must approach 1, we obtain the relationship

$$\sum_{k=1}^{\infty}\left[\frac{\varphi(P_{k-1})}{P_k}\right] = 1\,, \tag{14}$$

where we define $P_0 = 1$.

This result must be well known, although it has not been found in a cursory search of the literature. Its originality and significance will be the subject of a future study.

References

[1] Bajnok, Béla, 2013: *An Invitation to Abstract Mathematics.* Springer, 406pp. ISBN: 978-1-4899-9560-5.

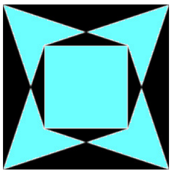
[2] Bromwich T. J. I'A, 1926: *Introduction to the Theory of Infinite Series*, 2nd Edn. (1926), Macmillan, London, 542pp. 1st Edn., 1908.

[3] Euler, Leonhard, 1737: *Variæ observationes circa series infinitas* (Various Observations about Infinite Series). Royal Imperial Academy, St Petersburg.

[4] Hardy, G. H. and E. M. Wright, 1960: *An Introduction to the Theory of Numbers*, 4th Edn., Oxford, Clarendon Press., xvi+418pp.

Peter Lynch is emeritus professor at UCD. His interests include all areas of mathematics and its history. He writes an occasional mathematics column in *The Irish Times* and has published three books of articles, entitled *That's Maths*, Vol. I (Gill Books), Vols. II and III (Logic Press). His blog is at `http://thatsmaths.com`.

(Author) School of Mathematics & Statistics, University College Dublin
E-mail address: `Peter.Lynch@ucd.ie`



Irish Math. Soc. Bulletin
Number 92, Winter 2023, 55–56
ISSN 0791-5578

Some Easy Inequalities for a Triangle

DES MACHALE

Abstract. We present some results about inequalities in a triangle which we have not been able to find elsewhere.

Even if a mathematical result is easy to prove, it still deserves to be recorded, if it has not appeared previously in the literature. In this note we present some pretty results about inequalities in a triangle which we have not been able to find elsewhere. Our notation is standard — ABC is a triangle with side-lengths a, b and c, with $2s = a+b+c$; R is its circumradius, r is its inradius and Δ is its area. We note that

$$16\Delta^2 = (a+b+c)(a+b-c)(b+c-a)(c+a-b),$$

an adaptation of Heron's formula. We need the following well-known preliminary results (see [1], for example).

Lemma 1.

$$4R = \frac{abc}{\Delta} \text{ and } r = \frac{\Delta}{s}.$$

Lemma 2 (Euler 1767)**.** *$R \geq 2r$, with equality if and only if the triangle is equilateral.*

Theorem 1.

$$R \geq \sqrt{\frac{abc}{a+b+c}} \geq 2r$$

Proof. By Lemma 1, $4Rrs = (abc/\Delta)(\Delta/s)s = abc$, so $2Rr = (abc)/(a+b+c)$.

By Lemma 2, this becomes $(abc)/(a+b+c) \geq 4r^2$, so $\sqrt{(abc)/(a+b+c)} \geq 2r$, as claimed.

Similarly, $2Rr \leq R^2$ and so $R^2 \geq (abc)/(a+b+c)$ and $R \geq \sqrt{(abc)/(a+b+c)}$.

Thus $R \geq \sqrt{(abc)/(a+b+c)} \geq 2r$, with equality if and only if the triangle is equilateral. □

Theorem 2. $(abc)(a+b+c) \geq 16\Delta^2$.

Proof. $R \geq 2r$ becomes

$$\frac{abc}{4\Delta} \geq \frac{2\Delta}{s} = \frac{4\Delta}{(a+b+c)}.$$

Thus $(abc)(a+b+c) \geq 16\Delta^2$. Again, by Lemma 2, we have equality if and only if the triangle is equilateral. □

Theorem 3. *$abc \geq (a+b-c)(b+c-a)(c+a-b)$.*

2020 *Mathematics Subject Classification.* 51-01.
Key words and phrases. triangle, inequalities.
Received on 19-6-2023.
DOI:10.33232/BIMS.0092.55.56.

Proof. By Theorem 2, $(abc)(a+b+c) \geq 16\Delta^2 = (a+b+c)(a+b-c)(b+c-a)(c+a-b)$. Since $a+b+c$ is non-zero, we may cancel it to get $(abc) \geq (a+b-c)(b+c-a)(c+a-b)$. □

Theorem 4. $(a+b+c)^3 \geq 27(a+b-c)(b+c-a)(c+a-b)$.

Proof. By the arithmetic mean/geometric mean inequality, we have $(a+b+c)^3 \geq 27abc$ which, by Theorem 3, is at least $27(a+b-c)(b+c-a)(c+a-b)$, and the result follows. Clearly, we have equality if and only if $a = b = c$. □

Topic For Investigation: What is the range of values of $(abc)/(a+b+c)$ if a, b and c are *positive integers* satisfying all three of the triangle inequalities?

References

[1] H.S. Hall and S.R. Knight. Elementary Trigonometry. Macmillan, London, 1955.

Des MacHale is Emeritus Professor of Mathematics at University College Cork where he taught for forty years. His main interests are in finite groups and rings, but he also dabbles in Number Theory, Euclidean Geometry, Trigonometric Inequalities, Combinatorial Geometry and Problem Posing and Solving. He has written several biographical books on George Boole but some would say his magnum opus is *Comic Sections Plus, the Book of Mathematical Jokes, Humour, Wit and Wisdom*, cf. `https://www.logicpress.ie/authors/machale`.

