# Irish Mathematical Society
# Cumann Matamaitice na hÉireann

## Bulletin

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**Irish Mathematical Society**
**Bulletin**

The aim of the *Bulletin* is to inform Society members, and the mathematical community at large, about the activities of the Society and about items of general mathematical interest. It appears twice each year. The *Bulletin* is published online free of charge.

The *Bulletin* seeks articles written in an expository style and likely to be of interest to the members of the Society and the wider mathematical community. We encourage informative surveys, biographical and historical articles, short research articles, classroom notes, book reviews and letters. All areas of mathematics will be considered, pure and applied, old and new.

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**Irish Mathematical Society Bulletin 93, Summer 2024**

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## EDITORIAL

I want to draw the attention of younger members to Róisín Neurerer's letter in this issue, about EMYA, the EMS Young Academy. There is a related deadline at the end of July.

The manuals of logic often follow the tradition of millenia, and illustrate the universal quantifier with the sentence: *All men are mortal.* A glance at the contents of this issue will reveal evidence that this sentence is more than hypothetical. What were for me the fixed features of the Irish mathematical landscape have fallen away. I find myself in sympathy with the veteran, hardened by the loss of so many friends in a long campaign, who hesitates to befriend fresh replacements. And yet the editor of this Bulletin should have his finger on the pulse of our national mathematical life, and I have decided that it is time to pass the baton, so a new editor will take over for the next issue. I want to thank the members of the Editorial Board, the website manager Michael Mackey, and David Malone for their unfailing help and support during my tenure.

In their obituary of Seán Dineen, Michael Mackey and Pauline Mellon mention his Cistercian teacher Father Emmanuel in Roscrea. This was my mother's brother, William Curtis. It would be remiss of me not to pay tribute to him. The Holy Rule of St. Benedict makes no mention of Mathematics, and it may well be that Father Emmanuel was surprised to find himself teaching the subject after his final profession. He also had a mulberry plantation, bred silkworms, spun wove and dyed silk, mainly for clerical vestments, although he also gave me a rather dashing tie. He wrote a biography of Oliver Plunkett, Martyr. He was best with strong pupils, and he ranked Seán as his best ever. He did much for the mathematical community, and was a stalwart of the IMTA and CESI. For many years he produced solution-books for the Leaving Certificate Maths papers, published by Folens for the use of teachers.

One hears much talk of artificial intelligence these days, and there is a stock-market frenzy about it. The claims are over-stated. It is true that machines running programs can perform many tasks that required human action up to now, but the best of what we do is still far beyond the capacity of such robots. The recent excitement results from the discovery that work hitherto deemed to require analytic and literary talent can be produced by a robot with access to a sufficiently-large database of pre-existing text and a program that matches text and predicts the most likely sequel. It remains the case that what our members do when they conceive, ponder and solve mathematical problems is of another order. I engaged in a little dialogue with ChatGPT:

**Q:** What problem did O'Farrell and Zaitsev pose about reversible formal maps?
**—:** The reply was ok, high-level, accurate as far as it went.
**Q:** How would you solve the problem?
**—:** The reply revealed a critical misunderstanding about the topic. The program considered only reversers tangent to the identity. Only involutions have reversers tangent to the identity.
**Q:** Must a reversible formal map have a reverser of finite order?
**—:** The reply tried to answer a slightly different (but definitely different) question.
**Q:** I'm not asking whether every reverser has finite order. Must there always be some reverser of finite order?
**—:** This time the answer was just wrong. A couple of days later, essentially the same question elicited a different, correct answer:

**Q:** Must a reversible formal map have a reverser of finite order? Please format your answer in LaTeX.

**—:** The program has now found a more sensible, but less exciting answer — the problem remains open — and came up with a competent LaTeX version.

The program is able to locate some data about our work, locate it correctly in relation to the spectrum of research areas, pick apart the question and find correct definitions of the terms, generate examples, and apply elementary logic reasonably accurately. It writes grammatically-correct English, and its output consists of a nicely-structured list of propositions, ending with a coherent summary. But it doesn't tell me any true thing I don't know, and it is just plain wrong in key assertions about the problem. It's like a hardworking first-year graduate student with a slightly loose screw, who will never make it unless I decide to write his thesis for him.

It must be admitted that there is a certain pleasure in conversation with the program. Its summaries of famous complex advanced topics resemble the coffee-room conversation of visiting colloquium speakers. As a substitute for talking to oneself while setting up a question to study, it is almost as useful as a well-motivated hardworking student who lacks a real flair for the business, and it may be kinder to dismiss the weak students sooner and talk to the program. On the other hand, the student who loves mathematics but lacks the talent to penetrate the real difficulties is a human being, with all the limitless value that implies, and with other talents and possibilities that, once discovered, point the way to the unique purposes for which he was created. One does not feel the same urge to help the program find its way in the world, and one has to be conscious of the damage that can result from its subtle errors[1]

The matter is nicely illustrated by a line in David E. Dunning's review of Stephen Budiansky'd biography of Kurt Gödel, in the January 2023 LMS Newsletter. He gives high praise to this account intended for the general public, but then says: "Readers of this *Newsetter* will find the mathematical content thin" (This adds a spicy ambiguity to his later advice: "Naturally anyone with an interest in Gödel or the history of logic ought to waste no time obtaining a copy of this book."). The brutal fact is that we differ from the general public in that we regard the fact that the great Gödel asserted something as interesting, but inconclusive until we have personally read through and checked the proof, or come up with our own. Arguments from authority, like arguments *ad hominem*, have no value to us. Many people recoil from the hard stuff, from the page that takes a day to digest, but we can't help digging in. It is not hard for us, no more than it is hard for the young to be beautiful.

There is a much more thorough examination of ChatGPT and its relatives, and the opportunities and dangers they pose, in the January 2024 issue of the AMS Notices. In particular, the article by Schmidt and Meir[2] tells an alarming tale. They demonstrate that there has already been some pollution of the mainstream research literature by nonsensical or false AI-generated content. I hope we have not as yet published any articles or abstracts written by generative AI tools, and just to be clear we expect our contributors to adhere to COPE guidelines (at least). See `publicationethics.org/cope-position-statements`.

[1]The program also exhibited a kind of stolid solemnity, when asked what is purple and commutative. It seems to know all the jokes, and to know about the anatomy of wit, but its own position is Victorian.

[2]Paul G. Schmidt and Amnon J. Meir. *Using generative AI for literature searches and scholarly writing: Is the integrity of the scientific discourse in jeopardy?* NAMS 71, no.1, pp93-104.

For a limited time, beginning as soon as possible after the online publication of this Bulletin, a printed (grayscale, not full-colour) and bound copy may be ordered online on a print-on-demand basis at a minimal price[3].

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*E-mail address*: ims.bulletin@gmail.com

[3]Go to www.lulu.com and search for *Irish Mathematical Society Bulletin*.

## LINKS FOR POSTGRADUATE STUDY

The following are the links provided by Irish Schools for prospective research students in Mathematics:

DCU: mailto://maths@dcu.ie
TUD: mailto://chris.hills@tudublin.ie
ATU: mailto://leo.creedon@atu.ie
MTU:
http://mathematics.mtu.ie/datascience
UG: mailto://james.cruickshank@universityofgalway.ie
MU: mailto://mathsstatspg@mu.ie
QUB:
http://web.am.qub.ac.uk/wp/msrc/msrc-home-page/postgrad_opportunities/
TCD: http://www.maths.tcd.ie/postgraduate/
UCC: https://www.ucc.ie/en/matsci/study-maths/postgraduate/#d.en.1274864
UCD: mailto://nuria.garcia@ucd.ie
UL: mailto://sarah.mitchell@ul.ie

The remaining schools with Ph.D. programmes in Mathematics are invited to send their preferred link to the editor.

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*E-mail address*: ims.bulletin@gmail.com

# Letters to the Editor

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Des MacHale
From Des MacHale

Dear Editor,

Dr. James Ward of the University of Galway tells me that Theorems 2 and 3 of my note *Some Easy Inequalities for a Triangle* (Irish Math. Soc. Bulletin Number 92, Winter 2023, 55-56) are known and have been published. He found them in a little booklet, *Ungleichungen*, by Edmund Hlawka, published in German by Manz-Verlag Wein, ISBN 3 214 91400 6 (1999) 80pp. They are to be found on pages 52 and 53, with proofs. My perhaps more significant result Theorem 1 of that note is not in this booklet and could very well be new. I apologise to your readers for this oversight. I rediscovered these results in good faith and consulted my extensive collection of old and new books on trigonometry as well as relevant online sites without finding them.
Yours sincerely

Des MacHale
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Received 5-4-2024
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EMYA: The European Mathematical Society Young Academy
From Róisín Neururer

Dear Editor,

The development of mathematics, and the promotion of research in mathematics, are primary aims of the European Mathematical Society (EMS) of which the Irish Mathematical Society (IMS) is a member. These echo the aims of the IMS on a national level. In order to shape future developments of mathematics, it is important to involve the voice of the next generation of mathematicians. This means creating opportunities for young mathematicians to contribute their ideas and share their perspectives. Unfortunately, at present much of the interaction within the EMS, and indeed the IMS, takes place at the level of established researchers and mathematicians. To address this, in 2022 the EMS established the European Mathematical Society Young Academy (EMYA).

Members of EMYA are young mathematicians (3rd year PhD students up to 5 years post PhD) who are selected from nominations provided by EMS member societies. I was honoured to be successfully nominated by the IMS. The inaugural cohort was installed in early 2023 and I am delighted to have been elected chair of EMYA.

EMYA mirrors the aims and responsibilities of the EMS, but with a specific focus on young mathematicians. It aims to give a voice to, and encourage participation of, young mathematicians within the EMS; to promote and support the work of young mathematicians across Europe; and to propose scientific activities of interest to our community. This community naturally encompasses early-career mathematicians and PhD students working in Ireland. To achieve its aims, EMYA hopes to organise a wide range of initiatives, both scientific and social, which will foster connections among young mathematicians in Europe and provide them with opportunities to develop research and academic skills, as well as create opportunities to share their views on what it means to be a young researcher in our academic system. At the upcoming European Congress of Mathematics in Seville, EMYA has organized a lightning talk session specifically for young mathematicians, a discussion panel on sustainability issues, and a networking session. More information can be found on the website, `https://www.ecm2024sevilla.com/index.php/programme/emya-activities`.

Since supporting young mathematicians is the primary aim of EMYA, as chair I would welcome ideas and suggestions from any young mathematicians in Ireland about how you can be supported in your careers and, more broadly, in your academic life. I strongly encourage you to follow EMYA on its social media channels [1] to stay up to date with its activities and contribute to discussions. I also encourage IMS members to share the news of EMYA with colleagues and PhD students, to ensure that our representative role is as effective as possible. Finally, as the call for nominating EMYA members for the period 2025-2028 will close on 31st July 2024, I urge any interested young mathematicians to put themselves forward as a candidate for the IMS nominations.

I look forward to seeing how EMYA continues to develop over the coming months and years, hoping that, in the future, it will become a reference point for young mathematicians aiming to build their life in this wonderful community.

Yours sincerely

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Received 31-3-2024
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[1]@EMSYoungAcademy, `https://www.facebook.com/ems.emya`, `linkedin.com/in/emsEMYA`

## NOTICES FROM THE SOCIETY

### Officers and Committee Members 2024

| | | |
|---|---|---|
| **President** | Dr Leo Creedon | ATU |
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## Applying for I.M.S. Membership

(1) The Irish Mathematical Society has reciprocity agreements with the American Mathematical Society, the Deutsche Mathematiker Vereinigung, the Irish Mathematics Teachers Association, the London Mathematical Society, the Moscow Mathematical Society, the New Zealand Mathematical Society and the Real Sociedad Matemática Española.

(2) The current subscription fees are given below:

| | |
|---|---|
| Institutional member | €250 |
| Ordinary member | €40 |
| Student member | €20 |
| DMV, IMTA, NZMS, MMS or RSME reciprocity member | €20 |
| AMS reciprocity member | $25 |
| LMS reciprocity member (paying in Euro) | €20 |
| LMS reciprocity member (paying in Sterling) | £20 |

(3) The subscription fees listed above should be paid in euro by means of electronic transfer, a cheque drawn on a bank in the Irish Republic, or an international money-order.

The subscription fee for ordinary membership can also be paid in a currency other than euro using a cheque drawn on a foreign bank according to the following schedule:

If paid in United States currency then the subscription fee is US$ 40.
If paid in sterling then the subscription is £30.
If paid in any other currency then the subscription fee is the amount in that currency equivalent to US$ 40.00.

The amounts given in the table above have been set for the current year to allow for bank charges and possible changes in exchange rates.

(4) Any member with a bank account in the Irish Republic may pay his or her subscription by a bank standing order using the form supplied by the Society.

(5) Any ordinary member who has reached the age of 65 years and has been a fully paid up member for the previous five years may pay at the student membership rate of subscription.

(6) Subscriptions normally fall due on 1 February each year.

(7) Cheques should be made payable to the Irish Mathematical Society.

(8) Any application for membership must be presented to the Committee of the I.M.S. before it can be accepted. This Committee meets three times each year.

