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Two Extensions of Cauchy's Double Alternant

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Abstract. Two extensions of Cauchy's double alternant are evaluated in closed form that may serve also as parametric generalizations of the remarkable determinant identity of a skew–symmetric matrix discovered by Schur (1911) and its multiplicative counterpart due to Laksov–Lascoux–Thorup (1989).

1. Introduction and Outline

There exist numerous determinant identities in the literature (cf. [9, 13]). For example, the determinants of Vandermonde and Cauchy's "double alternant"

$$\Lambda_m = \det_{1\le i,j\le m}\left[x_i^{m-j}\right] = \prod_{1\le i<j\le m}(x_i - x_j),$$

$$\det_{1\le i,j\le m}\left[\frac{1}{x_i+y_j}\right] = \frac{\prod_{1\le i<j\le m}(x_i-x_j)(y_i-y_j)}{\prod_{1\le i,j\le m}(x_i+y_j)};$$

play an important role in symmetric functions and group characters (cf. [5, 8, 10]). In general, a matrix $T(x_1, x_2, \cdots, x_m)$ of order $m\times n$ in m variables is called an alternant (cf. [9, §321]) when the elements of the first row of T are all functions of variable x_1, the elements of the second row the like functons of x_2, and so on. For example

$$\begin{bmatrix} e^x, \sin x, \cos x \\ e^y, \sin y, \cos y \\ e^z, \sin z, \cos z \end{bmatrix}.$$

Likewise, a matrix $\mathbb{T}(x_1, x_2, \cdots, x_m; y_1, y_2, \cdots, y_n)$ is a double alternant if $\mathbb{T}$ is an alternant respect to both rows in variables $\{x_1, x_2, \cdots, x_m\}$ and columns in variables $\{y_1, y_2, \cdots, y_n\}$. Suppose that $f(x, y)$ is a bivariate function, we have the following general double alternant

$$\Big[f(x_i, y_j)\Big]_{\substack{1\le i\le m\\ 1\le j\le n}} = \begin{bmatrix} f(x_1,y_1), & f(x_1,y_2), & \ddots, & f(x_1,y_n) \\ f(x_2,y_1), & f(x_2,y_2), & \ddots, & f(x_2,y_n) \\ \vdots & \vdots & \cdots & \vdots \\ f(x_m,y_1), & f(x_m,y_2), & \ddots, & f(x_m,y_n) \end{bmatrix}.$$

There exist several generalizations (cf. [1, 3, 6]) of the determinants for Cauchy's double alternant. By employing the calculus of divided differences, the second author [2] evaluated determinants for a large class of variants of Cauchy's double alternant. As a complements to the results appearing in [2], we shall examine, in this little article, the

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determinants of two particular matrices. Let T and $\{x_k, y_k\}_{1\le k\le m}$ be indeterminates. Define two matrices by

$$\mathrm{U}_m = \big[u_{i,j}\big]_{1\le i,j\le m} : u_{i,j} = \frac{x_i + Ty_j}{x_i + y_j}, \tag{1}$$

$$\mathrm{V}_m = \big[v_{i,j}\big]_{1\le i,j\le m} : v_{i,j} = \frac{x_i + Ty_j}{1 + x_i y_j}. \tag{2}$$

We shall prove the following two surprisingly elegant determinant identities.

Theorem 1 ($m \in \mathbb{N}$)**.**

$$\det \mathrm{U}_m = \frac{\prod_{1\le i<j\le m}(x_i - x_j)(y_i - y_j)}{(1-T)^{1-m}\prod_{1\le i,j\le m}(x_i + y_j)} \times \Big\{\prod_{i=1}^{m} x_i - T\prod_{i=1}^{m}(-y_i)\Big\}.$$

Theorem 2 ($m \in \mathbb{N}$)**.**

$$\det \mathrm{V}_m = \frac{\prod_{1\le i<j\le m}(x_j - x_i)(y_i - y_j)}{\prod_{1\le i,j\le m}(1 + x_i y_j)} \times \frac{1}{2}\Big\{\prod_{i=1}^{m}(x_i + \sqrt{T})(1 + y_i\sqrt{T}) + \prod_{i=1}^{m}(x_i - \sqrt{T})(1 - y_i\sqrt{T})\Big\}.$$