School of Mathematics, Applied Mathematics and Statistics, University College Cork, Cork.

E-mail address: d.machale@ucc.ie

Irish Math. Soc. Bulletin
Number 92, Winter 2023, 57–58
ISSN 0791-5578

The Beauty of Simultaneous Equations

M.R.F.SMYTH

Abstract. The old chestnut of "Two Trains and a Fly" is well known, but what happens in windy conditions? Simultaneous equations provide an attractive solution.

Two trains 90 miles apart are travelling towards each other along the same track. The first train goes 50 miles per hour; the second train trundles at 40 miles per hour. A fly hovers just above the front of the first train. It buzzes from the first train to the second train, turns around immediately, flies back to the first train, and turns around again. It goes on flying back and forth between the two trains until they collide. If the fly's speed is 60 miles per hour how far will it travel?

Nowadays it is generally accepted that there are two ways to solve this problem. The **pedestrian** method is to sum an infinite geometric series, but there is also a **smart** way which simply observes that the trains will collide after one hour and during that time the fly will have flown 60 miles which must therefore be the answer.

The story goes that when this question was addressed to von Neumann he thought for a couple of seconds before answering "60 miles". The questioner complimented him, "Well done. Most people try to sum the infinite series." to which he famously replied, "What do you mean? That's how I did it!"

The purpose of this note is to point out an apparent oversight. There is a third method of solution which is superior to both those above. Indeed it is one of the best-disguised applications of simultaneous equations that I have ever come across.

Consider a practical generalization by introducing a gentle breeze. How far does the fly travel if there is a constant wind blowing at 2 mph from the first train towards the second one?

A naïve attempt to answer this might place an observer on a parallel track travelling at the same velocity as the wind, namely 2 mph. The fly is always travelling at 60 mph relative to the observer and the two trains still meet after one hour. After that time the fly will have flown 60 miles in the observer's frame of reference and the observer himself will have moved 2 miles, so the "absolute" total distance travelled by the fly seems to be 62 miles.

Unfortunately this "solution" can be easily debunked by spotting that the fly is a 2-speed object which manages 62 mph downwind but only 58 mph in the other direction. In order to cover 62 miles within the hour it would have to maintain the higher speed throughout, which it clearly doesn't do. So this idea is flawed and we must try again.

A much better approach is to use simultaneous equations. Let x and y denote the time in hours the fly spends at 62 mph and 58 mph respectively.

$$\begin{array}{lll} \text{Then} & x + y = 1 & \text{(being the length of time until the trains meet)} \\ \text{and} & 62x - 58y = 50 & \text{(the distance between start point and collision point).} \end{array}$$

2020 *Mathematics Subject Classification.* 15Axx.
Key words and phrases. Simultaneous eqautions, linear algebra.
Received on 30-10-2023.
DOI:10.33232/BIMS.0092.57.58.

Straightforward computation yields $x = 0{\cdot}9$, $y = 0{\cdot}1$ and since the total distance flown during the hour in question is simply $62x + 58y$ the right answer is $61{\cdot}6$ miles. Anyone who doubts this may confirm it by following von Neumann's thought process and summing the infinite geometric series. Assume the trains are 459 units apart at the start of an iteration. After its out and back trip the fly will have covered 308 units and the trains will be 9 units apart. So each iteration is $\frac{9}{459} = \frac{1}{51}$ of the previous one, and the distance we want is

$$\frac{308 \times 90}{459}\left[1 + \left(\frac{1}{51}\right) + \left(\frac{1}{51}\right)^2 + \left(\frac{1}{51}\right)^3 + \dots\right] = \frac{3080 \times 51}{51 \times 50} = 61\tfrac{3}{5} \text{ miles}.$$

In fact if the wind speed is w miles per hour then $x = (110-w)/120$, $y = (10+w)/120$ and it is easily calculated that the total distance travelled by the fly is $60+w(50-w)/60$ miles. Naturally w must lie somewhere in the range -10 to 20 or else the fly will be unable to keep pace with one of the trains and so will happily miss the collision.

Finally return to the original question, in other words the special case when $w = 0$. The simultaneous equations become $x + y = 1$ and $60x - 60y = 50$ and the total distance flown is $60x + 60y$. For the purpose of deducing the latter the second equation is redundant. From the first equation alone we reach the smart conclusion that the fly travels exactly 60 miles before being squashed.

I have searched the World-Wide-Web in the expectation of finding this more general and (in my view) more satisfactory approach to Two Trains and a Fly. However I've found very few references to possible wind effects, and none that uses simultaneous equations to address the matter.

Roger Smyth studied at Cambridge University from 1965 to 1969 and then became a research student of Trevor West at Trinity College Dublin. He received his Ph.D. in 1972. Thereafter he moved into Information Technology and worked in Queen's University Belfast and the Northern Ireland Department of Health. His original mathematical interest was Fredholm theory in Banach algebras, but he maintains a recreational interest in simple Euclidean geometry.

(Roger Smyth) 15 Harberton Avenue, Belfast BT9 6PH
E-mail address: malcolm.smyth@hotmail.com

Irish Math. Soc. Bulletin
Number 92, Winter 2023, 59–61
ISSN 0791-5578

Desmond MacHale: The Poetry of George Boole, Logic Press, 2020. ISBN:978-1-71652-027-3, Euro 15, 138+vii pp.