(9) Please send the completed application form, available at
`http://www.irishmathsoc.org/links/apply.pdf`
with one year's subscription to:

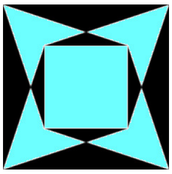
Dr Cónall Kelly
School of Mathematical Sciences
Western Gateway Building, Western Road
University College Cork
Cork, T12 XF62
Ireland

## Deceased Members

It is with regret that we report the deaths of members:

Seán Dineen, of UCD, who died on 22 January 2024.

*E-mail address*: subscriptions.ims@gmail.com



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# Seán Dineen 1944-2024

MICHAEL MACKEY AND PAULINE MELLON

FIGURE 1. Seán Dineen, 1944-2024

Seán Dineen, Professor of Mathematics at University College Dublin from 1979 until his retirement in 2009, passed away on the 18th of January 2024. Seán was a member of the Royal Irish Academy, a Head of Department and Head of School at UCD and a founding member and former president of the Irish Mathematical Society. He was a renowned expert in infinite dimensional holomorphy and leaves behind a canon of work, including scholarly papers, research monographs and undergraduate textbooks.

Seán was married to Carol and they had two children, Deirdre and Stephen, three grandchildren and three great-grand children.

*Key words and phrases.* Seán Dineen, Obituary.
Received on 14-6-2024.
DOI:10.33232/BIMS.0093.5.24.

In this article we attempt to paint a picture of the colourful character and life of Seán, treating not only his mathematics but also his contribution to the wider national and international academic community.

Seán was well-known in UCD, Ireland and beyond. He liked people and liked talking and his informal manner led to an easy rapport with many although he could be very resolute in arguments with those in authority. Like much about Seán, he was surprisingly deliberate in many of his approaches. He believed strongly in community, whether that was his community in Clonakilty (about which he wrote two books) or the UCD community (for example, his enthusiastic involvement with Erasmus students coming to UCD) or wider afield. He helped build a national mathematical community in the form of the Irish Mathematical Society (and its predecessor the Dublin Mathematical Group), of which he was a founding member and whose annual conferences he attended regularly, even in 2022 shortly after he had undergone major brain surgery. Indeed, the first of the IMS annual "September" meetings in 1988 took place during Seán's term as IMS president.

FIGURE 2. Some of the founding members of the Irish Mathematical Society, on the society's 40th anniversary in December 2016. L-R: Martin Newell, Ray Ryan, Seán, Ted Hurley, Tony O Farrell, Brendan Goldsmith, Tom Laffey, Peter Lynch and Murrough Golden.

Seán fostered an extensive international community of mathematicians with whom he maintained strong connections throughout his life. Some flew to Dublin to visit him not long before he died, knowing he was unwell. Others travelled for the funeral and many more watched it live from an analysis conference in Korea.

Seán played a large role in his local neighbourhood of Lakelands in Stillorgan, organising the publishing of a regular newsletter and campaigning to protect it during various local area developments. He reflected deeply on matters related to Irish education and corresponded with politicans regarding these. He was adept at negotiation and diplomacy, which by his own account he learned as a child on the streets of 'Clon': "*To organize ourselves for games, we children divided the town into natural regions and* *each area organized itself. These divisions were organized so that we had a balance between areas, none too powerful or dominant or too weak, otherwise why would we play one another? In all walks of life you have to have reasonable expectations of winning and fairness. All this natural training in negotiations and democracy went side by side with the skills we learned in sport, how to win, how to lose, how to play as a team and depend on one another, how to be patient and how to prepare.*"

On going through Seán's papers after his death, we were struck by the sheer meticulousness of his planning, scheduling and organisation. He often came across as spontaneous, even laid-back, but that was built on foundations of daily, monthly and yearly achievement lists. He described earlier in his writings how at secondary school "*I always gave my full concentration to my studies while I was studying and felt, when I completed a studies session, that I could then devote myself fully to my non-academic interests, and I have always had many of these, without any feeling of guilt. I have successfully followed this approach ever since.*"

## Early Life in Clonakilty

Figure 3. A young Seán

Seán felt a special mixture of belonging and pride for his home place of Clonakilty, a small town in West Cork with a population in the 1950s of about 2,500 people. He took many mathematical visitors there over the years, bringing them to his family home on

Emmet Square and showing them the local sights. In his retirement speech in 2009, he began by describing his early years in 'Clon' in some detail. We take Seán's lead in this obituary and borrow from his writings throughout.

Seán was born John Hugh Dineen on February 12, 1944 in Cork city, but from early on was known only as Seán. Shortly afterwards his family moved to Clonakilty. He was the eldest of six children born to Jeremiah Joseph Dineen and Margaret Jean Connaughton. Growing up in Clonakilty during the 1950s, Seán had a childhood that seemed ordinary to him at the time but that he realised later was quite special, "*Growing up one almost automatically assumes that everyone has a similar childhood and that while there are differences these are not important, more a matter of the luck of the draw.*" He described his household as close-knit, with himself and his siblings forming a "tribe" of sorts.

His father, a science graduate, taught mathematics, physics and chemistry, while his mother taught Irish and Commerce. Together they founded St. Mary's College in 1938, a secondary school for boys, at a time when there was no second-level education for boys available in the town.

His father passed away when Seán was nine years old, leaving his mother to manage the school and raise the six children. His mother's resilience and dedication left a lasting impression on him. She not only kept the school running but included her children in the decision-making process, something Seán believed fostered a key sense of responsibility and involvement in them from a young age. Seán recalled the challenges his mother faced, "*It was quite difficult for my mother to manage six children, be a full-time teacher, and manage a boys' school.*" (The school was incorporated into a new community school in 1979.)

Every summer, Seán and his siblings would help their mother maintain the school building, formerly an old fever hospital. Painting the school was an annual task, and this hands-on involvement taught the importance of community effort from a young age.

The family home was a large old Georgian house with an AGA cooker but lacking central heating and television. He remembered the practical and communal nature of their life, saying, "*The boys all wore short pants, and the socks were all knitted locally and did not, like nowadays, stay up. This necessitated the use of elastic garters, which, if too tight, left red marks on their legs. The struggle for the best seats by the fire led to the creation of a democratic rota system, an early lesson in fairness and organization.*"

Seán described life in Clonakilty at the time as simple and self-contained. The community was small and everyone knew everyone in town. The children organized their own society with its rules and seasonal activities, from marbles to chestnut collecting. Reflecting on this, Seán remarked, "*It seems that subconsciously we all knew, like salmon swimming up river, when we should do things.*"

The social structure of the town was pronounced, with clear class distinctions, and Seán and his peers accepted these as the natural order of things. Despite the small-town limitations, they found moments of excitement and curiosity: "*Religion got very intense during the mission by the Redemptorists from Limerick. Us kids would wait with bated breath for the personal visit by the missionaries to the so-called hard cases, those who were noted for not having been inside for 20 years or who were fond of the drop and we followed such sagas like people watch soap operas nowadays, would he go to mass, would he take the pledge, how long would it last?*"

In his writings, Seán recalled his upbringing with a blend of nostalgia and recognition of the profound impact it had on his life. His years in Clonakilty were marked by family support, a strong sense of community, and the challenges and joys of growing up in a small town in West Cork. These formative experiences shaped the man who would go on to have a distinguished career in mathematics.

## Primary and Second Level Education

Seán describes his father's early influence: "*When we children had childhood diseases such as measles and chicken-pox he would distract us by writing out pages and pages of sums and making some sort of game out of us children doing them.*" He also said, "*I'm not precisely sure when I became interested in mathematics but by the time I was twelve I was interested. The fact that I found mathematics honest and easy helped.*" Seán said of his primary school teacher, Con O'Ruairc. "*Mr. O'Ruairc had a unique way of teaching maths, especially mental arithmetic, which involved solving problems without pen and paper.*" Seán remarked, "*I found, then and ever since, that once one managed to firmly embed a mathematical idea or problem in the mind, it was effortlessly carried out and insight gradually appeared.*"

In primary school, Seán's sixth class classroom was shared with the "seventh class" students - those too young to legally leave school but who had already completed the official sixth class. This arrangement gave Seán an early exposure to algebra. He vividly recalled how Mr. O'Ruairc taught them to add, subtract, multiply, and divide polynomials.

After three years of second-level education at St. Mary's in Clonakilty, Seán's mother decided he should attend a boarding school to prepare for the Leaving Certificate, since St. Mary's then lacked a higher-level science teacher.

Seán went to Cistercian College Roscrea. Roscrea was a big shift for him. One of the most influential figures during this period was Fr. Emmanual, nicknamed "Rubber" for his frequent command to "*rub it out*" whenever student work was unsatisfactory. Seán's favourite subjects at Roscrea were mathematics, applied mathematics, and English. Fr. Emmanual introduced him to calculus, which he found fascinating and straightforward. Applied Mathematics, taught by Jack Murphy, and English, under Gus Martin (who later became Professor of English at UCD) also left lasting impressions on Seán.

## BSc and MSc at University College Cork

Seán entered UCC to study mathematics in 1961. He enjoyed his time there amongst a strong cohort of mathematicians. His influences included books by Cantor on his theory of transfinite numbers, and by Bromwich and Dienes on infinite series, and the excellent classroom notes on analysis given by Paddy Kennedy. He described a Hamilton-esque moment, probably shared by many mathematics students, when struggling with the notion of $\mathbb{R}^4$ "*...as I walked down the stairs I suddenly realised and almost fell as it hit me that dimension was a mathematical and not a physical concept and that mathematics had this tremendous freedom to use concepts anywhere and anyhow it wished, provided the rules of mathematics were followed*".

Seán continued in UCC for his MSc in 1964-1965 when he also acted as tutor to some of the undergraduate classes. He augmented his academic activities with organising student dances and acting as secretary to the student union. He had decided to do a PhD which, by default at that time, meant travelling abroad. Better financial support, and climate, meant that he accepted an offer to study at the University of Maryland.

The first week of September 1965 was a seminal one for Seán. He and several others from UCC travelled to Dublin to take the NUI exams for the MSc which also served as the examination for the travelling studentship. Staying in Gardiner Street, the roars from Croke Park for the All-Ireland hurling final on the Sunday could be heard as they tried to study for Monday's exams. This was trying for Seán who had lined out for the UCC hurling team earlier that summer, but it certainly would have been worse had Cork been playing.[1] At the end of that week, Seán travelled to America.

[1]Tipperary defeated Wexford in the 1965 final.

## PhD at Maryland, USA and Rio de Janeiro, Brazil

Seán's first night in the US in College Park, Maryland was spent in a police cell - after an officer took pity on him as he was too late to find somewhere else to sleep.

Maryland was a different scale to anything back home with some 250 graduate students in mathematics. The first two years of graduate training were mainly course work and reading, but there was a competitive element as well as students tried to secure a place with a thesis supervisor in their favoured area. Seán thrived academically, aided by the financial independence gained after being awarded the NUI Travelling Studentship in addition to his teaching assistantship at Maryland.

Seán met Carol Newbrough, a student of English and Education, at a friend's party, where they discovered that they lived in different apartments in the same house. They were married in December 1966 (and remained so until Carol passed away in 2021).

Seán was introduced to the work of Leopold Nachbin through a fellow overseas student, Tom Dwyer (who, in spite of the name, was Brazilian). Nachbin divided his time between his home institute, Instituto de Mathematica Pura e Applicada (IMPA) in Rio de Janeiro and the University of Rochester. Seán wrote to Nachbin saying he would like to work under his direction and would transfer to Rochester to facilitate this. Maryland professor, John Horvath, provided a letter of reference. Nachbin's suggestion was instead that Seán continue his registration in Maryland, with John Horvath as his official supervisor, but come to IMPA to work on his thesis. Nachbin was also able to arrange a teaching assistant position. This suited Seán except for the fact that his NUI Travelling Studentship precluded this employment. Seán dutifully wrote a letter to the NUI Senate explaining his intention to return home to Ireland to teach once his studies were finished and how the teaching assistant position would be of benefit in this regard. He eventually received in reply a formal typed letter advising that the Senate of the NUI had discussed the matter at great length and agreed that he could teach mathematics in Rio de Janeiro. Beneath the typed letter there was an unsigned scribbled note which said "*If this situation arises again, don't tell us about it.*" It is probably true that thereafter Seán took a slightly looser interpretation of academic rules and procedures.

Seán went to Rio in 1967 and began working under the guidance of Leopold Nachbin on infinite dimensional holomorphy, more specifically, on topologies on spaces of holomorphic functions over locally convex spaces. He returned to the US in February 1968 where his daughter Deirdre was born, after which the young family moved to Rio. Seán's PhD thesis entitled 'Holomorphy Types on a Banach Space' was submitted to the University of Maryland in June 1969 (incidentally his only publication under the name of John Hugh Dineen). He then spent the following year as an instructor at Johns Hopkins University in Baltimore.

## Dublin Institute for Advanced Studies

From 1970 to 1972, Seán worked at the Dublin Institute for Advanced Studies (DIAS) in the School of Theoretical Physics. He believed he landed the job because Professor McConnell thought his work on nuclear polynomials was related to Nuclear Physics, thanks to Grothendieck's terminology. DIAS, located then in Merrion Square, was an interesting and lively place. Seán recalled, "*The square itself was a private park in those days and we all had our own key to go in and play tennis and have our lunch there as we wished.*" During his time there, he interacted with staff, such as Synge, Lanczos, and O'Raifeartaigh.

