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Threshold Concepts as an Anchor in Undergraduate Mathematics Teaching

SINÉAD BREEN AND ANN O'SHEA

Abstract. In this short article, we consider the notion of Threshold Concepts and give some examples in mathematics. We will also explore the role of Threshold Concepts as potentially powerful transformative points in a student's learning of mathematics at University level and discuss the theoretical and practical approaches proposed by various educators to address such concepts at both modular and programme level.

1. Introduction

About 20 years ago, two economists (Erik Meyer and Ray Land) introduced the notion of a *Threshold Concept* while working on a project which aimed to enhance the teaching and learning in undergraduate courses. In a series of articles, [11], [12], [10], they defined the characteristics of such a concept and gave examples from various subjects including mathematics. Meyer and Land's core idea is that many academic disciplines have concepts that act as conceptual gateways or portals, and that while developing an understanding of these concepts students are led to engage in previously inaccessible ways of thinking [11]. These portals are places where students often 'get stuck', but when these concepts are fully mastered students are able to behave more like experts in the field [7], and crucially see the subject in a new way. Meyer and Land [11] gave the example of opportunity cost in economics to illustrate the idea. However one of their other examples, namely the ϵ - δ definition of the limit of a function, may resonate more with mathematicians. We will consider some examples of Threshold Concepts in the undergraduate mathematics syllabus shortly, but first let us define what we mean by this notion.

2. Characteristics of Threshold Concepts

Meyer and Land, [11], defined threshold concepts in terms of five characteristics. These characteristics are: transformative, irreversible, integrative, troublesome and bounded. The first of these characteristics has been alluded to above. A concept has this *transformative* character if an understanding of the concept not only changes the student's comprehension of the topic but their view of the subject. (You might stop to consider the impact of understanding the definition of a limit had on your own view of real analysis.) This transformation should also be *irreversible*, that is the change in perspective cannot easily be forgotten and so is usually permanent. This means that it is sometimes difficult for experts in the field to put themselves in the position of the students in their classes. The *integrative* nature of threshold concepts means that they allow students to make previously unseen connections between parts of the subject or

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links between 'isolated islands of knowledge' [13]. It is usually the case that Threshold Concepts are *troublesome* for students, indeed we would not expect such dramatic changes otherwise. Often the knowledge and understanding needed to master these concepts seems alien or counter-intuitive to novices, and may require a suspension of disbelief. Lastly, Meyer and Land added the *bounded* characteristic to their definition of a threshold concept. This characteristic conveys the idea that the understanding and use of a concept is specific to the particular discipline and may even define the boundaries of the discipline. Consider how the formal definition of a limit is a demarcation between calculus and analysis. Note that the five characteristics have some overlaps. For example, there is a deep connection between transformative, integrative and irreversible [7].

While the idea of a threshold concept is relatively new (introduced in 2003), some of its essential features have been described before in other ways. For example, in a 1990 AMS article on mathematics education, Thurston, the renowned mathematician and Fields medallist spoke about the 'compressibility' property of mathematics that arises once a concept or topic is completely understood:

> *Mathematics is amazingly compressible: you may struggle a long time, step by step, to work through some process or idea from several approaches. But once you understand it and have the mental perspective to see it as a whole, there is often a tremendous mental compression. You can file it away, recall it, quickly and completely when you need it, and use it as just one step in some other mental process. The insight that goes with this compression is one of the real joys of mathematics.* [19]

The struggle Thurston mentions speaks to the *troublesome* nature of particular mathematics concepts; the ability to see the process or concept 'as a whole', file it away and recover it efficiently at any future point, possibly just as one step in some other process, alludes to the *transformative, irreversible* and *integrative* nature of understanding such concepts. The fact that other researchers have come to similar conclusions or used related constructs to describe the learning of mathematical concepts lends weight to the characterisation and usefulness of a threshold concept for mathematics.