REVIEWED BY TOM CARROLL

To be clear from the outset, this is not a 'poetry collection' in the usual sense of, say, a collection of Seamus Heaney's poems. George Boole was, in MacHale's words, 'at best a recreational poet' (p. 28) and made no concerted effort to publish his poetry. Seventy or so poems have been painstakingly retrieved by MacHale from various of Boole's copybooks and sundry sources, beginning with translations of poems from classical Greek and Latin and the modern languages Italian, French and German. These translations date, for the most part, to when Boole was a teenager. Later chapters treat Boole's own poetry, organised under the headings of 'Sonnets' (Chapter 4), 'Family and Friends' (Chapter 5), 'Religious Verse' (Chapter 6) and 'Miscellaneous Poems' (Chapter 7). MacHale situates each set of poems in the context of Boole's life, both personal and professional, as Boole progressed from the self-taught student in Lincoln to the first Professor of Mathematics at Queen's College Cork, now UCC.

As is MacHale's intention in writing this book, Boole's poems provide a valuable insight into what was important to him and into what motivated him in his life and his work. This is possibly most apparent in the twenty or so surviving sonnets that MacHale collects and comments on in Chapter 4. Nature (with a capital 'N') and its restorative powers feature strongly in his work, (Sonnet 10, p. 41, for example):

> Yes! Though art med'cine to the weary mind,
> Dear Nature! They whom busy Life's affairs
> O'erburden - whom the wasting city wears,
> To Thee return and consolation find.

As a deeply religious man, Boole was keenly conscious of his responsibilities to the Almighty and of the duty and service that He is rightly owed (Sonnet 18, p. 48):

> Labour on Earth and rest in Heaven, fulfil
> Thy destiny below, each wish resign
> That leads thee adverse to the Master's will,
> Or casts oblivion on His work and thine;
> One thought the murmur of the breast thrusts still
> To be the co-worker in His high design.

Perhaps again steered by his religious beliefs, Boole felt morally obliged to make maximum use of his God-given intellectual talent (Sonnet 2, p. 35):

> Oh, what reproach is thine if thou remain
> Inactive now the golden hours invite,
> Sole loiterer in a world o'er which the bright
> Perpetual stars their ordered courses sustain.

Again, in a poem *Life in Earnest* dedicated to his sister Mary Ann Boole (p. 55) he writes:

Received on 27-09-2023; revised 31-10-2023.
DOI:10.33232/BIMS.0092.59.61.

Oh, leave thy desert lone recluse,
For all thy gifts the world hath use
Calm though thy hours and sweet thy prayers
Yet far the nobler aim is theirs
Who live the earnest life.

The 'earnest life' here is that of 'Faith, Hope and Love/Charity' or, again, 'Justice, Truth and God Himself'.

Even from these short extracts we glimpse the main forces acting on Boole and directing his life: a sense of duty to his family, his friends, his work, a sense of duty built on a strong religious faith. This is clearly evident in the poems gathered under 'Family and Friends', and especially 'Religious Verse'. The latter include meditations on episodes in the Bible with titles such as *Consider the Lilies of the Field* and *Paraphrase of the 137th Psalm*, a hymn *The Communion of Saints* (which was sung at the memorial service in Blackrock, Cork, to mark the 150$^{\text{th}}$ anniversary of his death on 8 December 2014), *I thank thee, O thou source of every good* and *Virtue*, to give some examples. MacHale positions each of Boole's poems in the arc of Boole's life and his experiences, a subject on which he is the undisputed expert. His goal of shedding light on 'Boole the Man' through the medium of his poetry is exceptionally well-achieved.

Boole's poetic style is also clear from these short extracts: the use of the archaic forms 'thee' and 'thou', old-fashioned poetic wordage such as 'o'er' instead of 'over'. The quotation from Sonnet 10 above, for example, continues with 'And hereunto wast thou by Heav'n designed:'. Some of Boole's poems were included by his wife Mary Everest[1] Boole in her 'Collected Works', even if she forbade Boole from ever again writing poetry after discovering, soon after their marriage, that he did so. This she did, ostensibly, so that Boole's energies would not be diverted from his more important scientific work. Some of his translations from the Greek were published in a local Lincoln newspaper, resulting in some controversy in that paper's letters page. Since Boole himself seems never to have claimed or sought recognition of his poetry from the literary establishment, it may therefore be somewhat unfair to judge his poetry from a literary perspective. Nevertheless, it is unavoidable that such an evaluation finds a place in MacHale's book. An appraisal of the young Boole's translation of Meleager's *Ode to the Spring* is given by Dr Patrick Cronin, Department of Ancient Classics, UCC, in Chapter 1. A brief, critical evaluation of a selection of Boole's poetry, dated to the early 1980's and written by Seán Lucey, formerly Professor of Modern English at UCC, is included in 'Critical Evaluation' (Chapter 9). This makes for entertaining reading: it begins 'Boole was a very able versifier who sometimes rose to poetry.' Lucey identifies Wordsworth, 'particularly - alas - the later Wordworth whose moralistic sonnets follow each other with thunderous monotony', and 'the massive loom of Milton and the ordered voices of the eighteenth-century philosophic poems' as influences, as well as the 'second-generation of Romantics'. Nevertheless 'Here and there a real feeling strikes through. Here and there in spite of imitation a fine image pleases the mind and transcends generalisation.' Boole's poetry is not entirely without literary merit.