## UCD in the 1970s

Seán joined UCD in 1972. Irish universities were transitioning to a modern research culture at the time and relied on the energy and expertise of new staff returning from PhDs abroad. Seán recalled, "*Maurice Kennedy felt that teaching was the top priority but also wanted research done.*" This led to evening seminars on operator theory with colleagues, followed by social gatherings at McCluskey's pub. These activities created a collaborative academic environment. In 1974, Seán was awarded a D.Sc. from the National University of Ireland for his published work.

Despite limited funding, Seán's connections with international mathematicians, like those from Nachbin's group in Brazil, helped to build a solid foundation for future growth in the department. One international collaborator wrote "*We were comrades in the early 70s when we struggled to become mathematicians researching in a new field and supporting each other by many letters and several meetings.*" Another colleague recalls "*I think I was most impressed by the numbers of visitors he had. This seemed to be entirely new for UCD. Trinity could offer accommodation to visitors... but finding money for visitors in UCD required Seán's talent.*" It was not only Seán and those in Analysis whose research programme benefitted from these visits. For example, Tom Laffey credits Seán for suggesting a swap arrangement whereby Tom Dwyer visited UCD from Northern Illinois University for a year, while Tom Laffey spent the year working with the strong algebra group in Illinois. Tom (Laffey) said he was delighted to avail of this opportunity and that it greatly helped his career.

Seán became a professor in 1979 and held the Chair in Analysis. The story of his appointment is a colourful one, which he told at his retirement. Suffice to say, the rule 'no canvassing' came shortly afterwards! Seán recalled, "*The whole process took 3 months, every day going to see people, making a lot of phone calls, and at the same time teaching*". He even remembered trying to explain his research to a veterinary professor who was giving a horse an injection at the time!

## UCD in the 1980s, 90s and beyond

During the 1980s Seán taught first-year, final year honours and MSc courses, including several engineering courses. He collaborated with Richard Timoney then at TCD on the topic of Bounded Symmetric Domains, a research interest sparked by a course (given by Wilhelm Kaup) that Seán had attended in Rio in 1978. This collaboration lasted for 12 years and produced 11 research papers. Seán was elected a member of the Royal Irish Academy in 1984.

In the 1990s, Seán contributed to developing the Erasmus exchange program in UCD, bringing many students from Europe to UCD. He noted that these students raised the standard and the atmosphere in the department. He also helped create the Mathematical Studies degree in Arts, which became important for teacher training. He was instrumental in ensuring that the Economics and Finance degree had a strong and rigorous mathematical content that challenged its students. His work in tailoring mathematics for these different student cohorts led to three undergraduate textbooks.

Seán served as Head of the Department of Mathematics at UCD from 1982 to 1986 and again from 1990 to 1994. He later served as Head of the School of Mathematical Sciences from 2006 to 2009. He also served on UCD's giverning authority where he fought hard for academic integrity in a changing university landscape. He retired in 2009 and this was marked with a conference attended by over 80 people from all over the world.

FIGURE 4. Seán's UCD retirement conference in 2009.

## RESEARCH ACHIEVEMENTS

Seán's research approach emphasized collaboration and support and while he would advocate strongly for his own area of research he expected others to do likewise. He asked one slightly confused new postgraduate in the department "*So, are you studying maths or algebra?*". Seán strongly believed in creating an expansive mathematical community, stating "*None of this is possible without the support and interest of local academic colleagues*".

In the 1970s and 80s Seán, with several of his academic colleagues, worked to deepen the mathematical research culture at UCD. He collaborated with and hosted many international scholars, from Brazil, Spain, the US, Japan and elsewhere. He prioritised the development of a PhD programme in the department. His efforts contributed to a very active research atmosphere despite limited resources. He emphasized, "*We had very little money here at the time, but you don't need very much to do mathematics*". "Very little" might well have been none were it not for a scheme of producing problem books containing exercises for all modules for sale to students.

Seán's research interests were broad, including Infinite Dimensional Holomorphy, Functional Analysis, Complex Analysis, Jordan Triple Systems, Spectral Theory, Geometry of Bounded Symmetric Domains, Probability Theory, Financial Mathematics, History of Mathematics, and Mathematical Education. He collaborated with 40 co-authors from all over the world. His 540 page book "*Complex Analysis on Infinite Dimensional Spaces*" in the Springer Monograph series is a standard reference in the area with currently over 650 citations.

Seán organized regular seminars and was adept at encouraging others to speak. With Richard Timoney he co-organised and ran the UCD-TCD Analysis seminar series for close to three decades. Seán himself gave 126 research seminars in Dublin in the period 1972-2015, even maintaining a list of all his seminar titles. He organised several major conferences in Dublin during his career. It was remarkable to discover his detailed records of a holomorphy meeting organised in Dublin in 1978 and the volume of handwritten letters exchanged in organising a meeting in those pre-digital days. The

FIGURE 5. Conference uniform in Fukuoka Japan, 1999.

advent of email meant that the next large meeting in 1994 was somewhat easier to organise, although Seán still drove the conference bus! UCD changed its policy on staff requisition of university vehicles shortly after that.

Between conferences and invited lectures he records his attendance at 209 events from 1969 to 2015, mostly abroad. He also gave 15 intensive research courses over the period 1971-2000 ranging in length from 3 to 15 lectures, including five different courses at Universidade Federal Do Rio de Janeiro, three in Coimbra, three in UCD and one each in Cork, Wuppertal, Madrid and Kent State.

FIGURE 6. Ray Ryan, Seán and Pauline Mellon in 2018.

**Publications.** Since 1970 Seán published over 120 research articles in international mathematics journals, working with 40 co-authors. These papers are detailed below in the appendix listing his published work. His most productive collaboration was with Richard Timoney with whom he published 11 papers. Poignantly, one of Seán's last mathematical publications was his obituary for Richard that appeared in Issue No. 83 of the Bulletin.

**PhD Students.** Seán supervised 11 Ph.D. students (and more than 20 MSc students):

Paul Berner (1974, University of Rochester, co-supervisor Leopoldo Nachbin)
Raymundo Alencar (1982, Universidade de São Paulo, co-supervisor Jorge Mujica)
Thomas Barton (1984, Kent State University, co-supervisor Joe Diestel)
Fergus Gaughran (1990, UCD)
Pauline Mellon (1990, UCD)
Christopher Boyd (1992, UCD)
Ciaran Taylor (2000, UCD)
Milena Venkova (2001, UCD)
Pablo Sevilla Peris (2001, Universitat de Valecia, co-supervisor Domingo Garcia Rodriguez and Manuel Maestre Vera)
Adriano Lima Aguiar (2003, UFRJ, co-supervisor Luiza Moraes)
Cristina Radu (2008, UCD).

**Books.**

*Complex Analysis on Locally Convex Spaces*, North Holland Mathematical Studies Vol. 57 (1981)
*The Schwarz Lemma*, Oxford University Press (1989)
*Functions of Two Variables*, Chapman and Hall (1995)
*Multivariate Calculus and Geometry*, Springer Verlag (1998)
*Complex Analysis on Infinite Dimensional Spaces*, Springer Verlag Monographs in Mathematics (1999)
*Probability Theory in Finance—A Mathematical Guide to the Black-Scholes Formula*, American Mathematical Society (2005)
Co-edited *Vector Space Measures and Applications I and II*, Springer Verlag Lecture Notes in Mathematics (1977)
Co-edited *Functional Analysis Proceedings of the First Trier Workshop on Frechet Spaces*, Functional Analysis de Gruyter (1994)

In addition to the texts above on mathematics, Seán also published *Tres problemas en Analysis Infinite Dimensional*, Publicaciones del Universidad de Santiago de Compostela, Spain (1979). He also self-published two non-mathematical texts centered around Clonakilty; one "*Stones in the Wood*" was a book of (mostly) fictional short stories while "*The Texture of West Cork*" described the history, make-up and uniqueness of the place he called home.

## Teaching

Seán believed that first-year teaching sets the standard for later years and helps students develop perseverance. He stated, "*We have students who become enthused by maths when they are pushed.*" One example he shared was an economics and finance student who became so engaged with maths in first year that he pursued a higher diploma in the subject. Seán also highlighted the importance of communicating effectively, recalling his advisor, Nachbin's words: "*You could prove Fermat's Last Theorem* *in your bathroom, but if you didn't tell anyone, nobody would know. You have to communicate it.*" His approach was to challenge students and foster a genuine interest in mathematics. He was generous with his time and supportive of students at any level who were struggling with mathematics. He happily gave his time to children of friends and neighbours who needed help in their school mathematics and this led to an interest in the secondary level mathematics curriculum and serving on a national committee for its review. He was an active and highly effective lobbyist for the introduction of bonus points for mathematics and changed the minds of several politicians and university administrators on this issue. He was awarded a President's Teaching Award from UCD in 2002.

A defining characteristic of Seán's approach was that he was interested in all students, unlike many mathematicians interested only in stronger students. Students seemed able to sniff out his innate sense of decency and he advocated for many who found themselves academically tripped up for whatever reason. When Irish comedian Dara Ó'Briain was honoured as the recipient of the 2021 Maths Week Ireland Award for his contribution to raising public awareness of maths, he dedicated the award to Seán. It might tickle Dara to know that (years prior to Dara's gesture) as a minor act of protest at the commercial nature of the university naming its new buildings after wealthy sponsors, Seán, in one of his research articles, deliberately gave his address as the Dara Ó'Briain Centre for Science rather than the (Denis) O'Brien Centre for Science. Seán had a chuckle showing us that. (See Studia Mathematica 222 (1) (2014) 'Distances between Hilbertian operator spaces' by Seán Dineen (Dublin) and Cristina Radu (Rio de Janeiro).)

While Seán was renowned for his mathematics across the world, any chat with mathematicians who had visited here inevitably steered towards stories of his driving, or his smoking, or his smoking while driving. He was unfailingly generous though in driving his increasingly apprehensive mathematical visitors to and from the airport. Indeed one who came for his funeral mentioned that Seán had collected and dropped him back to the airport on something like 17 visits. Seán had a flexible interpretation of most rules of the road (including one-way road signs) and considered several feet out from the footpath a perfectly good place to park. One former student wrote on his passing that "*Seán encouraged and supported my PhD work with wit, deep insight, and friendship. He even loaned me his car on one occasion, my first ever attempt to drive on the left side of the road (it was not entirely disastrous). I recall his advice to me about it: you only need to watch out for what is in front of you. You'll be to the front of other drivers who are behind you or to your side, so no need to worry about them! Logical, a bit irreverent, and a reminder to not take things too seriously. That was just like him.*" Seán didn't always call it right though. He was the last to hold out against UCD's no-smoking rules introduced in the 1990s, to the particular chagrin of David Tipple in the maths department who was vehemently anti-smoking. When caught smoking once again by an irate UCD porter who demanded his name, Sean mischievously answered "I'm David Tipple".

Shortly before he died, Seán described himself as a problem-solver. That was certainly true in the realm of pure mathematics but it was also true for many other aspects of his life. On the wet blustery day of Seán's funeral in Clonakilty, a woman told the story of how some years earlier Seán had 'gotten' a mortgage for her, when even she admitted that no sensible institution would consider lending to her, by arranging that she meet a "*fellow in a pub one evening*" (the fellow being a former student of Seán's who worked in banking). That sense of community that Seán attributed to his childhood growing up in 'Clon' was something that he cultivated to full effect in both his mathematical and his non-mathematical life.

Ní fheicfimid a leithéid arís. Ar dheis Dé go raibh a hanam.

## Published Work of Seán Dineen


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# Donal Patrick O'Donovan 1945–2019

ROBIN HARTE, DAVID MALONE AND ELIZABETH OLDHAM

Donal O'Donovan.

## 1. Overview

Donal O'Donovan was born in Dublin on the 31st of July 1945. He grew up in Mount Merrion in Dublin and attended St Mary's Boys National School on Haddington Road; he then moved to boarding school in Roscrea. He completed the B.Sc. in mathematics in UCD in 1966, and followed on to do the Master's in 1967. After a brief appointment as a research assistant in statistics in TCD in 1967, having received a National University of Ireland Traveling Studentship, he moved to the University of California, Berkeley to do a PhD in 1968 [18]. At Berkeley he worked with William Arveson on operator algebras, graduating in 1973 with a thesis titled *Weighted Shifts and Covariance Algebras*.

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While a student there, he acted as a teaching associate and instructor. After graduation, he went to Stony Brook University, New York as an instructor in 1973, before moving in 1974 to Dalhousie University, Canada. He returned to a temporary lecturing job in TCD in 1976 and secured a permanent position in 1979 in the Department of Pure Mathematics. This was around the time of Victor Graham formally finishing as a part-time lecturer in the department; Donal's arrival was between that of familiar names such as Richard Aron and Richard Timoney.