Since their introduction in 2003, the threshold concepts that have been discussed in different domains have adopted one of three distinct forms [4]. The first form is that of a key idea in a knowledge field (e.g. opportunity cost in economics); the second is a key skill for professional practice in the area (e.g. constructing researcher identity for a PhD student, procedural decomposition in computer programming); and the third, is a stance (e.g. learner-centredness in education).

Although the idea of a threshold concept is relatively new in education, it is an idea that has been embraced by many researchers and educationalists. A wealth of concepts, skills or stances have been identified as Threshold Concepts in the literature. Some early examples of these, from different disciplines, are: the laws of motion and heat transfer in physics, equal temperament in music, sampling distribution in statistics [11].

3. Examples and Non-Examples in Mathematics

Let us look in more detail at some examples from mathematics.

3.1. Limits. The $\epsilon - \delta$ definition of the limit of a function is often troublesome for students and many mathematicians will naturally think of it when they hear about threshold concepts and remember the problems that students encounter. It is *troublesome* for a number of reasons. Previous studies on the learning of limits have found, for instance, that pre-existing images and understandings students have of the phrases 'limit' and 'tends to' cause problems for them [3]. The structure of the definition itself is also often a stumbling block for students [17] because of the inequalities and quantifiers used in the definition and more specifically the combination of both 'there exists' and 'for all' quantifiers in the same mathematical statement. However, coming to an understanding of the limit definition can be argued to be both *transformative* and *irreversible*. One indication of this is that mathematicians often remember the precise moment when they reached a clear understanding of the formal definition of a limit and report it as a real 'aha' moment. Other studies have found that students' way of speaking about limits changes when they begin to understand the definition, providing further evidence that the concept is transformative and irreversible. In addition, the concept of a limit can be thought of as *bounded* as it sits on the boundary between calculus and analysis and it acts in effect as a gateway to mathematical analysis. It is also *integrative* for this reason as it links differential and integral calculus to each other and to mathematical analysis.

3.2. **Proving.** Proof is a key component of mathematics and proving a key competence of mathematicians. The act of proving has many functions including verification, explanation, communication, and systematization. Easdown [8] argues that the ability to not only understand but also to construct proofs is *transformative* for students – not only in terms of how they perceive old ideas but also how they receive new and exciting mathematical discoveries. Mastering the act of proving acts as a 'rite of passage' to membership of the mathematical community and is often accompanied by a 'road to Damascus' effect [8]. As such, it is *irreversible*.

However, proving can be *troublesome* for students due to their uncertainty about how to start and about what is, or is not, allowed when constructing a proof [22]. Insufficient knowledge of the rules of logic and different proving strategies can also cause difficulties. Not only this but the notion that mathematics is **de**ductive rather than **in**ductive like other sciences can be counter-intuitive for students. However, proving can be thought of as *integrative* in the sense that it allows various results to be organised into a system of axioms, concepts and theorems. Proof is *bounded* since its use and meaning is specific to the subject of mathematics, and indeed is one of the defining characteristics of the discipline itself.

3.3. **Non-examples.** To illustrate the difference between 'key' or 'core' concepts and skills and 'threshold concepts', consider the processes of the chain rule for differentiation and integration by parts. Students often find these processes troublesome, and it can be challenging for them to use the techniques correctly. However, while these processes may be key to the learning of calculus, there is no evidence that the mastering of these processes is transformative or irreversible.

For a further discussion of the threshold concepts of limits and proving, along with other threshold concepts in mathematics, see [1], [15], [22], [21].