As well as commentary on Boole's poetry and life, MacHale includes some personal essays on the Arts and Sciences and on beauty in mathematics. As MacHale describes in the 'Critical Evaluation' chapter, this was prompted by a criticism, or perhaps a reservation, by a 'dear friend' from a literary background but with little understanding of or interest in mathematics. There is an implication here, and in Chapter 1's discussion on the Arts and Sciences, that the Arts look askance on the Sciences and fail to appreciate the inherent beauty of good mathematics: 'The notion that mathematics could be in any sense beautiful would be met with incredulity and even scorn by the

[1]George Everest, after whom Mount Everest is named, was her paternal uncle.

average poet. The artistic world, especially poets and artists, feels it has a monopoly on beauty, and would feel threatened by any such claims by outsiders' (p. 3). It is unclear how widespread this attitude may be. In my own experience, poets and artists display, at worst, a benign commiseration for my lot as a mathematician. I doubt that they would feel threatened by any artistic claims of beauty I might make on behalf of mathematics.

It would be remiss of me not to mention Sonnets 15 and 17 written in mid-November 1849 shortly after Boole moved to Cork. The aftermath of the famine was evident on the streets of Cork. Boole addresses Ireland in Sonnet 15:

> . . . though want be bold
> And clamours in thy streets, and where the gold
> Of plenteous harvests waved, lie plashy plains,
> O'er which the bulrush towers, the ragweed reigns,
> Yet thou in wisdom still art young, though old
> In misery and tears. . . .

These two sonnets on the subject of Ireland display, as MacHale recognises, a political and historical naivety that is difficult to ignore. Boole wishes that Ireland would forget its 'store of bitter thoughts, which brood upon the past'. If only that could be achieved, 'the brightness of thy coming Morn' would emerge. Sonnet 17 follows a similar vein. It concludes with:

> In noble breasts past injuries are tame;
> Courage distains them. Patience makes them light,
> The wise forget them, and the good forgive.
> O ye who seek the Patriot's holy aim!
> Teach Ireland this - the self-sustaining might
> of Duty teach - instruct her how to live.

We should not, perhaps, be overly critical of Boole on account of these private musings. We are not to know how Boole's views on Ireland, its past and its future, evolved over the ensuing years in Cork. Moreover, Boole's opinions would presumably have been widely shared at the time. A further consideration is whether one should take for granted that a brilliant mathematician will necessarily have equally keen insights into the political or the human sphere.

At worst, Boole's appeal to the Irish to show courage, patience, wisdom and goodness in forgetting and forgiving 'past injuries' displays an ingenuousness on his part when it comes to the politics of the time. In contrast, taken as a whole, his poetry is evidence of a man who is faithful, rigorous, religious and dutiful in all aspects of his personal and professional life. In any case, and this is key, he should be judged on his mathematics above all else, in which case the poems herein and the accompanying commentary by MacHale provide novel insight into the working mindset of the mathematician George Boole.

Tom Carroll is Senior Lecturer in Mathematics at UCC, and works for the most part in one variable complex analysis. As an undergraduate student at UCC, he had the pleasure of having Des MacHale as a lecturer. He studied under Walter Hayman and Phil Rippon in the UK, then spent two years at Purdue University in the US before gratefully returning to Cork. He was President of the Irish Mathematical Society 2021-2022.

School of Mathematical Sciences, University College Cork
E-mail address: t.carroll@ucc.ie

Irish Math. Soc. Bulletin
Number 92, Winter 2023, 62–64
ISSN 0791-5578

Ron McCartney: A Gentle Path into General Topology, Bookboon, 2022. ISBN:978-87-403-4370-0, GBP 00.00, 256 pp.

REVIEWED BY AISLING MCCLUSKEY

This book, e-published and currently freely available by Bookboon, comes in three parts encompassing a total of eight chapters of introductory material. Its content emerged from an undergraduate course developed in Asia by the author over an eight-year period. The author indicates a supplementary *What you need to know for Chapter n* (WYNTK n) feature that is available on the Bookboon website. He provides this as a useful gauge for the reader to assess their level of preparedness for a course based on the book. Forewarned is forearmed!

The book is pitched at the level of undergraduates who are meeting general topology for the first time, typically within the last two years of their degree programme. It is written in a style that recognises and embraces the step-up required in handling abstraction. The book is aptly named in that regard as the author scaffolds the learning in a gentle, well-paced manner borne of long experience in careful, thoughtful teaching.

Each of the book's three parts is prefaced by a call for the reader to reinforce the basic skills needed to fare well in the course – these are of course the usual suspects such as basic set theory notation including functions and their inverses, limits in $\mathbb{R}$, manipulation of absolute value inequalities, intervals in $\mathbb{R}$, and knowledge of proof methods. The cognitive and pedagogical piece in terms of advising how the students should undertake the 'gentle path' ahead is also comprehensively captured and repeated thrice over in each part's Introduction, with mild customisation towards the book part it heralds. As mathematicians and educators, this advice is ingrained in us – but we cannot say it often enough to our students. It is useful to have this core advice integrated and reinforced throughout. A further pedagogical aide is the peppering of reflections throughout the book under the guise of *Think about . . . .*