Those who met Donal know that he was approachable and friendly, and he conveyed that in lectures, sometimes sharing classic mathematical jokes such as "What's purple and commutes?" He had an interactive style of lecturing that was probably somewhat ahead of its time. For students in their first year in college, he made a point of helping them come to terms with university mathematics, in particular with the way in which its focus differs from that typically experienced in schools — a culture shock for students who chose the subject for the pleasure of doing exercises and solving problems rather than engaging with the theoretical aspects. At one point he resorted to offering bonus examination marks for first-years students to encourage them to engage in class discussions such as "What does $\mathbb{R}[X]/\langle x^2+1\rangle$ look like?" He was also quite unflappable, and had no trouble dealing with larger (occasionally high-spirited) groups, such as the first year engineers.

Donal became a senior lecturer in 2002, and served as the School's director of teaching and learning from 2005 to 2007, getting involved with university-level initiatives such as the Working Group on Interdisciplinary and Service Teaching. He began a second term in the same role, but took up the position of Head of the School of Mathematics in 2008. While Head of School, he introduced a year-long module on Mathematics Education as an option for third- and fourth-year undergraduate mathematics students. A key aspect of the module design was that, after attending lectures in the first semester, the undergraduates would spend time in classrooms acting as assistants to teachers. Donal's vision was that the scheme would make a contribution to Irish society by helping to enhance the learning of mathematics, ideally or especially in areas of disadvantage, while also giving undergraduate students a chance to encounter classrooms "from the other side" and perhaps think about teaching as a career [8, 9]. He invited Elizabeth Oldham, who was on the verge of retirement, to teach the lecture course[1]; initially, he himself arranged the classroom placements, though later many students were able to find convenient ones themselves. The classroom element broadened over the years, with some students helping in homework clubs, some working in the Maths Help Room, and some tutoring for the Trinity Access Programme, while a few have supported patients in hospital — all areas of interest and concern to Donal.

While dealing with the administrative jobs of the school, he was also happy to stay connected with the students. He took on the (ceremonial) position of President of the student Mathsoc. He also addressed the University Philosophical Society's "Liferaft Debate", to convince them that it would be worth bringing a mathematician on a liferaft to a new world. He argued that mathematics could use abstraction and rigour while also solving practical problems. This made it an indispensable tool, but also an art in its own right.

He remained Head of School for a little over one term, until retiring in 2012. He remained an adjunct member of the department after his retirement.

[1]She undertook to do it on a one-off basis and is finally passing the baton to others after fourteen iterations, a measure of how enjoyable she found it.

## 2. Tutor

The *tutor* role, for academic staff in TCD, involves being an advisor to students on their interactions with the college and helping with their general wellbeing. Donal's approachable and sympathetic nature made him a good tutor. Students remember him providing the right mix of support, encouragement and, where necessary, sceptical looks to help them get through their degrees. For example, Paul McNicholas (now at McMaster University), described his experience as follows.

> Donal was a dedicated and generous tutor who routinely went the extra mile to ensure that his tutees were taken care of. I fondly remember the radio playing in his office when we would meet but, mostly, I remember his miraculous ability to provide hope in situations where there had seemed to be none. Through his deeds, he profoundly improved my life and I have no doubt that the same is true for many others. Donal was a champion for underprivileged students and had a passion to help them succeed. Although none of us will ever know the extent to which he helped students in need, his support for initiatives such as the Maths Help Room and the Trinity Access Programmes speaks volumes.

Another former student, Suzanne Wylde, also remembers Donal's support fondly.

> Donal was my Tutor (and Lecturer) in TCD in the late 80s and he was amazing. He recognised, encouraged and personally connected with students who were incredibly talented *and* those who struggled academically and with our personal lives. He went out of his way to assist us in his quiet, unassuming, caring and positive way. For some of us, he was life changing. He put us on the path to success and happiness. I am forever grateful and indebted to him. I still think of him often. Thank you Donal, you truly were a special Tutor.

These are just two of many students that Donal helped negotiate the sometimes strange world of higher education.

## 3. Research

Donal's pure mathematical interests were around operator algebras, and $C^*$ algebras in particular. Consequently, he was a supporter of Trevor West's *Thursday evening seminar*, with other regulars such as Richard Timoney, Mícheál Ó Searcóid and Robin Harte. Much of his work arose from finding clever embeddings to understand operators or giving explicit examples to help better understand more abstract work.

Maybe his best-known paper is *Weighted Shifts and Covariance Algebras* [10], where he considers the $C^*$ algebra generated by a weighted shift on a Hilbert space, as a subalgebra of the bounded operators on that space. He shows that, under some conditions on the weighted shift, the subalgebra will be isomorphic to a covariance algebra. This allows results from the study of covariance algebras to be used to understand representations of the subalgebra.

His paper, *A Tale of Three $C^*$ Algebras* [3] even came to the attention of the undergraduates, due to its clever title. It tells the story of good and bad $C^*$ algebras and even gives a link between the algebras and continued fraction approximations to irrational numbers. It was not just the title of the paper that was clever, with George A. Elliott describing the embeddings used as ingenious.

As mentioned above, Donal also had a strong interest in education, publishing the results of a survey of graduates conducted with Richard Timoney [15]. Following on from his later introduction of the Mathematics Education module, with its component of classroom experience, he published with Elizabeth and one of the students in this area [8, 9].

## 4. Personal Life

Donal was very sociable. He enjoyed and was good at organising events, like dinners and concert trips, both at work and with friends and his family, Joan, Caoimhe and Róisín. He loved traveling, cooking, gardening, reading, listening to music, and going to concerts.

Donal passed away on November $9^{\text{th}}$ 2019 in the Blackrock Clinic surrounded by his family. His funeral was at St. Mary's Church, Haddington Road and included an entertaining eulogy detailing some of Donal's many non-mathematical exploits.

## Acknowledgments

We would like to thank Donal's family, particularly Joan, Caoimhe and Róisín, for providing details that we were missing and offering feedback.

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**Robin Harte** taught for twenty years at University College, Cork, and retired a long time ago, but somehow never goes away; his ghost can sometimes be seen at the TCD-UCD Analysis Seminar.

**David Malone** received B.A. (mod), M.Sc. and Ph.D. degrees in mathematics from Trinity College Dublin. He is a professor at Hamilton Institute and Department of Mathematics & Statistics, Maynooth University.

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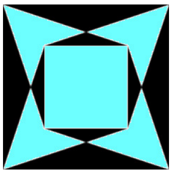

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# Seán Tobin, 1930-2023

JOHN MCDERMOTT AND MARTIN NEWELL

Seán Tobin

Seán was born in Dublin on the 16th of January, 1930. His family moved to Carrick-on-Suir in Tipperary, where he received most of his schooling from the Christian Brothers. He spent the final two years of secondary school in New Ross, as a boarder at the

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Irish-speaking A-school run by the Augustinians. As a result of his excellent performance in the Leaving Certificate examination in 1947 he was granted one of the new and prestigious Aiken scholarships to study through Irish in UCG. Following a very successful undergraduate career he enrolled for the MSc in Mathematical Science, and he graduated wih first-class honours in 1951. His performance was such that the college awarded him a special grant to enable him to work for a doctorate at the University of Manchester. His supervisor there was the eminent group-theorist Graham Higman who later became Waynflete Professor of Mathematics at Oxford. Seán received his PhD in 1954 for his thesis on groups of exponent four. And, to quote himself, 'even more importantly, in Manchester I met Lois Brown, a Yorkshire girl then studying at the College of Domestic Science, who later became my wife'.

Having spent a post-doctoral year (1954-5) in Ohio working with the renowned group-theorist Marshall Hall, Seán returned to UCG as an Assistant in Mathematics. The following year he was appointed to a Statutory Lectureship and was elected to the Academic Council. He became the Professor of Mathematics and head of the department in 1961, and held and enjoyed both positions until his retirement in 1995.

Many warm tributes have been paid by former colleagues and students, in appreciation of the encouragement, support and advice that he gave them. Seán loved lecturing, and despite the extremely heavy teaching loads that were common at the time he was able to produce courses that challenged and inspired his students.

Bhí dúil daingean diongbháilte i gcónaí aige sa Ghaeilge, mar go bhfuair sé a chuid scolaíochta féin tré Ghaeilge, seans. Bhí sé in a bhall den Coiste Gnó as a d'fhás an Foclóir Eolaíochta, foilsithe ag an Gúm. Thug sé tacaíocht ar leith don nós lena linn léachtaí Matamaitice san árd-chúrsa a chur ar fáil tré mheán na Gaeilge. Bhí sé de phribhléid againn beirt – agus ag a lán lán eile – céim a bhaint amach faoin a chúram agus a threoir.

The main focus of Seán's research throughout his career was on groups of exponent four. We recall that the group $G$ has exponent $m$ if $x^m = 1$ for all $x$ in $G$ and $m$ is minimal with this property. If such a group has $n$ generators then it is a homomorphic image of $B(n, m) = F/F^m$. Here $F$ is the free group of rank $n$ and $F^m$ is the subgroup generated by all the $m$th powers of elements of $F$. $B(n, m)$ is the Burnside group of rank $n$ and exponent $m$. The name and notation[1] reflect the fact that in a 1902 paper Burnside posed the following question (later called the Burnside problem): for which $n$ and $m$ is $B(n, m)$ finite? He also asked for the order when the group is finite.

In the 1902 paper Burnside noted that $B(n, 2)$ is elementary abelian of order $2^n$ and he showed that $B(n, 3)$ is finite, giving an upper bound for its order. He also proved that $B(2, 4)$ is finite and claimed that its order is $2^{12}$, but his argument only showed that this is a multiple of the order.

In 1933 Levi and van der Waerden found the exact order of $B(n, 3)$, and in 1940 Sanov proved that $B(n, 4)$ is finite for all values of $n$.

In his thesis (1954) Seán studied the structure, the derived length and the class of groups of exponent four. In particular, he found an unexpected connection between solvability and nilpotency for such groups. Here is a sample consequence :

*A group $G$ of exponent four is solvable only if $G^2$ is nilpotent.*

He also gave a construction for $B(2, 4)$ and showed that its order is indeed $2^{12}$. Furthermore, he corrected a (mis)calculation that had been made by Philip Hall.

Seán published several papers on groups of exponent four and related topics. His expertise was acknowledged by an invitation to give a course of lectures at the Groups–St Andrews 1981 conference. The corresponding survey article [6] was a major contribution

[1] *Caveat lector:* some authors use $B(m, n)$ for our $B(n, m)$!

in the field, including as it did many new results as well as significant improvements on older ones. Seán's research certainly sparked renewed interest in the use of commutator calculations and their application in combinatorial group theory.

Other contributions to the study of the Burnside problem include, for example, the following theorem of Lysenok (1996) : *$B(2,n)$ is infinite if $n > 8000$.* A nice short account of such contributions is given in a paper by M.F.Newman in Volume I of The Collected Papers of William Burnside (edited by Peter M. Neumann et al, OUP 2004).

Seán was always keen to extend the research profile of the Mathematics Department, initially in algebra (particularly group theory) and later in other areas. He instituted a regular research seminar, and the limited seminar budget was more productive than one might expect because he and Lois frequently hosted the visiting speaker in their home. Seán always encouraged colleagues to take sabbatical leave when it was due. He supported the many mathematical conferences held in UCG, notably a major international meeting on 'Group Theory and Computing' in 1973 and the 'Galway–St Andrews International Group Theory Conference' in 1993.

Seán was an active member of many academic bodies both within the college and nationally. He served several terms on the Governing Body of UCG, and was also a member of the Governing Body of the School of Theoretical Physics at the Dublin Institute of Advanced Studies. He regularly attended the Mathematical Science meetings held in DIAS every Christmas and Easter, and was a long-time member and supporter of the Irish Mathematical Society. Moreover, he was a very active member of the Irish Federation of University Teachers throughout his career and well into his retirement.

Seán and Lois were wonderful hosts and great company. They were also inveterate travellers: in addition to spending sabbaticals in several parts of Europe and North America, they visited family in England, Peru and Australia. And they continued to travel after Seán retired, making a memorable journey along the Silk Road and taking several trips with the Retired Staff Association.

Seán died on the 11th of July, 2023. Rinne sé éacht ina shaol.

Several parts of this obituary are informed by the splendid essay [9] that Seán wrote as part of a history of what is now University of Galway.

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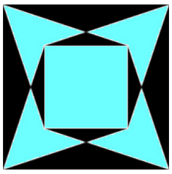

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# Some Aspects of Brendan Goldsmith's Contributions to Abelian Groups

PETER DANCHEV

*Dedicated to Brendan Goldsmith on the occasion of his* **75***th birthday )*

Abstract. We give a brief overview of the contribution to Abelian group theory by Brendan Goldsmith on the occasion of his 75th birthday.

## 1. Some Short Preliminary Historical Details

The well-known mathematician Brendan Goldsmith was born in Belfast, Ireland, on 22 January 1949. He attended Queen's University and later Oxford University from where he graduated with a D. Phil. in Mathematics in 1978 under the supervision of Professor Anthony L. S. Corner from Oxford writing a dissertation entitled "*An Investigation in Abelian Group Theory*". His thesis was concerned with realising endomorphism rings of certain classes of Abelian groups and $p$-adic modules modulo the ideal of so-called *inessential endomorphisms*, an approach which became important in subsequent work of R. Göbel and others on Abelian groups.