4. Why are Threshold Concepts Important?

We have seen that threshold concepts are present in the undergraduate mathematics curriculum and that they present both difficulties and opportunities to students. It seems sensible then to pay attention to them when designing courses and to help students navigate the consequent blocked spaces "by, for example, redesigning activities and sequences, through scaffolding, through provision of support materials and technologies or new conceptual tools, through mentoring or peer collaboration" [10, p.62-63]. Traditionally, undergraduate mathematics syllabii have been described solely in terms of mathematical content (such as techniques, theorems, etc.) and this approach has been seen as sometimes leading to rote learning rather than deep understanding and as inhibiting mastery of the subject. Davies and Mangan assert that putting an emphasis on the threshold concepts in a course can help:

> *Threshold concepts offer potential help to lecturers in higher education who are grappling with ... students who struggle with underpinning theory and resort to verbatim learning of isolated aspects of the subject that they seem unable to use effectively.*[6, p.711]

Cousin [5] has some advice for lecturers. She asserts that threshold concepts should be seen as 'jewels in the curriculum' as they are central to students' efforts to master the subject. Because of this centrality and also the problems that these concepts pose for students, she advises that lecturers adopt a recursive approach; that is, that we should not assume that once we have 'covered' the material that students now understand it, rather we should revisit these topics multiple times during a course to give students a chance to view the concept in a variety of ways. The need for active student engagement with, and manipulation of, the conceptual material is also emphasised [10] in order to enable students to experience the 'ways of thinking and practising' that are expected of practitioners within a given discipline, and to facilitate them to join that community of practice. Timmermans and Meyer [20] reinforce this view and advocate that activities used by teachers should deliberately confront learners with the 'troublesomeness' of threshold concepts causing them to 'get stuck'.

The word *threshold* was chosen by Meyer and Land to convey the idea that these concepts act as gateways into an expert-like appreciation of a topic. Because of the nature of a threshold concept, and the difficulty it presents to students, it often seems that mastery of a threshold concept involves the occupation of a liminal space, that is, learners oscillate between old and new understandings and (hopefully) emerge transformed. (This is reminiscent of adolescence which can be seen as a liminal state between childhood and adulthood, where adolescents often oscillate between child-like and adult-like behaviour.) This transformation can entail letting go of an earlier, comfortable position to enter a sometimes disconcerting new territory [13]. Therefore we, as lecturers, should not only expect that students will experience confusion when struggling with these concepts but we should appreciate this confusion as an opportunity for developing deep understanding. Indeed, Cousin [5] recommends that lecturers support students while in this liminal space, and explain to them that the feeling of confusion is normal and even necessary. If knowledge is to have a transformative effect, it probably should be troublesome, but that does not mean that it should be overly stressful or anxiety-inducing for students [13]. Lecturers need to walk a fine line between allowing students to struggle for too long (in which case students may resort to rote-learning to succeed) and shielding them from difficult topics (and thus denying them some valuable learning opportunities).

Cousin suggests that a key component of good lecturing is the ability to listen to students and to understand what their misunderstandings are. This is difficult for experts in the field, since by the nature of threshold concepts, once you have grasped one it can be hard to remember what it was like not to understand it. It can therefore be very challenging for lecturers to see things from the students' point of view. We also need to be aware of students' mathematical backgrounds and how this might help or hinder their development of understanding. For example, it may be that students are hampered by some previous understanding or tacit knowledge (such as the notion that a limit is something that can never be reached), and lecturers may need to help students develop new intuitions and break previous rules. Moreover, it may be that students' prior educational experiences have taught them to value being correct and, thus, they may be avoiding crossing thresholds in a classroom or a discipline [9]. Giving them an understanding and appreciation of the importance of truly mastering a particular concept may be necessary to encourage them to willingly enter a liminal space.

5. From Theory to Practice

Many lecturers and educators have endeavoured to take on board the advice above in relation to the teaching and learning of threshold concepts. We are not aware of such studies pertaining to the teaching of mathematics in particular and so we will consider some findings below for subjects other than mathematics.

Olaniyi [14] focussed on incorporating a number of Meyer and Land's guidelines ([13]) for overcoming barriers to students' understanding of threshold concepts - namely, engaging students, providing for recursive and excursive learning journeys through a topic, and including peer assessment as a means of students sharing their difficulties and anxieties as they inhabit the liminal space. She chose a flipped classroom approach to fit her needs as it facilitates active learning which she asserts should be the cornerstone of any pedagogical approach adopted for teaching threshold concepts. Active learning is important because it encourages deep rather than superficial learning. Indeed, in the US, the Conference Board of the Mathematical Sciences (of which the AMS is a member) called for the incorporation of active learning into university mathematics classrooms [2]. Olaniyi carried out an action research study into her own implementation of a flipped classroom approach to teaching the threshold concept of thermodynamics in physics and reported positively on the results both in terms of improvements in students' understanding and their study skills.