Part 1 introduces metric spaces (Chapter 1) and their open sets (Chapter 2). Chapter 1 is chock-full of examples, non-examples and exercises which serve to strengthen an understanding of the cornerstone definition of a metric space. One or two of the examples are a little too informal or too vague for my taste (for example, the set S in Example 1.3 is introduced as 'quite a large set of people'; Example 1.6 refers to the imagery of mangoes in the interpretation of adding a new point to the set $\mathbb{N}$!) but the slow, worked-out solutions provided will be invaluable to the conscientious learner. Interestingly the taxicab metric (or Manhattan metric) is called the postal metric in Exercise 1.3 – this may be a contextual decision. Chapter 2 then introduces the concept of an open ball leading to an open set. Enroute the term *neighbourhood* is used without prior definition – so there is a sequencing issue here. (In fact, neighbourhood in a topological space is defined later in Chapter 3.) An open ball of radius r is referred to as an r-open ball which is not a turn of phrase I have come across before. There may be some potential here for confusion given how open sets are subsequently introduced

Received on 09-06-2023; revised 04-07-2023.
DOI:10.33232/BIMS.0092.62.64.

as 'd-open sets' where d is the given metric. Part 1 ends appropriately by setting out the properties of open sets that signpost the imminent arrival of a topological space.

Part 2 right on cue proceeds with an introduction to a topology (Chapter 3) and to continuity (Chapter 4). It wends its way through a variety of standard 'training' topologies and the occasional non-topology, presenting a strategy for students to check when a collection of sets forms a topology. Interestingly the author includes a section (3.2) on the ordering by inclusion of topologies on a given set. Section 3.3 raises the important and classic question concerning whether a given topology arises from a metric. This is a nice prelude to the introduction of the Hausdorff property. Further sections are devoted to the concepts of basis for a topology, subspaces, closed sets and closures of sets. Chapter 3 concludes with an investigative nod towards the equivalent way of defining a topology via neighbourhoods and of a characterisation of closed sets via sequential convergence. Chapter 4 takes on the concept of continuity, building up from its meaning for real-valued functions on $\mathbb{R}$ through metric spaces to topological spaces and paying close attention to detail as it does so. It is tried out on various functions and the role of a basis is usefully brought into play in the process. Section 4.3 on homeomorphisms is particularly important in this chapter. Section 4.5 on product spaces is also important but it is let down somewhat by some typesetting errors (for example, pages 91 and 94). While the author does indicate that Section 4.7 on quotient spaces borrows heavily from Jänich's *Topology* (Springer), the key definition of a quotient map on page 96 is unclear (a_k is not defined). Part 2 is quite substantial, running to 101 pages, but there are standalone sections that can be omitted on first reading/study under the guidance of an instructor.

Finally, Part 3 clips along at a steady pace with four chapters on compactness, connectedness, completeness and separation axioms respectively. The treatment of compactness is nice, involving its interaction with Hausdorffness, its behaviour under continuity and on taking subspaces and referencing other authors for investigation of more advanced topics such as compactifications. Connectedness follows a similar approach with some nice discussion on components and a lovely selection of examples to illustrate non-homoeomorphic spaces. Completeness of necessity requires some preliminary work on sequences and subsequences but quickly ramps up to consider the property alongside compactness. A reasonably extensive investigation section on topics for further study is also provided. The final section on separation axioms then comes somewhat as light relief but it nonetheless does get to a discussion of normality and its waywardness (by comparison with the weaker axioms). The cohesion of compact Hausdorff in this section is pleasing. Appropriately the investigative add-on for this section covers issues of metrizability and products and subspaces of normal spaces.

As a more general comment, the book is not prepared in LateX and so the mathematical notation and general formatting is non-standard which can detract from the overall presentation. For example, $\mathbb{R}^n$ is rendered as R^N in a capitalised title. In Part 1, the title in 2.3 featuring 'r-open balls in $\mathbb{R}$' is rendered "R-open balls in R". The inevitable typos, the bane of every author, also carry an impact (for example in Part 3, sequence notation depicted as xn on page 50, two different fonts for ϵ on page 56); use of 'equivalent' in Section 7.5.2 (page 65) where 'compatible' might be a better choice given the significance of the word in an earlier discussion on equivalent metrics. Other idiosyncracies occur such as the reference to a circle when the intention is the entire disk (so circle and its interior) in the plane. Professionals can readily pass over these, understanding what is meant - but they can trip up the beginner.

Fundamentally the book constitutes a thoughtful and student-centred companion to similar undergraduate texts in the market and will be a useful resource for student and instructor alike.

Aisling McCluskey was educated at Queen's University Belfast and is a Personal Professor in Mathematics at the University of Galway. She is currently a governor at the University and Head of the School of Mathematical and Statistical Sciences there. Her research interests lie within analytic topology fuelled by a lifetime interest in and commitment to undergraduate mathematics education.

School of Mathematical and Statistical Sciences, University of Galway
E-mail address: aisling.mccluskey@universityofgalway.ie

Irish Math. Soc. Bulletin
Number 92, Winter 2023, 65–67
ISSN 0791-5578

Y.A. Gonczarowski and N. Nisan: Mathematical Logic through Python, Cambridge University Press, 2022. ISBN:978-110-8949-47-7, GBP 22.99, 284 pp.

REVIEWED BY TIM SWIFT

Mathematical logic is one of the most significant sub-areas of mathematics for computer science. For many working mathematicians, the subject is perhaps less important, even though, of course, an important everyday task in mathematics is to show the truth of a proposition by correctly deriving it from appropriate axioms using the rules of logic. Thus, a course on mathematical logic is a core component of most undergraduate computer science programmes, but only sometimes appears in undergraduate mathematics programmes, and then usually as an option module that might also include other foundational aspects.