He was appointed in 1974 to a lectureship at the College of Technology, Kevin Street, and worked for ten years in that rôle. During this period he was a founding member of the Irish Mathematical Society and subsequently served as President of the Society. In 1983 he was appointed Head of School of Mathematics and introduced new programmes in mathematics and computer science. Ten year later, in late 1993, he was appointed President of DIT, becoming the first president of the new institute. In that same year he worked with Tom Laffey and the late David Simms on the organisation for the Royal Irish Academy of a conference to celebrate the sesqui-centennial of Hamilton's discovery of quaternions; the special issue of the Academy's Proceedings dedicated to the conference was edited by the three of them.

He often recalls the great challenges he experienced as the new president of DIT in those early days and, in particular, the difficulty in acquiring the new campus at Grangegorman; the campus is now home to a central part of the recently established Technological University, Dublin and Goldsmith presently continues to work from there as an emeritus professor. According to the existing current on-line databases, Professor Goldsmith has 6 students and 9 descendants. They are as follows: one from Trinity College (Dublin), three from Dublin Institute of Technology (DIT) and two from University of Duisburg-Essen, Germany, respectively; the latter two students had Goldsmith as external examiner for their doctorates.

Goldsmith's scientific interests are mainly in the areas of Abelian groups and modules over (possibly commutative) rings. He has published widely in journals including Proceedings and Transactions of AMS, Journal and Bulletin of LMS, Quarterly

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J. Mathematics, J. Algebra, J. Pure and Applied Algebra, Commun. Algebra, Forum Mathematicum, Mediterranean J. Math., J. Physics A, J. Commut. Algebra, Illinois J. Math., Archiv Math., J. Group Theory, Rendiconti Sem. Mat. Univ. Padova and, of course, IMS Bulletin. He has also been an editor of several books in these areas: Models, Modules and Abelian Groups (de Gruyter 2008) - a volume in memory of A.L.S. Corner; Groups, Modules and Model Theory; Surveys and Recent Developments (Springer 2017) - a volume in memory of R. Göbel; a guest editor of a special volume honoured to the 95th birthday of László Fuchs in the Rend. Sem. Mat. Univ. Padova **144** (2020); and he is also currently involved as a guest editor in the production of two issues of the Journal of Commutative Algebra to celebrate the forthcoming 100th birthday of László Fuchs in June 2024.

An interesting feature of Goldsmith's research work has been his extensive collaborations: to date he has some 30 collaborators, including multiple collaborations with myself, R. Göbel, A.L.S. Corner, L. Salce, L. Strüngmann, P. Vámos and P. Zanardo.

## 2. Recent Papers on Abelian Groups and Modules

Here I will concentrate initially on the joint cooperation between Goldsmith and myself. Our collaboration began in 2009. To date we have succeeded to write and publish jointly ten very interesting papers in prestigious international journals and our joint work is still active (see, e.g., [11] and [12]) by involving two other well-established specialists in Abelian groups like Andrey R. Chekhlov from Tomsk State University of Russia and Patrick W. Keef from Whitman College in Walla Walla, WA, United States.

(1) In a series of papers [1], [2] and [3], we introduced the notions of *socle-regularity* and, respectively, *strong socle-regularity* for Abelian $p$-groups. The notions arise because of the difficulty in describing completely the fully invariant and characteristic subgroups of an arbitrary $p$-group; we restricted attention to consideration of the socle of groups (i.e., the subgroup of elements of order $p$). An interesting outcome was the surprising connection between the two notions: a group $G$ is socle-regular if, and only if, its square $G \oplus G$ is strongly socle-regular; the result has a clear resemblance to an earlier result of Files and Goldsmith published in PAMS solving an old problem of Kaplansky on the connection between the concepts of transitivity and full transitivity.

(2) In paper [4] we used a similar approach replacing fully invariant subgroups with the well-known projection-invariant subgroups. Paper [5] concerns strengthening the classical concept of fully transitive group to groups which we named *projectively fully transitive* Abelian $p$-groups. The final paper in this initial period of collaboration works in the same vein but focuses on subgroups $C$ which are *commutator invariant* in the sense that $f(C) \leq C$ for all endomorphisms $f$ of the form $f = [\phi, \psi] = \phi\psi - \psi\phi$ for endomorphisms $\phi, \psi$.

After a reloaded break of seven years, we renewed our scientific collaboration with a series of successful papers in another two areas of Abelian group theory; these were written together with the Russian algebraist A. R. Chekhlov.

(4) Common generalizations to the standard notions of fully invariant and characteristic subgroups are, respectively, fully inert and characteristically inert subgroups and the latter ones allow us to define in [7] and [8] *fully inert socle-regular* and *characteristically inert socle-regular* Abelian $p$-groups. They are, certainly, closely related to the aforementioned articles [1] and [2], being their natural expansions.

(5) In our three further papers [9], [10] and [11], we study arbitrary (possibly mixed) Abelian groups which are devoted to the classification of *Bassian* and *generalized Bassian* groups as well as of some their derivations. Unfortunately, we were able to

classify only Bassian groups and some large types of generalized Bassian groups conjecturing that the second kind is the direct sum of a Bassian group and an elementary group. This problem seems to be quite difficult and is **not** answered yet.

During this period, Goldsmith also had a number of other important collaborations. In particular, he and Luigi Salce have published a series of interesting papers relating to Abelian $p$-groups revisiting and progressing ideas going back to Pierce's fundamental work in this area. He has also collaborated with his former student, Ketao Gong, on ideas relating to quotient and cyclic subgroup transitivity. He has also worked with his student, Noel White, on ideas deriving from an old unpublished work of Corner; for instance, one of these has appeared recently in this Bulletin.

Goldsmith's most recent publication in Forum Mathematicum has just appeared in February 2024 and is a joint work with Fuchs, Salce and Strüngmann on *Cellular Covers of Modules over Valuation Domains*. As an aside, the joint age of the authors is over 300, perhaps giving the lie to the notion that mathematical research is only for the young!

In conclusion, I personally would like to emphasize that I really have had the personal privilege to enjoy working with this well-known algebraist as the expert work with him considerably influenced on my own scientific interests definitely!

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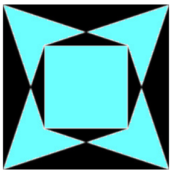

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# The genesis of a conjecture in number theory

ROBERT HEFFERNAN AND DES MACHALE

Abstract. We discuss how a knowledge of commutativity in finite groups and conjugacy classes in the symmetric group leads to a conjecture in number theory.

Conjectures play an important role in the development of mathematics at all levels and it is sometimes a mystery where they come from. The example to follow convinces us that some conjectures at least come from speculation, experimentation and the invaluable practice of examining as much numerical evidence as you can lay your hands on. First, some background.

If $n$ is a natural number, the (integer) partition function $p(n)$ is the total number of ways of writing $n$ as the sum of natural numbers, without regard to order. For example, since

$$5 = 4+1 = 3+2 = 3+1+1 = 2+2+1 = 2+1+1+1 = 1+1+1+1+1,$$

we have $p(5) = 7$. A great deal is known about the function $p(n)$ but there are still many unanswered questions; for example, we do not know when $p(n)$ is odd or even. Here is a table of values of $p(n)$ for small $n$:

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $p(n)$ | 1 | 2 | 3 | 5 | 7 | 11 | 15 | 22 | 30 | 42 | 56 | 77 | 101 | 135 | 176 | 231 |

We note that $p(n)$ increases rather quickly; for example $p(50) = 204226$ and $p(100) = 190569292$.

Our aim is to motivate the following conjecture:

**Conjecture 1.** *The values of $n$ for which $p(n)$ divides $n!$ all belong to the finite set*

$$S = \{1, 2, 3, 7, 9, 10, 11, 12, 14, 15, 16, 17, 18, 19, 20, 21, 24, 28, 32, 33, 39\}.$$

The symmetric group $S_n$ is the group of all permutations on the symbols $\{1, 2, 3, \cdots, n\}$. $S_n$ has order $n!$ and is a non-commutative group for $n \geq 3$. It is well-known that $S_n$ has exactly $p(n)$ conjugacy classes and since a group $G$ is commutative if and only if it has $|G|$ conjugacy classes, we immediately get

**Theorem 1.** *For $n \geq 3$, $p(n) < n!$.*

This is a crude bound, much weaker than best possible, but a proof involving number theory alone might be quite tricky, and the reader is invited to find one.

In general, if $G$ is a finite group with exactly $k(G)$ conjugacy classes, we form the ratio

$$\Pr(G) = \frac{k(G)}{|G|},$$

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so that $\Pr(G) = 1$ if and only if $G$ is a commutative group. Note that $\Pr(G)$ is the probability that two elements of the finite group $G$, selected at random with replacement, commute. There is an extensive literature on $\Pr(G)$. (See [2], [5], [3] and [1]). In particular, we have the following results which are of relevance here

**Result 1.** *If $G$ is a non-commutative group, then* $\Pr(G) \leq \frac{5}{8}$.

**Result 2.** *If $H$ is a subgroup of $G$, then* $\Pr(G) \leq \Pr(H)$.

**Result 3.** *If $G$ is an insoluble group, then* $\Pr(G) \leq \frac{1}{12}$.

Applying Result 1 to $S_n$ we get

**Theorem 2.** *For $n \geq 3$, $p(n) \leq \left(\frac{5}{8}\right) n!$.*

Since $S_3$ is a subgroup of $S_n$ for each $n \geq 3$, applying Result 2 we see that, for each $n \geq 3$, $\Pr(S_n) \leq \Pr(S_3) = \frac{1}{2}$. This gives the following improvement on Theorem 2:

**Theorem 3.** *For $n \geq 3$, $p(n) \leq \left(\frac{1}{2}\right) n!$.*

Since, for $n > 1$, $S_{n+1}$ has a subgroup isomorphic to $S_n$, by Result 2 we have $\Pr(S_{n+1}) \leq \Pr(S_n)$, so that

$$\frac{p(n+1)}{(n+1)!} < \frac{p(n)}{n!}.$$

Thus we get

**Theorem 4.** *For $n \geq 2$, $p(n+1) < (n+1)p(n)$.*

Since for $n \geq 5$, $S_n$ is an insoluble group, we have, by Result 3,

**Theorem 5.** *For $n \geq 5$, $p(n) \leq \left(\frac{1}{12}\right) n!$.*

Many other such results are easily deduced. For example we have:

**Theorem 6.** (1) *For $n \geq 4$, $p(n) < \left(\frac{5}{24}\right) n!$.*
(2) *For $n \geq 5$, $p(n) < \left(\frac{7}{20}\right) n!$.*
(3) *For $n \geq t$, $p(n) < \frac{p(t)n!}{t!}$.*

Incidentally, we know of no purely number theoretic solutions to the following pretty problems in number theory, but there are easy solutions using the properties of the symmetric group.

**Problem 1.** *Show that for each $n$, $n!$ can be written as*

$$n! = \sum_{i=1}^{p(n)} c_i,$$

*where $p(n)$ is the partition function and each $c_i$ is a positive integer divisor of $n!$.*

*Solution.* Just take the class equation of $S_n$ which has $p(n)$ conjugacy classes each of whose cardinalities is a divisor of $n!$. Thus $6 = 1+2+3$, $24 = 1+3+6+6+8$, etc. These representations are clearly not unique, since $6 = 2+2+2$ and $24 = 2+4+6+6+6$, etc. □

**Problem 2.** *Show that, for each $n$, $n!$ can be written as*

$$n! = \sum_{i=1}^{p(n)} d_i^2$$

*where $p(n)$ is the partition function and each $d_i$ is a positive integer divisor of $n!$.*

*Solution.* Here we use the degree equation of a group $G$, which states that $|G|$ is the sum of the squares of the degrees of the irreducible complex matrix representations of $G$, which are $k(G)$ in number, i.e.

$$|G| = \sum_{i=1}^{k(G)} d_i^2$$

and each $d_i$ is a divisor of $|G|$. Applying this result to $S_n$ we get

$$n! = \sum_{i=1}^{p(n)} d_i^2$$

as desired. □

Thus $6 = 1^2 + 1^2 + 2^2$, $24 = 1^2 + 1^2 + 2^2 + 3^2 + 3^2$, etc.

And now to our number-theoretic conjecture. Originally, we were trying to prove that for each $t \in \mathbb{N}$ one can find a group $G_t$ with $\Pr(G_t) = \frac{1}{t}$. (Actually, this turns out to be rather easy using direct products of dihedral groups and is left as an exercise for the reader). Looking at $S_n$, we were struck by the number of instances for small $n$ where this phenomenon occurred. The number-theoretic version of this is, of course, "find all $n$ for which $p(n)$ divides $n!$".

One can easily work out the first few values of $n$ for which this happens. They are

$$1, 2, 3, 7, 9, 10, 11, 12, 14, 15, 16, 17, 18, 19, 20, 21, 24, 28, 32, 33, 39, \cdots .$$

This is now sequence A046668 of Sloane's Online Encyclopedia of Integer Sequences [4] but we could not find any more terms greater than 39.

On July 6th, 2018, Vaclav Kotesovec posted the following on [4] after extensive computer calculations: the next term, if it exists, is greater than 2000000.