Rodgers et al [16] adopted the perspective of foregrounding threshold concepts as 'jewels in the curriculum'. They describe an action research approach through which they systematically identified a set of five threshold concepts that are encountered by students on their occupational therapy programme. These threshold concepts were then used to underpin the redesign of their curriculum so that the concepts were encountered and re-encountered at various points in the programme. Benefits of this curriculum redesign included a more consistent approach to the threshold concepts from staff and a more coherent integration of concepts overall, making learning less confusing for students. For staff, an awareness of threshold concepts helped them to develop a whole-of-programme view and use this perspective when designing and structuring content, learning activities and assessment tasks. For students, the focus on threshold concepts and the process of making troublesome knowledge explicit helped to 'capture the essence of the programme' (p.552). It also facilitated their development of a professional identity.

One way of gathering information about student thinking is to set assessment tasks in which students are asked to explain or represent a concept in a new way, and to make connections to other parts of their knowledge. To this end, Scott, Peter & Harlow [18] advocate the construction and use of concept inventories in the teaching and learning of threshold concepts. A concept inventory is a set of questions designed to gauge the depth of students' conceptual understanding of a given topic. Such an inventory is pedagogically desirable and powerful due to a two-fold function: firstly, it affords the measurement of true conceptual understanding or correct thinking on the part of students; and secondly, it enables an evaluation of the effectiveness of instruction (or an increase in student understanding) through pre- and post-testing. Scott et al. [18] believe there is a natural marriage between the theory of threshold concepts and that of concept inventories, and they acted upon this by designing and validating a concept inventory for threshold concepts arising in electronics engineering.

6. Conclusion

To an expert, a threshold concept is an idea that gives shape and structure to their subject, but such concepts can be inaccessible to a novice. From the students' point of view, grappling with threshold concepts is often a rite of passage. If they succeed in their struggle and cross the threshold, learners may find it easier to gain entry into the community of practice in their subject [4]. There is also evidence that students who do not develop a good understanding of these concepts may end up resorting to a rote-learning approach [6] or even withdraw from the study of that subject [10]. If our goal is to help students to experience the compression that Thurston described, then it is important for the mathematical community both to identify threshold concepts in our undergraduate modules, and to think carefully about the teaching methods that we use in relation to teaching these concepts. There may also be a hidden benefit for us as teachers. Timmermans and Meyer [20] have observed that some teachers experience a transformation in their conceptualization of their own disciplines, their teaching and their understanding of their students' learning during the work of identifying threshold concepts. We hope that this article will provide some food for thought on this front.

A comprehensive bibliography on the Threshold Concept Framework is maintained at the website `https://www.ee.ucl.ac.uk/mflanaga/thresholds.html` which itself acts as a portal to this area of Threshold Concepts!

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Sinéad Breen is a lecturer in DCU's School of Mathematical Sciences. While her PhD was in asymptotic analysis, she is now involved in mathematics education research. She is particularly interested in exploring the means by which (higher order) mathematical thinking skills can be developed, through the careful design and use of mathematical tasks.

Ann O'Shea has a PhD in Complex Analysis from the University of Notre Dame. She has been a member of the Department of Mathematics and Statistics at Maynooth University since 1992. Her research interests lie in Mathematics Education.

(S. Breen) School of Mathematical Sciences & CASTeL, Dublin City University.

(A. O'Shea) Department of Mathematics & Statistics, Maynooth University
E-mail address, S. Breen: sinead.breen@dcu.ie
E-mail address, A. O'Shea: ann.oshea@mu.ie