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PROBLEMS

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Problems

A polygon is said to be *inscribed* in a simple closed curve in the plane if all the vertices of the polygon lie on the curve.

Problem 93.1. Find a simple closed curve in the plane that does not have an inscribed regular pentagon.

The same problem but with a square instead of a pentagon is unsolved; it is known as the *inscribed square problem*.

Problem 93.2. Determine the least positive integer n for which a continued fraction

$$\cfrac{1}{b_1+\cfrac{1}{b_2+\cfrac{1}{b_3+\cdots+\cfrac{1}{b_n}}}}$$

has value ∞, where b_i are Gaussian integers each of modulus greater than 1.

The third problem was passed on to me by Andrei Zabolotskii of the Open University, who encountered it at the Moscow Mathematical Olympiad in 2005. There are recent publications on the problem.

Problem 93.3. Dissect a disc into a finite number of congruent connected pieces (reflections allowed) in such a way that at least one piece does not contain the centre of the disc inside it or on its boundary.

Solutions

Here are solutions to the problems from *Bulletin* Number 91.

The first problem was solved by the North Kildare Mathematics Problem Club and the proposer, Toyesh Prakash Sharma of Agra College, India. We present an abridged version of the solution of the proposer, which begins with an attractive equation for the cosines of the angles of a triangle.

Problem 91.1. Prove that the angles α, β, and γ of a triangle satisfy

$$\big(1+\sec^2\alpha+\sec^2\beta+\sec^2\gamma\big)(1-\cos\alpha\cos\beta\cos\gamma)\geqslant 8$$

We ought to assume here that none of the angles are right-angles, so that none of the cosines of the angles vanish.

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Solution 91.1. Observe that

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma + 2\cos\alpha\cos\beta\cos\gamma - 1 = 0.$$

An elementary if tedious way to establish this equation is to use the cosine rule to express $\cos\alpha$, $\cos\beta$, and $\cos\gamma$ in terms of the lengths of the sides of the triangle; we omit the details.

It follows immediately that

$$1 - \cos\alpha\cos\beta\cos\gamma = \frac{1}{2}(1 + \cos^2\alpha + \cos^2\beta + \cos^2\gamma),$$

and hence

$$\begin{aligned}\left(1+\sec^2\alpha+\sec^2\beta+\sec^2\gamma\right)&(1-\cos\alpha\cos\beta\cos\gamma)\\ &= \frac{1}{2}\left(1+\sec^2\alpha+\sec^2\beta+\sec^2\gamma\right)\left(1+\cos^2\alpha+\cos^2\beta+\cos^2\gamma\right).\end{aligned}$$

By applying the arithmetic-geometric mean inequality (twice) we obtain

$$\begin{aligned}\left(1+\sec^2\alpha+\sec^2\beta+\sec^2\gamma\right)&(1-\cos\alpha\cos\beta\cos\gamma)\\ &\geqslant \tfrac{1}{2} \times 4\sqrt[4]{\sec^2\alpha\sec^2\beta\sec^2\gamma} \times 4\sqrt[4]{\cos^2\alpha\cos^2\beta\cos^2\gamma} = 8. \quad \square\end{aligned}$$

The second problem was solved by Henry Ricardo of the Westchester Area Math Circle, USA, Seán Stewart of the King Abdullah University of Science and Technology, Saudi Arabia, the North Kildare Mathematics Problem Club, and the proposer Des MacHale of University College Cork. We provide the solution of Henry Ricardo which, like some (but not all) of the other solutions, uses Brahmagupta's elegant formula for the area of a cyclic quadrilateral.

*Problem 91.2***.** Prove that the perimeter P and area A of a cyclic quadrilateral satisfy

$$P^2 \geqslant 16A,$$

with equality if and only if the cyclic quadrilateral is a square.

Solution 91.2. The area A of a cyclic quadrilateral with sides a, b, c, and d is given by Brahmagupta's formula

$$A = \sqrt{(s-a)(s-b)(s-c)(s-d)},$$

where $s = P/2 = (a+b+c+d)/2$. By applying the arithmetic-geometric mean (AGM) inequality we obtain

$$\begin{aligned}A &= \left(\sqrt[4]{(s-a)(s-b)(s-c)(s-d)}\right)^2\\ &\leqslant \left(\frac{4s-P}{4}\right)^2\\ &= \frac{P^2}{16}.\end{aligned}$$

Equality holds in the AGM inequality if and only if $s-a = s-b = s-c = s-d$, that is, if and only if the quadrilateral is a square. $\square$

The third problem was solved by the North Kildare Mathematics Problem Club and the proposer, Tran Quang Hung of the Vietnam National University at Hanoi, Vietnam. We present the solution of the problem club.

Problem 91.3. Let $A_0, A_1, \dots, A_n$ be the vertices of a simplex in n-dimensional Euclidean space for which the edges $A_0A_1, A_0A_2, \dots, A_0A_n$ are mutually perpendicular. Let B_i be the centroid of the set of points $\{A_0, A_1, \dots, A_n\} \setminus \{A_i\}$, for $i = 0, 1, \dots, n$. Consider any point C other than A_0 for which the line through A_0 and C is perpendicular to the hyperplane spanned by $A_1, A_2, \dots, A_n$, and let P be the midpoint of the segment B_0C. Prove that all distances PB_i are equal, for $i = 1, 2, \dots, n$.

Solution 91.3. We may assume that A_0 is the origin and $A_j = a_je_j$, for $j = 1, 2, \dots, n$, where $(e_1, e_2, \dots, e_n)$ is an orthonormal basis for $\mathbb{R}^n$, and $a_j > 0$. Then

$$B_i = \frac{1}{n}\sum_{j\neq i} a_je_j.$$

The plane spanned by $A_1, A_2, \dots, A_n$ has equation

$$\frac{x_1}{a_1} + \frac{x_2}{a_2} + \cdots \frac{x_n}{a_n} = 1,$$

so the perpendicular vector C takes the form

$$C = \lambda\left(\frac{1}{a_1}, \frac{1}{a_2}, \dots, \frac{1}{a_n}\right),$$

for some nonzero real number λ. Then

$$B_0 = \frac{1}{n}\sum_{j=1}^{n} a_je_j \quad\text{and}\quad P = \frac{1}{2}\left(\frac{1}{n}\sum_{j=1}^{n} a_je_j + \lambda\sum_{j=1}^{n}\frac{1}{a_j}e_j\right),$$

so

$$\begin{aligned}|PB_i|^2 &= \frac{1}{4}\left(\left(\frac{a_i}{n} + \frac{\lambda}{a_i}\right)^2 + \sum_{j\neq i}\left(\frac{a_j}{n} - \frac{\lambda}{a_j}\right)^2\right)\\ &= \frac{1}{4}\sum_{j=1}^{n}\left(\frac{a_j^2}{n^2} + \frac{\lambda^2}{a_j^2}\right) + \frac{\lambda(2-n)}{2n},\end{aligned}$$

which is independent of the index i, as required. □

We invite readers to submit problems and solutions. Please email submissions to `imsproblems@gmail.com` in any format (we prefer LaTeX). Submissions for the summer Bulletin should arrive before the end of April, and submissions for the winter Bulletin should arrive by October. The solution to a problem is published two issues after the issue in which the problem first appeared. Please include solutions to any problems you submit, if you have them.

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