Hence we are led to formulate Conjecture 1. Currently we do not have a proof of this conjecture but would be pleased to hear from anyone who does.

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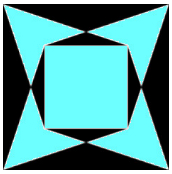

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# A note on a class of Fourier transforms

J.B. TWOMEY

Abstract. We consider functions $f \in L^2(\mathbf{R}^n)$ for which

$$\int_{\mathbf{R}^n} |\hat{f}(t)|^2 (1 + \log^+ |t|)^{2\beta} dt < \infty,\ \beta > 0,$$

where $\hat{f}$ is the Fourier transform of $f$, and we identify a kernel $\mathcal{K}_\beta$ such that $f$ satisfies this integral condition if, and only if,

$$f(x) = (\mathcal{K}_\beta * F)(x) = \int_{R^n} \mathcal{K}_\beta(x - t)\, F(t)\, dt$$

for some function $F \in L^2(\mathbf{R}^n)$. We also address the question of 'Fourier inversion' for this class by showing that certain Bochner-Riesz means of the transforms of $f = \mathcal{K}_\beta * F$ converge to $f$ outside small exceptional sets of points in $\mathbf{R}^n$ of capacity zero.

## 1. Introduction

It was conjectured by Lusin in 1915 that the Fourier series of a periodic function $f \in L^2(-\pi, \pi)$ converges almost everywhere, that is, if $c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-ikx} f(x) dx, k \in \mathbf{Z}$, denote the Fourier coefficients of $f$, then the partial sums

$$s_n(f)(x) = \sum_{k=-n}^{k=n} c_k e^{ikx}$$

converge almost everywhere to $f(x)$ as $n \to \infty$. The conjecture remained unproven for several decades and, as doubts began to arise regarding its veracity, some research was directed towards constructing a counterexample. It came as a major surprise therefore when, in a famous and very difficult paper [2], Lennart Carleson proved Lusin's conjecture in 1966. This result was widely celebrated within mathematics and particularly, perhaps, by those analysts who (like this author) had been nurtured mathematically on Zygmund's *Trigonometric Series*! Carleson's result was extended to $L^p$ functions, $p > 1$, by Hunt [6].

We are concerned with Fourier transforms, and the question that arises in this context is whether Carleson's result has an analogue in $\mathbf{R}^n$, specifically whether for a function $f \in L^2(\mathbf{R}^n)$, the spherical partial integral

$$S_R f(x) = \int_{|t| \le R} \hat{f}(t) \exp(2\pi i x{\cdot}t) dt,\ R > 0,\ x \in \mathbf{R}^n,\ n \ge 2, \tag{1}$$

converges almost everywhere to $f(x)$ in $\mathbf{R}^n$ as $R \to \infty$, where $\hat{f}$ is the Fourier transform of $f$. This question remains open, but by analogy with partial results established for Fourier series prior to Carleson's paper (see [14, 1.13, p. 163]), it is natural to begin

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seeking answers by investigating functions $f$ which satisfy conditions such as

$$\int_{\mathbf{R}^n} |\hat{f}(t)|^2 (1+\log^+|t|)^{2\beta} dt < \infty, \ \beta > 0, \tag{2}$$

a stronger requirement than $f \in L^2(\mathbf{R}^n)$. We note that it has been shown by Carberry and Soria [4] (see also [5]) that if $f$ satisfies (2) with $\beta = 1$ then $S_R(f) \to f$, as $R \to \infty$, almost everywhere. We focus in this article on providing a characterisation of functions $f$ for which (2) holds and, to this end, we define a kernel $\mathcal{K}_\beta$ which, as we shall prove in section 2, has the property that $f$ satisfies (2) if, and only if, $f = \mathcal{K}_\beta * F$ for some $F \in L^2(\mathbf{R}^n)$. A *kernel* $K$ is a non-negative, unbounded, and integrable function on $\mathbf{R}^n$ which is radially symmetric and decreasing, i.e. $K(x) = K(t)$ if $|x| = |t|$ and $K(x) \le K(t)$ if $|x| \ge |t|$. We write $L^2_K(\mathbf{R}^n)$ to denote the class of potentials

$$(K * F)(x) = \int_{R^n} K(x-t)\, F(t)\, dt\,,$$

where $K$ is a kernel on $\mathbf{R}^n$ and $F \in L^2(\mathbf{R}^n)$, with $n \ge 2$. (From here on we shall write $L^2_K$ for $L^2_K(\mathbf{R}^n)$, and $L^2$ for $L^2(\mathbf{R}^n)$.) We note [11, p.3] that if $f \in L^2_K$ then $f \in L^2$ and hence has a Fourier transform $\hat{f} \in L^2$ by the Plancherel theorem. It follows that $\hat{f}$ is integrable in $\{x : |x| \le R\}$ for every fixed $R > 0$, and the integral for the mean $S_R f$ in (1), and the mean $T^\lambda_R f$ in (3) below, are thus well-defined for $f \in L^2_K$.

An important alternative summability method to the one based on the mean $S_R f$ is *Bochner-Riesz summability* ([11, pp.170-172], [8], [13]) with

$$T^\lambda_R f(x) = \int_{|t|\le R} (1 - \frac{|t|^2}{R^2})^\lambda\, \hat{f}(t) \exp(2\pi i x{\cdot}t) dt, \ \lambda > 0, \tag{3}$$

a more amenable mean than the spherical partial integral. In section 3, using the characterisation $f = \mathcal{K}_\beta * F$, we derive a result on the convergence of $T^\lambda_R f$ means, outside sets of capacity zero, for functions satisfying (2).

## 2. The main theorem

We begin with the definition of the kernel $\mathcal{K}_\beta$. We set

$$\mathcal{K}_\beta(x) = \int_0^1 \frac{P_s(x)}{s\left(\log \frac{2}{s}\right)^{\beta+1}}\, ds, \quad x \in \mathbf{R}^n,\ \beta > 0,$$

where, for $n \ge 1$ and $s > 0$,

$$P_s(x) = \frac{\lambda_n\, s}{(s^2 + |x|^2)^{(n+1)/2}}\,, \quad \lambda_n = \Gamma(\frac{n+1}{2})/\pi^{(n+1)/2},$$

is the Poisson kernel for $\mathbf{R}^{n+1}_+ = \{(x,s) : x \in \mathbf{R}^n, s > 0\}$. Since $\int_{\mathbf{R}^n} P_s(x) dx = 1$ for each $s > 0$ [11, p. 9], and $P_s$ is radially symmetric and decreasing, it follows that $\mathcal{K}_\beta$ is a kernel.

To prepare for our theorem we present three lemmas, the first two of which provide estimates for $\mathcal{K}_\beta$ and $\hat{\mathcal{K}}_\beta$, and the third establishes an equivalence relation for the classes $L^2_K$ which is central to the proof of the theorem. We will not use Lemma 1 in the proof but, as it answers obvious questions, we include the lemma for the sake of completeness.

**Lemma 2.1.** *We have*

$$\frac{c_\beta}{|x|^n \left(\log \frac{2}{|x|}\right)^{\beta+1}} \le \mathcal{K}_\beta(x) \le \frac{c'_\beta}{|x|^n \left(\log \frac{2}{|x|}\right)^{\beta+1}}\,, \quad 0 < |x| \le 1. \tag{1}$$

*We also have* $\mathcal{K}_\beta(x) \le c_\beta |x|^{-(n+1)}$ *for* $|x| > 1$.

In Lemmas 2.1 and 2.2, $c_\beta$ and $c'_\beta$ denote positive quantities which depend on $\beta$ or $n$ or both, but are not necessarily the same at each occurrence.

*Proof of Lemma 2.1.* For notational simplicity we write $\gamma$ for $\beta+1$ throughout this proof. Since $a^2+b^2 \le (a+b)^2 \le 2(a^2+b^2)$ for $a, b \ge 0$, we note that it is enough to show that the inequalities in (1) are satisfied by

$$\begin{aligned}\int_0^1 \frac{ds}{\left(\log\frac{2}{s}\right)^\gamma (|x|+s)^{n+1}} &= |x|^{-n}\int_0^{1/|x|} \frac{dr}{\left(\log\frac{2}{r|x|}\right)^\gamma (1+r)^{n+1}} \\ &= |x|^{-n} I(x),\end{aligned}$$

say. Next, if $0<|x|\le 1$ and we write $\varphi_x(r)$ for the integrand in $I(x)$, we have, since $r \ge |x|/2$ implies $2/r|x| \le 4/|x|^2$,

$$I(x) \ge \int_{|x|/2}^{1/|x|} \varphi_x(r)dr \ge 2^{-\gamma}\left(\log\frac{2}{|x|}\right)^{-\gamma}\int_{1/2}^1 \frac{dr}{(1+r)^{n+1}} \ge 2^{-(\gamma+n+2)}\left(\log\frac{2}{|x|}\right)^{-\gamma}.$$

This gives the lefthand inequality in (1). To obtain the second inequality we note that $1/|x|^{1/2} \le 1/|x|$ when $0<|x|\le 1$, and write $I(x) = I_1(x)+I_2(x)$, where in $I_1$ we integrate over $(0, 1/|x|^{1/2})$ and in $I_2$ over $(1/|x|^{1/2}, 1/|x|)$. Since $r \le 1/|x|^{1/2}$ implies $2/r|x| \ge (2/|x|)^{1/2}$,

$$I_1(x) = \int_0^{1/|x|^{1/2}} \varphi_x(r)dr \le 2^\gamma\left(\log\frac{2}{|x|}\right)^{-\gamma}\int_0^\infty \frac{dr}{(1+r)^{n+1}} \le 2^\gamma\left(\log\frac{2}{|x|}\right)^{-\gamma}.$$

We have $r|x| \le 1$ in $I_2(x)$, so

$$I_2(x) \le (\log 2)^{-\gamma}\int_{1/|x|^{1/2}}^{1/|x|} \frac{dr}{r^2} \le (\log 2)^{-\gamma}|x|^{1/2} \le c_\beta\left(\log\frac{2}{|x|}\right)^{-\gamma}$$

since $|x|\le 1$ implies $\left(\log\frac{2}{|x|}\right)^\gamma \le c'_\beta(2/|x|)^{1/2}$, for a big enough constant $c'_\beta$. Inequality (1) follows. The estimate for $|x|>1$ is easily obtained and the Lemma is proved.

**Lemma 2.2.** *If $\mathcal{K}_\beta$ is the kernel defined as above for $\beta>0$, then the Fourier transform $\hat{\mathcal{K}}_\beta(x) = \int_{\mathbf{R}^n}\mathcal{K}_\beta(t)\exp(-2\pi i x\cdot t)dt$ satisfies*

$$\frac{c_\beta}{(1+\log^+|x|)^\beta} \le \hat{\mathcal{K}}_\beta(x) \le \frac{c'_\beta}{(1+\log^+|x|)^\beta}, \quad x\in\mathbf{R}^n. \tag{2}$$

*Proof of Lemma 2.2.* We note to begin with, since $\hat{P}_s(x) = \exp(-2\pi s|x|)$ [11, p. 5], that, by an interchange of integrals,

$$\begin{aligned}\hat{\mathcal{K}}_\beta(x) &= \int_{\mathbf{R}^n}\left(\int_0^1 \frac{P_s(t)}{s\left(\log\frac{2}{s}\right)^{\beta+1}}\,ds\right)\exp(-2\pi i x\cdot t)dt \\ &= \int_0^1 \frac{\exp(-2\pi s|x|)\,ds}{s\left(\log\frac{2}{s}\right)^{\beta+1}} = \int_0^{1/|x|}\frac{\exp(-2\pi r)\,dr}{r\left(\log\frac{2|x|}{r}\right)^{\beta+1}}.\end{aligned} \tag{3}$$

Assume that $|x|>1$. Then

$$\begin{aligned}\hat{\mathcal{K}}_\beta(x) &\ge \int_{1/4|x|^2}^{1/2|x|}\frac{\exp(-2\pi r)\,dr}{r\left(\log\frac{2|x|}{r}\right)^{\beta+1}} \\ &\ge \mathrm{e}^{-2\pi}3^{-\beta-1}(\log 2|x|)^{-\beta-1}\int_{1/4|x|^2}^{1/2|x|}\frac{1}{r}\,dr \\ &= \mathrm{e}^{-2\pi}3^{-\beta-1}(\log 2|x|)^{-\beta} \ge c_\beta(1+\log|x|)^{-\beta} = c_\beta(1+\log^+|x|)^{-\beta}.\end{aligned}$$

This gives the lower bound in (2). To obtain the upper bound for $|x| > 1$, we note from the second equality in (3) that

$$\beta \hat{K}_\beta(x) = \int_0^1 \mathrm{e}^{-2\pi s|x|} d\left(\log \frac{2}{s}\right)^{-\beta} = \mathrm{e}^{-2\pi|x|}(\log 2)^{-\beta} + 2\pi|x|\, J(x), \tag{4}$$

where

$$J(x) = \int_0^1 \mathrm{e}^{-2\pi s|x|}(\log \frac{2}{s})^{-\beta} ds.$$

We write $J$ as $J_1 + J_2$ where $J_1 = \int_0^{1/|x|^{1/2}}$, $J_2 = \int_{1/|x|^{1/2}}^1$. Then

$$\begin{aligned} J_1(x) &\leq (\log 2|x|^{1/2})^{-\beta} \int_0^{1/|x|^{1/2}} \mathrm{e}^{-2\pi s|x|} ds \\ &= \frac{2^{\beta-1}(\log 4|x|)^{-\beta}}{2\pi|x|} \int_0^{2\pi|x|^{1/2}} \mathrm{e}^{-u} du \leq c_\beta\, |x|^{-1}(1 + \log^+ |x|)^{-\beta}, \end{aligned}$$

since $|x| > 1$. Next, by a similar argument,

$$J_2(x) = \int_{1/|x|^{1/2}}^1 \leq (\log 2)^{-\beta} \frac{1}{2\pi|x|} \int_{2\pi|x|^{1/2}}^{2\pi|x|} \mathrm{e}^{-u} du \leq c_\beta\, (1 + \log^+ |x|)^{-\beta}/|x|,$$

choosing $c_\beta$ large enough. Since $J = J_1 + J_2$, and $\mathrm{e}^{-2\pi|x|} \leq c_\beta(1 + \log^+ |x|)^{1-\beta}$, $|x| \geq 1$, the required upper bound follows from (4) when $|x| > 1$. The proof of the inequalities (2) for the case $|x| \leq 1$, i.e. that $c_\beta \leq \hat{\mathcal{K}}_\beta(x) \leq c'_\beta$, is easy and is omitted. The proof of Lemma 2.2 is complete.