When $T = 0$, the corresponding identities in both theorems are equivalent to the Cauchy double alternant. For $T = -1$ and even $m = 2n$, these identities reduce, in the case $x_k = y_k$ for all k, to the following remarkable Pfaffian formulae discovered by Schur [11] and Laksov–Lascoux–Thorup [7] (see also [12]), respectively:

$$\det_{1\le i,j\le 2n}\left[\frac{x_i - x_j}{x_i + x_j}\right] = \prod_{1\le i<j\le 2n}\left(\frac{x_i - x_j}{x_i + x_j}\right)^2,$$

$$\det_{1\le i,j\le 2n}\left[\frac{x_i - x_j}{1 + x_i x_j}\right] = \prod_{1\le i<j\le 2n}\left(\frac{x_i - x_j}{1 + x_i x_j}\right)^2.$$

2. Proof of Theorem 1

For the matrix U_m, by subtracting the last row from the other rows, we can check that the resulting matrix becomes

$$\mathrm{U}'_m = \big[u'_{i,j}\big]_{1\le i,j\le m} :\ u'_{i,j} = \begin{cases} \dfrac{(1-T)(x_i - x_m)y_j}{(x_i + y_j)(x_m + y_j)}, & 1 \le i < m; \\[2mm] \dfrac{x_m + Ty_j}{x_m + y_j}, & i = m. \end{cases}$$

By extracting the common row factor $(1-T)(x_i - x_m)$ for $1 \le i < m$ and the common column factor $\dfrac{y_j}{x_m + y_j}$ for $1 \le j \le m$, we find the following determinant equality

$$\det \mathrm{U}_m = \det \mathrm{U}'_m = \det \mathrm{U}''_m \times (1-T)^{m-1}\prod_{i=1}^{m-1}(x_i - x_m)\prod_{j=1}^{m}\frac{y_j}{x_m + y_j}, \tag{3}$$

where the matrix U''_m is given by

$$\mathrm{U}''_m = \big[u''_{i,j}\big]_{1\le i,j\le m} : u''_{i,j} = \begin{cases} \dfrac{1}{x_i + y_j}, & 1 \le i < m; \\[2mm] \dfrac{x_m + Ty_j}{y_j}, & i = m. \end{cases}$$

Expanding the determinant along the last row leads us to the equality

$$\det \mathrm{U}''_m = \sum_{k=1}^{m} (-1)^{m+k} \frac{x_m + Ty_k}{y_k} \det \mathrm{U}''_m[m,k], \tag{4}$$

where $\mathrm{U}''_m[m,k]$ is the sub-matrix of U''_m with the mth row and the kth column being removed. By applying the Cauchy double alternant, we can evaluate

$$\begin{aligned}\det \mathrm{U}''_m[m,k] &= \frac{\prod_{1\le i<j<m}(x_i - x_j) \prod_{1\le i<j\le m\ (i,j\neq k)}(y_i - y_j)}{\prod_{1\le i<m} \prod_{1\le j\le m\ (j\neq k)}(x_i + y_j)} \\ &= (-1)^{m-k} \frac{\prod_{i=1}^{m}(x_i + y_k)}{\prod_{i\neq m}(x_m - x_i)} \prod_{j\neq k} \frac{x_m + y_j}{y_k - y_j} \frac{\prod_{1\le i<j\le m}(x_i - x_j)(y_i - y_j)}{\prod_{1\le i,j\le m}(x_i + y_j)}.\end{aligned}$$

Here and henceforth for simplicity, $\prod_{\ell\neq k}$ stands for the product with the index ℓ running from 1 to m except for $\ell = k$. Substituting this into (4) gives rise to the following expression

$$\det \mathrm{U}''_m = \frac{\prod_{1\le i<j\le m}(x_i - x_j)(y_i - y_j)}{\prod_{1\le i,j\le m}(x_i + y_j)} \frac{\prod_{j=1}^{m}(x_m + y_j)}{\prod_{\imath\neq m}(x_m - x_\imath)} \sum_{k=1}^{m} \frac{x_m + Ty_k}{y_k} \frac{\prod_{i\neq m}(x_i + y_k)}{\prod_{j\neq k}(y_k - y_j)}.$$