The book under review, which has been written mainly for computer science students, but also contains much of interest for mathematics students, is an impressive contribution to a literature that already includes many excellent texts on mathematical logic. Although written from a modern viewpoint, it might be said that the book also tips a hat at some of the twentieth-century roots of the subject, in particular the work of Alonzo Church and Alan Turing, so important for the development of computer science. Via an imaginative pedagogical approach based on the Python programming language, the book covers a one-semester course on propositional and first-order predicate logic, up to and including Gödel's Completeness Theorem. The underlying idea of the book is that the student develops their knowledge and understanding by writing a carefully structured sequence of working programs in Python that implement both logical concepts and mathematical proofs. The authors' intention is that such an approach should appeal to computer science students, equipped with their existing programming expertise, more than would a traditional mathematical development of the subject via the proving of theorems. The authors carefully explain their pedagogical approach and why they have developed this method of teaching logic, which is an outcome of their experience of delivering the subject to undergraduate computer scientists over several years.

The book covers the standard topics in propositional logic, namely syntax, semantics, proof, the Tautology Theorem, and completeness, and then follows a similar path for predicate logic. The final chapter provides a preview of what a second course along the same lines would cover, and, in particular, an introduction to Gödel's Incompleteness Theorem is provided. The authors estimate that the 150+ programming exercises cover about 95% of the mathematical content of a standard first course in mathematical logic. The remaining content, mainly results that involve infinite sets, is treated in the traditional mathematical fashion. Given that, according to several indices, Python is currently the world's most popular and fastest-growing programming language, the use in the book of this open-source language is entirely appropriate and should help to ensure widespread appeal. Although the text is aimed primarily at computer science students, it could also be used as a text for mathematics undergraduates, perhaps used

Received on 18-10-2023; revised 01-11-2023.
DOI:10.33232/BIMS.0092.65.67.

in parallel with a more traditional book on mathematical logic. Another use of the book might be as a resource for a final-year undergraduate mathematics project that combines developing an account of mathematical logic with associated investigations based around Python programming.

Beyond the innovative development of the material, there are several other features that should make the book useful from the learning and teaching perspectives. These include: an introductory overview chapter, which contains an explanation of how to use the book, in particular the Python exercises; a useful roadmap; footnotes that make connections with other areas of mathematics; optional reading sections at end of each chapter; the summary list of axioms and inference rules at the end of the book; a comprehensive index; extra online resources including skeleton code and unit tests for all of the programming exercises. In addition, it is good to see the mathematical examples that are used to illustrate proofs in predicate logic.

This reviewer, as a mathematician who is far from being an expert in Python programming, worked through some - but not all! - of the exercises, which was very instructive; indeed, because more Python skills had to be acquired, my engaging with the book certainly had an additional flavour of 'Python through mathematical logic'. I feel that the pre-knowledge of Python required could be better described: the authors state only that '. . . basic proficiency in Python is assumed', and perhaps a brief list of particular skills could have been provided. On the other hand, the Python exercises are carefully selected, enabling the traditional mathematical development to be followed, and I found the authors' suggestion to consult the examples within the test code for a given task before attempting a particular exercise to be very good advice.

On the whole, the notation and terminology are standard, and, where differences occur, these are pointed out. A mathematics student using the text should take note of a few differences between mathematics and computer science terminology, e.g., operator, rather than operation. Also, it was slightly disconcerting to see the plus symbol used for the binary operation in a possibly non-abelian group. The method of numbering the environments, e.g., there are Definition 12.1, Lemma 12.1 and Theorem 12.1, is not to this reviewer's taste, and makes navigation a little more difficult.

In summary, I feel that this publication is a very useful addition to the existing library of mathematical logic textbooks. It should certainly be helpful, as intended, to computer science undergraduates, but also to mathematics students, and, indeed, to mathematics lecturers who wish to integrate Python coding in a meaningful and important way in their teaching.

Finally, it is worthwhile noting the appropriate choice of cover illustration, namely Wassily Kandinsky's wonderful painting 'Serious- Fun'. The authors conclude the Preface by remarking that they hope that the name of this picture will describe the reader's experience as they work through the book. It is interesting to note that Kandinsky made the painting in 1930, the start of a crucial decade for mathematical logic, during which workers such as Gödel, Tarski, Carnap, Church, and Turing made radical contributions to the development of the subject. For this reviewer, at least, 'Serious-Fun' might be a brief, but accurate, description of engaging with the mathematics game itself.

Tim Swift Tim Swift is a senior lecturer in mathematics at the University of the West of England, Bristol. Born in the West Riding of Yorkshire, he was educated at the Universities of Cambridge and Southampton. He has worked in general relativity theory and differential geometry, and currently has interests in graph theory, stochastic processes and mathematical

culture. As undergraduate programme leader, he is interested in curriculum development and in increasing diversity in mathematics.

Mathematics and Statistics, University of the West of England, Bristol
E-mail address: tim.swift@uwe.ac.uk

Irish Math. Soc. Bulletin
Number 92, Winter 2023, 68–71
ISSN 0791-5578

PROBLEMS

IAN SHORT

Problems

The first problem this issue was suggested by Des MacHale of University College Cork.

Problem 92.1. Show that the infinite cyclic group is not the full automorphism group of any group.

Des comments that readers may wish to consider the more diffficult (and only partially solved) problem of determining when the finite cyclic group of order n is the full automorphism group of (a) a finite group and (b) an infinite group.