*Remark* If we apply the result $(\exp(-2\pi s|x|))^\wedge = P_s(x)$ [11, p. 6] to the middle integral in (3) we see that $(\hat{\mathcal{K}}_\beta)^\wedge = \mathcal{K}_\beta$.

**Lemma 2.3.** *Let $K$ be a kernel with $\hat{K} > 0$ and suppose that $f \in L^2$. Then*

$$\int_{\mathbf{R}^n} \hat{K}(t)^{-2}|\hat{f}(t)|^2 dt < \infty \Longleftrightarrow f \in L^2_K.$$

*Proof.* Note first, by $L^2$ transform theory, that if $g \in L^2$ then the transform $\hat{g} \in L^2$ and $(\hat{g})^\wedge(-t) = g(t)$.

Assume that $\int_{\mathbf{R}^n} \hat{K}(t)^{-2}|\hat{f}(t)|^2 dt < \infty$ and set $F(t) = \hat{K}(t)^{-1}\hat{f}(t)$, so that $F \in L^2$. Then

$$\hat{f}(t) = \hat{K}(t)F(t) = \hat{K}(t)(\hat{F})^\wedge(-t) = \hat{K}(t)\hat{H}(t) = (K * H)^\wedge(t)$$

where $H(t) = \hat{F}(-t)$, so $H \in L^2$, and we have applied the multiplication formula from [7, theorem 5.8], with $p = 1$ and $q = 2$, for the last equality. Hence

$$(\hat{f})^\wedge = ((K * H)^\wedge)^\wedge, \quad i.e.\ f(-t) = (K * H)(-t),$$

or $f(t) = (K * H)(t)$, $t \in \mathbf{R}^n$. This proves the first part of the Lemma.
For the second part assume that $f \in L^2_K$ so that $f = K * Q$ where $Q \in L^2$. Then $\hat{f} = \hat{K}\hat{Q}$ and $\hat{Q}(t) = \hat{K}(t)^{-1}\hat{f}(t)$. Since $\hat{Q} \in L^2$, the converse implication follows and the proof is complete.

Our main theorem now follows immediately from Lemmas 2.2 and 2.3 (with $K = \mathcal{K}_\beta$).

**Theorem 2.4.** *If $f \in L^2$ then*

$$\int_{\mathbf{R}^n} |\hat{f}(t)|^2(1 + \log^+ |t|)^{2\beta} dt < \infty,\ \beta > 0, \tag{5}$$

*if, and only if, $f \in L^2_{\mathcal{K}_\beta}$.*

## 3. Bochner-Riesz summability

We obtain our final theorem on the convergence of certain Bochner-Riesz means in $L^2_K$ by simply combining two known results. We begin by noting from [11, Corollary 4.16 (b), p.172] that

*if $f \in L^2$ has a Lebesgue point at $x \in \mathbf{R}^n$, that is, if*

$$\lim_{r\to 0} \frac{1}{m(B(x,r))} \int_{B(x,r)} |f(t) - f(x)|\, dt = 0,$$

*where $m(B(x,r))$ denotes the Lebesgue measure of the ball $B(x,r)$, then the Bochner-Riesz mean $T^\lambda_R f(x)$ in (1.3) converges to $f(x)$ for all $\lambda > (n-1)/2$ as $R \to \infty$.*

Since $L^2_K \subset L^2$, as noted above, and $L^2$ functions have Lebesgue points almost everywhere, it follows that for $f \in L^2_K$ the set of $x \in \mathbf{R}^n$ for which (1.3) fails to converge for $\lambda > (n-1)/2$ has Lebesgue measure zero. We strengthen this by combining the Stein-Weiss result with the following consequence of [12, Theorem 1]:

*if $f \in L^2_K$ then $f$ has a Lebesgue point at all points in $\mathbf{R}^n$ except possibly for a set of points of $C_{K,2}$-capacity zero.*

The capacity referred to here is the particular case $p = 2$ of the $L^p$-capacities of Meyers ([9], [1, Chapter 2]). For a brief summary of the basic properties of these capacities see [10, pp. 341-2]. If $C_{K,2}(E) = 0$, then $E$ has measure zero, and Meyers' capacities provide a way of differentiating between sets of measure zero.

Taking $K = \mathcal{K}_\beta$ we immediately deduce the following convergence result for functions in $L^2_{\mathcal{K}_\beta}$.

**Theorem 3.1.** *If $f \in L^2_{\mathcal{K}_\beta}$, or equivalently if (2.5) holds, and $\lambda > (n-1)/2$, then $\lim_{R\to\infty} T^\lambda_R f(x) = f(x)$ for all $x \in \mathbf{R}^n$ outside an exceptional set of $C_{\mathcal{K}_\beta,2}$-capacity zero.*

It has been shown in [3, Theorem A] that if $f \in L^2$ then $\lim_{R\to\infty} T^\lambda_R f(x) = f(x)$ *almost everywhere* in $\mathbf{R}^n$ for *all* $\lambda > 0$, and an obvious question here therefore is whether the range of $\lambda$ in Theorem 3.1 can be extended to all positive values. There is also the question of whether the characterisation $f = \mathcal{K}_\beta * F$ can be used to obtain convergence results in $L^2_{\mathcal{K}_\beta}$ for the spherical partial integral $S_R f$. These questions are open.

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# Des MacHale: Lateral Solutions to Mathematical Problems, A K Peters/CRC Press. Taylor & Francis Group, 2023. ISBN: 978-100-3341-46-8, GBP 26.99, 123+xiv pp.

REVIEWED BY PETER LYNCH

This book is a rich source of delightful puzzles in school and first-year undergraduate mathematics: number theory, geometry, trigonometry, calculus, probability and logic, as well as miscellaneous other topics. The problems often look daunting, but solutions can usually be found by means of lateral reasoning. Lateral thinking (LT) is a method of solving problems by adopting a new perspective.

The book will be of interest to students and also to teachers and lecturers seeking new material to spark the imaginations of their students. There are ten problems in each of the twelve sections, giving a total of 120 problems. These vary greatly in difficulty. In most cases, some clever mental gymnastics can reduce an apparently intractable problem to a simple calculation or line of logic. In a smaller number, the solution is tricky even after the answer is presented.

Problems in Section 1 are on Number Theory, beginning with this: If you multiply all the primes less than one million together, what is the final digit of your answer? A moment's reflection gives the answer but, as for all the problems, the solution is augmented by suggestions for "further investigation", which enrich the text.

Another easy problem asks for the final digit of $2^{999}$? While the line of reasoning is correct, the answer 6 on p. 35 is not. The number has 301 decimal digits, ending in 688. Problem 1.8, to determine the sum of the first $n$ Fibonacci numbers, is a clear and simple application of LT. Likewise, to show that $(n!)^2$ divides $(2n)!$, which is easily done by some "thinking outside the box".

Problem 2.9, a variation on a theme of Goldbach, is easy and is amusing to solve. The last problem in Section 2 is to show that the sum of reciprocals of the first $n$ primes is never an integer. This is elementary but subtle and requires lateral imagination.

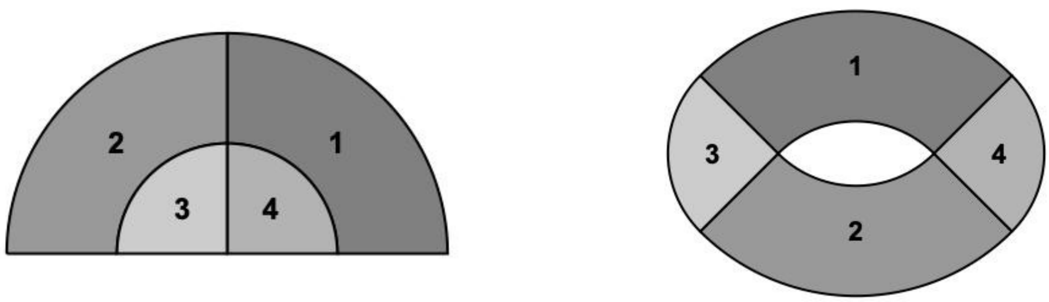


FIGURE 1. Rearranging a semicircle (left) to form an oval (right).

In Section 3, on Geometry, one problem requires us to cut a semicircle into four pieces and reassemble them into an ellipse. What we are not told is that the solution may

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have a hole in the middle (Figure 1). Is this a cheat, or a legitimate solution? I suppose it depends on the latitude of the lateral thinking! Moreover, the outer boundary is not an ellipse but a set of four circular arcs.

Problem 3.7 asks for the area of the smaller square in Figure 2, given the area of the larger one. It is a delightful puzzle and the simple solution, requiring no calculation, should produce a warm glow of satisfaction. Spoiler alert: no peeping ahead to Figure 3!

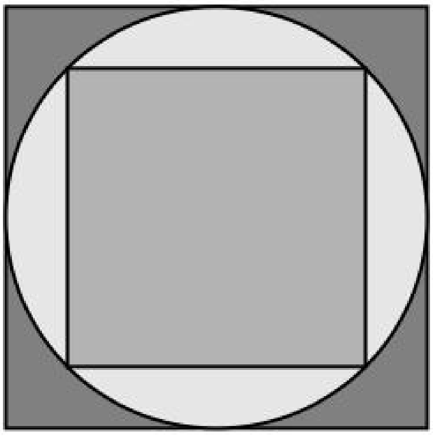

FIGURE 2. Circle with circumscribed and inscribed squares.

How many observations of the position of a planet are needed to determine its orbit? Three suffice for a circular orbit, so we might guess four for an ellipse. But we need the direction as well as the extent of the broken circular symmetry, so five observations are required. The solution of Problem 3.10 has a diagram confirming this.

On Trigonometry, Problem 4.1 is solved by noting that $\tan 45^\circ = 1$ and using the addition formula for tangents. This is described as a "lovely lateral twist". Another problem asks for a triangle such that the tangents of all three angles are positive integers. The (unique) answer is quite surprising. Problem 4.3, on an infinite sum of inverse sines of inverse radicals, looks hopeless. I got nowhere by staring at it. Readers will need inspiration, whether lateral, dorsal, ventral or even sagittal, to solve this without a peep at page 52.

Probability problems can be quite counter-intuitive, and ambiguous if not carefully formulated. Problem 5.4 illustrates this: a thin rod is broken at random into three pieces. What is the probability that the pieces can be used to form a triangle? There is a subtlety here: how are the break-points chosen? (1) By choosing two random break-points on the rod or (2) by breaking once, then breaking the longer piece. The solution given is for the first method but the second seems more natural. The probability depends on the method of breaking the rod. This is analogous to Bertrand's paradox, which reappears in Problem 5.9.

Problem 5.7 asks: what is the probability that your father and mother have birthdays six months or less apart? I am ashamed to say that my initial guess for this simple problem was wrong. Most readers should not fall so easily into the trap.

Combinatorial problems can be fiendishly difficult, and you will need all your LT skills. Problem 6.9 where we must show that a set with precisely $n$ elements has exactly $2^n$ subsets, is described by the author as "one of the most important facts in mathematics, central to a whole lot of theory and applications." Three distinct proofs are given, and also a discussion of mathematical induction.

Section 7 (on Dissections) has several problems on cutting up and rearranging polygons. Some are simple, some more challenging. As an example: Can you cut a $30 \times 30$ square piece of carpet to cover a $25 \times 36$ floor exactly? Surprisingly, a single cut suffices (provided it is the right one!).

Section 8, on "Matchsticks and Coins", has several minor but irksome errors. The figures are of poor quality and, in some cases, misleading. In Problem 8.2, the figure already shows the solution. In 8.5 there are 14 matches in the problem (p. 19) but 15 in the solution (p. 80). In 8.6, the added match is not the same length as the originals.