Denote by $\Delta[y_1, y_2, \cdots, y_m]f(y)$ the divided difference (cf. [2]) of the function $f(y)$ at the points $\{y_k\}_{k=1}^m$, which can be expressed by Newton's symmetric sum

$$\Delta[y_1, y_2, \cdots, y_m]f(y) = \sum_{k=1}^{m} \frac{f(y_k)}{\prod_{j\neq k}(y_k - y_j)}.$$

Then we can evaluate the last sum (cf. [4]) as

$$\begin{aligned}\sum_{k=1}^{m} \frac{x_m + Ty_k}{y_k} \frac{\prod_{i\neq m}(x_i + y_k)}{\prod_{j\neq k}(y_k - y_j)} &= \Delta[y_1, y_2, \cdots, y_m]\Big\{\frac{x_m + Ty}{y} \prod_{i\neq m}(x_i + y)\Big\} \\ &= \Delta[y_1, y_2, \cdots, y_m]\Big\{Ty^{m-1} + \frac{\prod_{i=1}^m x_i}{y}\Big\} \\ &= T - (-1)^m \prod_{i=1}^{m} \frac{x_i}{y_i},\end{aligned}$$

which leads us to the following simpler formula

$$\det \mathrm{U}''_m = \frac{\prod_{1\le i<j\le m}(x_i - x_j)(y_i - y_j)}{\prod_{1\le i,j\le m}(x_i + y_j)} \frac{\prod_{j=1}^{m}(x_m + y_j)}{\prod_{\imath\neq m}(x_m - x_\imath)} \Big\{T - (-1)^m \prod_{i=1}^{m} \frac{x_i}{y_i}\Big\}.$$

Substituting this into (3) and then simplifying the resulting expression, we confirm the determinant identity stated in Theorem 1. □

3. Proof of Theorem 2

By following exactly the same procedure as done in the last section, we can explicitly evaluate the determinant for the matrix V_m. Subtracting the last row from the other rows transforms V_m into the following one:

$$\mathrm{V}'_m = \big[v'_{i,j}\big]_{1\le i,j\le m}: \ v'_{i,j} = \begin{cases} \dfrac{(1 - Ty_j^2)(x_i - x_m)}{(1 + x_i y_j)(1 + x_m y_j)}, & 1 \le i < m; \\ \dfrac{x_m + Ty_j}{1 + x_m y_j}, & i = m. \end{cases}$$

By extracting the common row factor $x_i - x_m$ for $1 \leq i < m$ and the common column factor $\dfrac{1-Ty_j^2}{1+x_m y_j}$ for $1 \leq j \leq m$, we find the following determinant equality

$$\det \mathrm{V}_m = \det \mathrm{V}'_m = \det \mathrm{V}''_m \times \prod_{i=1}^{m-1}(x_i - x_m) \prod_{j=1}^{m} \frac{1-Ty_j^2}{1+x_m y_j}, \tag{5}$$

where the matrix V''_m is explicitly given by

$$\mathrm{V}''_m = \left[v''_{i,j}\right]_{1 \leq i,j \leq m} : v''_{i,j} = \begin{cases} \dfrac{1}{1+x_i y_j}, & 1 \leq i < m; \\ \dfrac{x_m + Ty_j}{1-Ty_j^2}, & i = m. \end{cases}$$

Expanding the determinant along the last row leads us to the equality

$$\det \mathrm{V}''_m = \sum_{k=1}^{m} (-1)^{m+k} \frac{x_m + Ty_k}{1-Ty_k^2} \det \mathrm{V}_m[m,k], \tag{6}$$

where $\mathrm{V}_m[m,k]$ is the sub-matrix of V''_m with the mth row and the kth column being crossed out. By making replacements $x_i \to x_i^{-1}$, we can reformulate Cauchy's double alternant as

$$\det_{1 \leq i,j \leq m} \left[\frac{1}{1+x_i y_j}\right] = \frac{\prod_{1 \leq i<j \leq m}(x_j - x_i)(y_i - y_j)}{\prod_{1 \leq i,j \leq m}(1+x_i y_j)}.$$