The second problem is from Andrei Zabolotskii of the Open University.

Problem 92.2. Let A be a symmetric square matrix of even order over the ring of integers modulo 2. Suppose that all entries on the leading diagonal of A are 0. Let B be the square matrix obtained from A by replacing each 0 entry with 1 and replacing each 1 entry with 0. Prove that $\det A = \det B$.

Readers who solve Problem 92.2 might care to consider the more challenging question of whether or not the characteristic polynomials of A and B are equal.

The third problem was proposed by Finbarr Holland of University College Cork.

Problem 92.3. For $x > 0$, let $\mu(x)$ denote the ℓ_∞-norm of the sequence

$$u_n(x) = \frac{x^n}{n^n}, \quad n = 1, 2, \dots .$$

Determine

$$\lim_{x\to\infty} \frac{\log \mu(x)}{x}.$$

Solutions

Here are solutions to the problems from *Bulletin* Number 90.

The first problem was solved by Brian Bradie of Christopher Newport University, USA, Elshan Huseynov of ADA University, Azerbaijan, Seán Stewart of the King Abdullah University of Science and Technology, Saudi Arabia, and the North Kildare Mathematics Problem Club. All solutions gave the same correct formula; we provide commentary derived from Seán Stewart's contribution.

Problem 90.1. Let M be any 3-by-3 matrix

$$M = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

Received 4-1-2024
DOI:10.33232/BIMS.0092.68.71.

over a field, where $e \neq 0$, and let A, B, C, D be the four submatrices of M given by

$$A = \begin{pmatrix} a & b \\ d & e \end{pmatrix}, \quad B = \begin{pmatrix} b & c \\ e & f \end{pmatrix}, \quad C = \begin{pmatrix} d & e \\ g & h \end{pmatrix}, \quad D = \begin{pmatrix} e & f \\ h & i \end{pmatrix}.$$

Find an expression for $\det M$ in terms of $\det A$, $\det B$, $\det C$, $\det D$, and e.

Solution 90.1. A straightforward computation shows that

$$\det M = \frac{1}{e}(\det A \det D - \det B \det C). \qquad \square$$

This is a special case of a more general identity for n-by-n matrices known as the *Desnanot–Jacobi identity* or sometimes *Dodgson's identity* (see Matrix Analysis (2nd edition), by Horn and Johnson, Section 0.8.11). The method for computing determinants based on these identies is known as the *Dodgson condensation method*.

The second problem was solved by Brian Bradie, Seán Stewart, the North Kildare Mathematics Problem Club, and the proposer Finbarr Holland. Each solution was different; we provide that of Brian Bradie.

*Problem 90.2***.** Prove that

$$\sum_{n=0}^{\infty} a_n \sum_{k=0}^{n} \frac{a_k a_{n-k}}{(2k+1)(2(n-k)+1)} = \frac{2}{\pi}\int_0^{\pi/2} \frac{x^2}{\sin^2 x}\,dx = \log 4,$$

where

$$a_n = \frac{\Gamma(n+\frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(n+1)}, \quad n = 0, 1, 2, \dots .$$

Solution 90.2. Let $f(x) = \log(\sin x)$. Then $f'(x) = \cot x$ and $f''(x) = -\mathrm{cosec}^2 x$. Observe that

$$\int_0^{\pi/2} \frac{x^2}{\sin^2 x}\,dx = -\int_0^{\pi/2} x^2 f''(x)\,dx.$$

Integrating by parts twice gives

$$\int_0^{\pi/2} \frac{x^2}{\sin^2 x}\,dx = 2\big[xf(x)\big]_0^{\pi/2} - \big[x^2 f'(x)\big]_0^{\pi/2} - 2\int_0^{\pi/2} f(x)\,dx = -2\int_0^{\pi/2} f(x)\,dx.$$

Now,

$$\int_0^{\pi/2} \log(\sin x)\,dx = \int_0^{\pi/2} \log(\cos x)\,dx,$$

so

$$\int_0^{\pi/2} f(x)\,dx = \tfrac{1}{2}\int_0^{\pi/2} (\log(\sin x) + \log(\cos x))\,dx = -\tfrac{1}{4}\pi \log 2 + \tfrac{1}{2}\int_0^{\pi/2} \log(\sin 2x)\,dx.$$

After making the substitution $y = 2x$ and rearranging we obtain

$$\int_0^{\pi/2} f(x)\,dx = -\frac{\pi}{2}\log 2.$$

Hence

$$\frac{2}{\pi}\int_0^{\pi/2} \frac{x^2}{\sin^2 x}\,dx = \frac{2}{\pi} \times \pi \log 2 = \log 4.$$

Next, observe that

$$a_n = \frac{\Gamma(n+\frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(n+1)} = \frac{1}{2^{2n}}\binom{2n}{n}.$$

Using the generating function

$$\arcsin x = \sum_{n=0}^{\infty} \frac{a_n}{2n+1} x^{2n+1},$$

we see that the convolution

$$\sum_{k=0}^{n}\left(\frac{a_k}{2k+1}\right)\left(\frac{a_{n-k}}{2(n-k)+1}\right)$$

is the coefficent of x^{2n+2} in $(\arcsin x)^2$. But

$$(\arcsin x)^2=\frac{1}{2}\sum_{n=1}^{\infty}\frac{(2x)^{2n}}{n^2}\binom{2n}{n}^{-1},$$

so

$$\sum_{k=0}^{n}\left(\frac{a_k}{2k+1}\right)\left(\frac{a_{n-k}}{2(n-k)+1}\right)=\frac{1}{2}\frac{2^{2n+2}}{(n+1)^2}\binom{2n+2}{n+1}^{-1}$$

and

$$\begin{aligned}a_n\sum_{k=0}^{n}\frac{a_k a_{n-k}}{(2k+1)(2(n-k)+1)}&=\frac{1}{2^{2n}}\binom{2n}{n}\times\frac{1}{2}\frac{2^{2n+2}}{(n+1)^2}\binom{2n+2}{n+1}^{-1}\\&=\frac{2}{(2n+1)(2n+2)}\\&=2\left(\frac{1}{2n+1}-\frac{1}{2n+2}\right).\end{aligned}$$

Finally,

$$\begin{aligned}\sum_{n=0}^{\infty}a_n\sum_{k=0}^{n}\frac{a_k a_{n-k}}{(2k+1)(2(n-k)+1)}&=2\sum_{n=0}^{\infty}\left(\frac{1}{2n+1}-\frac{1}{2n+2}\right)\\&=2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}=\log 4.\end{aligned}$$ □

The third problem was solved by Brian Bradie, Finbarr Holland, Kee-Wai Lau of Hong Kong, China, the North Kildare Mathematics Problem Club, and the proposer, Seán Stewart. We present the solution of Finbarr Holland.

Problem 90.3. Evaluate

$$\sum_{m,n=0}^{\infty}\binom{2m}{m}^2\binom{2n}{n}\frac{1}{2^{4m+2n}(m+n+1)}.$$

Solution 90.3. Let

$$a_n=\binom{2n}{n}\frac{1}{2^{2n}},\quad n=0,1,\ldots.$$

It is easy to show by induction that

$$a_n=\frac{\Gamma(\frac{1}{2}+n)}{\Gamma(\frac{1}{2})\Gamma(n+1)},\quad n=0,1,\ldots.$$

Let $f(z)=1/\sqrt{1-z}$, for $|z|<1$. Then

$$f(z)=\sum_{n=0}^{\infty}a_n z^n.$$

With this notation, the sum S that we are to evaluate satisfies

$$\begin{aligned}
S &= \sum_{m,n=0}^{\infty} \frac{a_m^2 a_n}{m+n+1} \\
&= \int_0^1 \sum_{m,n=0}^{\infty} a_m^2 a_n t^{m+n}\, dt \\
&= \int_0^1 \left(\sum_{m=0}^{\infty} a_m^2 t^m\right)\left(\sum_{n=0}^{\infty} a_n t^n\right) dt \\
&= \sum_{m=0}^{\infty} a_m^2 \int_0^1 t^m f(t)\, dt.
\end{aligned}$$

Observe that

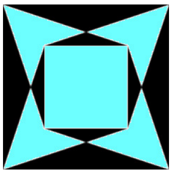
$$\int_0^1 t^m f(t)\, dt = \int_0^1 t^m (1-t)^{-1/2}\, dt = \frac{\Gamma(\frac{1}{2})\Gamma(m+1)}{\Gamma(m+\frac{3}{2})} = \frac{1}{a_m(m+\frac{1}{2})}.$$

Hence

$$S = \sum_{m=0}^{\infty} \frac{a_m}{m+\frac{1}{2}} = 2\int_0^1 \sum_{m=0}^{\infty} a_m t^{2m}\, dt = 2\int_0^1 \frac{1}{\sqrt{1-t^2}}\, dt = \pi. \qquad \square$$

We invite readers to submit problems and solutions. Please email submissions to `imsproblems@gmail.com` in any format (we prefer LaTeX). Submissions for the summer Bulletin should arrive before the end of April, and submissions for the winter Bulletin should arrive by October. The solution to a problem is published two issues after the issue in which the problem first appeared. Please include solutions to any problems you submit, if you have them.

School of Mathematics and Statistics, The Open University, Milton Keynes MK7 6AA, United Kingdom



Instructions to Authors

Papers should be submitted by email to the address:

`ims.bulletin@gmail.com`

In the first instance, authors may submit a pdf version of their paper. Other formats such as MS/Word or RTF are not acceptable. The *Bulletin* is typeset using PDF files produced from LaTeX source; therefore, authors must be ready to supply LaTeX source files (and any ancillary graphics or other files needed) should their paper be accepted. Source files should be suitable for processing using `pdflatex`.

Once a paper is accepted in final form, the author(s) will be responsible for producing an output according to the *Bulletin's* standard layout. Standard template files for articles, abstracts and reviews, and the necessary class and style files may be downloaded from the IMS website `http://www.irishmathsoc.org`, or obtained from the editor in case of difficulty.

Since normally no proofs are sent out before publication, it is the author's responsibility to check carefully for any misprints or other errors.

The submission of a paper carries with it the author's assurance that the text has not been copyrighted or published elsewhere (except in the form of an abstract or as part of a published lecture or thesis); that it is not under consideration for publication elsewhere; that its submission has been approved by all coauthors and that, should it be accepted by the *Bulletin*, it will not be published in another journal. After publication, copyright in the article belongs to the IMS. The IMS will make the pdf file of the article freely available online. The Society grants authors free use of this pdf file; hence they may post it on personal websites or electronic archives. They may reuse the content in other publications, provided they follow academic codes of best practice as these are commonly understood, and provided they explicitly acknowledge that this is being done with the permission of the IMS.