Then we can evaluate $\det \mathrm{V}_m[m,k]$ by the following product expression

$$\begin{aligned} \det \mathrm{V}_m[m,k] &= \frac{\prod_{1 \leq i<j<m}(x_j - x_i) \prod_{1 \leq i<j \leq m \ (i,j \neq k)}(y_i - y_j)}{\prod_{1 \leq i<m} \prod_{1 \leq j \leq m \ (j \neq k)}(1+x_i y_j)} \\ &= (-1)^{m-k} \frac{\prod_{i=1}^{m}(1+x_i y_k)}{\prod_{i \neq m}(x_i - x_m)} \prod_{j \neq k} \frac{1+x_m y_j}{y_k - y_j} \frac{\prod_{1 \leq i<j \leq m}(x_j - x_i)(y_i - y_j)}{\prod_{1 \leq i,j \leq m}(1+x_i y_j)}. \end{aligned}$$

Substituting this into (6) yields that

$$\det \mathrm{V}''_m = \frac{\prod_{1 \leq i<j \leq m}(x_j - x_i)(y_i - y_j)}{\prod_{1 \leq i,j \leq m}(1+x_i y_j)} \frac{\prod_{j=1}^{m}(1+x_m y_j)}{\prod_{i \neq m}(x_i - x_m)} \sum_{k=1}^{m} \frac{x_m + Ty_k}{1-Ty_k^2} \frac{\prod_{i \neq m}(1+x_i y_k)}{\prod_{j \neq k}(y_k - y_j)}.$$

By decomposing into partial fractions

$$\frac{1}{1-Ty^2} = \frac{1}{2\sqrt{T}} \times \left\{\frac{1}{y+\sqrt{T^{-1}}} - \frac{1}{y-\sqrt{T^{-1}}}\right\},$$

we can evaluate the rightmost sum again by divided differences

$$\begin{aligned}
&\sum_{k=1}^{m}\frac{x_m+Ty_k}{1-Ty_k^2}\frac{\prod_{i\neq m}(1+x_iy_k)}{\prod_{j\neq k}(y_k-y_j)}\\
&=\Delta[y_1,y_2,\cdots,y_m]\Big\{\frac{x_m+Ty}{1-Ty^2}\prod_{i\neq m}(x_i+y)\Big\}\\
&=\frac{1}{2\sqrt{T}}\Delta[y_1,y_2,\cdots,y_m]\Big\{\frac{x_m-\sqrt{T}}{y+\sqrt{T^{-1}}}\prod_{i\neq m}(1-x/\sqrt{T})\Big\}\\
&-\frac{1}{2\sqrt{T}}\Delta[y_1,y_2,\cdots,y_m]\Big\{\frac{x_m+\sqrt{T}}{y-\sqrt{T^{-1}}}\prod_{i\neq m}(1+x/\sqrt{T})\Big\}\\
&=\frac{1}{2}\prod_{i=1}^{m}\frac{x_i+\sqrt{T}}{1-y_i\sqrt{T}}+\frac{1}{2}\prod_{i=1}^{m}\frac{x_i-\sqrt{T}}{1+y_i\sqrt{T}}.
\end{aligned}$$

Consequently, we derive the closed form expression

$$\begin{aligned}
\det \mathrm{V}''_m=&\frac{\prod_{1\le i<j\le m}(x_j-x_i)(y_i-y_j)}{\prod_{1\le i,j\le m}(1+x_iy_j)}\frac{\prod_{j=1}^{m}(1+x_my_j)}{\prod_{i\neq m}(x_i-x_m)}\\
&\times\frac{1}{2}\Big\{\prod_{i=1}^{m}\frac{x_i+\sqrt{T}}{1-y_i\sqrt{T}}+\prod_{i=1}^{m}\frac{x_i-\sqrt{T}}{1+y_i\sqrt{T}}\Big\}.
\end{aligned}$$

Finally, substituting this into (5) and then simplifying the resulting expression, we find that

$$\begin{aligned}
\det \mathrm{V}_m=&\frac{\prod_{1\le i<j\le m}(x_j-x_i)(y_i-y_j)}{\prod_{1\le i,j\le m}(1+x_iy_j)}\\
&\times\frac{1}{2}\Big\{\prod_{i=1}^{m}(x_i+\sqrt{T})(1+y_i\sqrt{T})+\prod_{i=1}^{m}(x_i-\sqrt{T})(1-y_i\sqrt{T})\Big\}.
\end{aligned}$$

This completes the proof of Theorem 2. □

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