In Section 9, on logic, warm-up questions such as *Is "no" the answer to this question?* will sharpen up the reasoning. Several "old chestnuts" appear, but they are probably unfamiliar to younger readers. One problem I had not seen before (Problem 9.6) is intriguing: *You are given two ropes, each of which takes exactly 60 minutes to burn. They are made of different materials and they burn at different rates and inconsistently. How can you measure 45 minutes exactly by burning the ropes?* Here you really need to think laterally. It is worthwhile taking a little time on this, to experience an Aha-moment.

In Section 10, on Maxima and Minima, some problems that would normally be solved by calculus are tackled without it using LT. Section 11 has some harder questions, leading up to the Painter's Paradox, dealing with a finite volume having infinite surface area. Problem 11.4 asks for a proof of the Arithmetic Mean-Geometric Mean inequality. The proof presented is attributed to the great Hungarian-American mathematician George Pólya. A typo in Problem 11.6 may confuse readers.

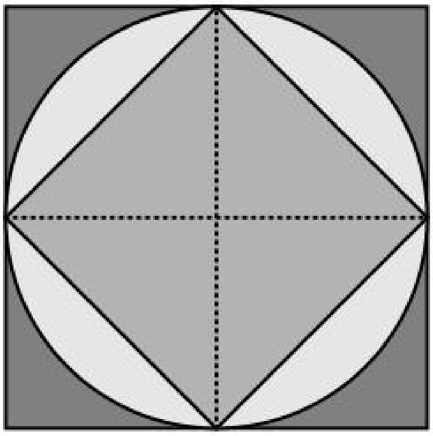

FIGURE 3. Solution to problem in Fig. 2.

Section 12 contains a miscellany of puzzles to round off the collection. The mnemonic rhyme for $\pi$ has sixth word "remember" while the relevant digit of $\pi$ is 9, and there are better *piems*, or poems about pi. In Problem 12.5, a quadratic is solved without completing the square, but I suspect that something is hidden here.

*Summary.* It is more than fifty years since LT was popularised by Edward De Bono's book *The Use of Lateral Thinking*, but the concepts and methods that he solidified and made practical are of lasting value. It is good to see these methods employed effectively to solve simple and not-so-simple mathematical problems. Despite some flaws, and the regretted absence of an index, this book is a nice collection of puzzles, problems, paradoxes and enigmas. It would also serve as an ideal gift for a mathematically-inclined friend.

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# Teo Banica: Introduction to Quantum Groups, Springer, 2023. ISBN:978-3-031-23816-1, EUR 50.28, 425 pp.

REVIEWED BY J.P. MCCARTHY

For each non-zero $q \in [-1, 1]$, Woronowicz [9] defined a universal unital C*-algebra $C(\mathrm{SU}_q(2))$ generated by elements $\alpha$, $\gamma$ subject to a number of relations, including $\alpha\gamma = q\gamma\alpha$. The algebra admits a *comultiplication* into the minimal tensor product

$$\Delta : C(\mathrm{SU}_q(2)) \to C(\mathrm{SU}_q(2)) \otimes C(\mathrm{SU}_q(2)),$$

and the crucial point is that at $q = 1$, the algebra is commutative, $C(\mathrm{SU}_1(2)) \cong C(\mathrm{SU}(2))$, the algebra of continuous functions on the matrix group SU(2), and, via $C(\mathrm{SU}(2)) \otimes C(\mathrm{SU}(2)) \cong C(\mathrm{SU}(2) \times \mathrm{SU}(2))$, the comultiplication is a transpose

$$\Delta f(u, v) = f(uv) \qquad (u, v \in \mathrm{SU}(2),\ f \in C(\mathrm{SU}(2))).$$

Thus $C(\mathrm{SU}_q(2))$ is said to be a $q$-deformation of $C(\mathrm{SU}(2))$, a quantum SU(2). This exhibits the very basic philosophy of quantum groups: they are given by an algebra $A$ that satisfies some specific axioms such that whenever an algebra satisfying those same axioms is commutative, it is an algebra of functions on a group, and the comultiplication is the transpose of the group law. When such an algebra is non-commutative, it can be viewed as an algebra of functions on a quantum group, $\mathbb{G}$, and the algebra written $A = C(\mathbb{G})$, but this quantum group $\mathbb{G}$ is a so-called *virtual object*, i.e. it is not a set.

The quantum groups of Drinfeld and Jimbo [3, 4] are also $q$-deformations, and while $\mathrm{SU}_q(2)$ fits naturally into their framework, a 1987 paper of Woronowicz [10] split the field into two schools Woronowicz vs Drinfeld–Jimbo, and the book author is firmly of the Woronowicz school (often called the C*-algebraic approach). Woronowicz proved foundational theorems for compact quantum groups, but it was the 1990s examples of Wang [7, 8] of quantum $\mathrm{U}_N$, quantum $\mathrm{O}_N$, and quantum $\mathrm{S}_N$ that gave a real impetus to the field.

In this book, Banica concentrates largely on quantum subgroups of the quantum unitary group $\mathrm{U}_N^+$ and the quantum orthogonal group $\mathrm{O}_N^+$. These compact quantum groups are *free* or *liberated* versions of the classical orthogonal and unitary groups, not deformations in the sense of $\mathrm{SU}_q(2)$. Consider an invertible matrix $u \in M_N(A)$ with entries in a unital C*-algebra $A$. If its entries are self-adjoint $u_{ij}^* = u_{ij}$, and the inverse $u^{-1}$ is given by the transpose $u^T$, it is said to be a quantum orthogonal matrix. The algebra of continuous functions on $\mathrm{O}_N$ may be realised as a universal commutative C*-algebra

$$C(\mathrm{O}_N) = \mathrm{C}^*_{\mathrm{comm}}(u_{ij} \,|\, u \text{ an } N \times N \text{ quantum orthogonal matrix}).$$

The generators $u_{ij} \in C(\mathrm{O}_N)$ are coordinate functions $u_{ij}(g) = g_{ij}$. If the algebra is *liberated* from commutativity, the relations still define a unital C*-algebra

$$A_o(N) = \mathrm{C}^*(u_{ij} \,|\, u \text{ an } N \times N \text{ quantum orthogonal matrix}),$$

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and this algebra admits a $*$-homomorphism $\Delta : A_o(N) \to A_o(N) \otimes A_o(N)$

$$\Delta(u_{ij}) = \sum_{k=1}^{N} u_{ik} \otimes u_{kj},$$

giving $(A_o(N), u)$ the structure of a compact quantum group $A_o(N) = C(\mathrm{O}_N^+)$.

In the 25 years since Wang's examples, Banica and co-authors have studied these quantum groups over the course of 99 papers, the highlight of which might be the seminal work of Banica and Speicher [1]. This 25-year study has culminated in Banica beginning to write a series of tomes, of which the one under review is the first to be published. It is structured in four parts as follows.

*Part I: Quantum Groups* The preliminaries begin with the definition of a Hilbert space, and then neatly summarises the basics of C*-algebras, including the important examples of the group C*-algebras $\mathrm{C}^*(G)$ and their reduced versions $\mathrm{C_r}(G)$. In my opinion, the existing literature on compact quantum groups does not go far enough in discussing axioms. In contrast, Banica spends a good ten pages carefully teasing out why compact quantum groups are defined as they are. One of the reasons that compact quantum groups have been so well-studied is that they have a very rich representation theory, which extends the theories of Peter–Weyl and Tannaka–Krien. This book gives a comprehensive treatment of the representation theory, and presents the relatively recent, more concrete version of Tannaka duality, that is due to Malacarne [6].

*Part II: Quantum Rotations* Where $u \in M_N(C(\mathbb{G}))$ is the so-called fundamental representation, the intertwiner spaces $\hom(u^{\otimes k}, u^{\otimes l})$, which form a tensor category, are crucial in the representation theory of compact quantum groups. In the classical setting, for $\mathrm{U}_N$ and $\mathrm{O}_N$, Brauer's work shows that the elements of $\hom(u^{\otimes k}, u^{\otimes l})$ are linear combinations of intertwiners $T_\pi$ parameterised by coloured partitions in a partition category, $\pi \in D(k, l)$. In fact, any such partition category $D$ yields a tensor category $C_D$, which we learned in Part I yields a compact quantum group, a so-called *easy quantum group* (Theorem 2.7). If the partition category is crossing (non-planar partitions), then $fg = gf$ for all $f, g \in C(\mathbb{G})$, the algebra of functions is commutative, and the associated compact quantum group is classical. On the other hand, if the partition category is non-crossing (planar partitions), the algebra of functions is noncommutative, and the associated compact quantum group is genuinely quantum. That intertwiners $T_\pi$, $T_\sigma$ parameterised with partitions $\pi$, $\sigma \in D$ respect the tensor category structure of $C_D$ is given in terms of a diagrammatic calculus (e.g. horizontal and vertical concatenation,"upside-down turning", "the semi-circle", etc.), but unfortunately there are no pictures accompanying these descriptions. Of particular interest here is the study in relation to free probability. Where $u \in M_N(C(\mathrm{O}_N^+))$ is the fundamental representation of the quantum orthogonal group, and $\int_{\mathrm{O}_N^+} : C(\mathrm{O}_N^+) \to \mathbb{C}$ the Haar state, the moments

$$\int_{\mathrm{O}_N^+} \mathrm{tr}(u)^k = C_k,$$

the Catalan numbers, and thus the law of the 'main character' $\mathrm{tr}(u)$ with respect to the Haar state is the famous semi-circle law of Wigner.

*Part III: Quantum Permutations* Banica defines the quantum permutation groups $\mathrm{S}_N^+$ as discovered by Wang in the 1990s [8], which, in the case of finite sets, answered a famous question of Alain Connes: "*What is the quantum automorphism group of a space?*". Of particular importance in the study of quantum permutation groups was the author's development and study of the quantum automorphism group $G^+(X)$ of a finite graph $X$, which has since met applications in quantum information [5]. This is a very active field, with a long-standing problem being settled as recently as November 2023.

However, while this book introduces quantum automorphism groups, it concentrates mostly on the relationship between $S_N^+$ and other easy compact quantum groups (there is a tome focussed on quantum permutation groups in production).

*Part IV: Advanced Topics* In Part I, the author chose 'lighter' axioms

> *"...with the aim of focussing on what is beautiful and essential..."*

Firstly, the author assumes that the antipode $S : C(\mathbb{G}) \to C(\mathbb{G})$ is a *-antihomomorphism. This assumption, stronger than Woronowicz's, forces the algebras of functions to be of Kac type, satisfying $S^2 = \text{id}$ (so that $\text{SU}_q(2)$ falls outside the author's formalism). Secondly, algebras of continuous functions $C(\mathbb{G})$ have a dense Hopf*-algebra $\mathcal{O}(\mathbb{G})$, but in general this algebra admits more than one completion to a C*-algebra. Banica chooses to work in one particular completion (the universal one): the counit $\varepsilon : \mathcal{O}(\mathbb{G}) \to \mathbb{C}$ extends to this completion, but on the other hand the Haar state is not necessarily faithful. For certain applications, for example quantum automorphism groups of finite graphs, what completion (if any) is used is often irrelevant. Part IV looks at this (and other assorted topics) in more detail (with our own G.J. Murphy involved in the treatment in the full formalism [2]).

This book has fantastic character, is out to inspire, well-written, and is full of what the author calls "philosophy". The author claims the presentation is elementary: "*a standard first year, graduate level textbook*". It is an excellent reference for many facets of the theory, but is not encyclopaedic, with sometimes only brief proofs, and with the intentional avoidance of getting stuck in technical mud. The locally compact case is not treated (despite the fact that many of the latest results in the compact case are actually proven in this more technically demanding setting). The book has very interesting exercises: a mix of calculations (for self-study), and some much more open-ended problems that would be ideal for group discussion and development (and perhaps undergraduate projects).

This book is for anyone with an interest or use for compact quantum groups, but the breadth of the exposition probably means that there is something interesting in there for everyone.

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# A. B. Sossinsky: Knots, Links and Their Invariants: An Elementary Course in Contemporary Knot Theory, American Mathematical Society, 2023.
# ISBN:978-1-4704-7151-4, USD 59, 128+xvii pp.

REVIEWED BY ASHLEIGH WILCOX

Despite its small size, the book holds an abundance of information and provides an introduction to knot theory as a series of thirteen lectures. The author assumes no prerequisites of the reader, other than basic plane and spatial Euclidean geometry. Having never previously studied topology, I was interested to see whether this book would be accessible for myself, and therefore the readership: undergraduate and graduate students interested in knot theory.

Originally, Sossinsky presented this course online to students across the world and then published a small print run of the lecture notes. This book is a revised version of those lecture notes, complete with numerous illustrations and exercises for the reader to complete, although without solutions.

This book provides an excellent framework for an introductory course into knot theory. It contains the important theorems of the last 40 years, with proofs using mostly elementary methods. All necessary preliminary material is included.

The first eleven lectures are essentially self-contained, with the only references being to tables contained in [1] and [2]. The final two chapters provide a rather brief overview of the history of knot theory and introduce further important topics which an interested reader could explore after the course.

Chapter 1 discusses what will be learnt and how examples will reappear throughout the course. This drip-feeding of information is very helpful, as it allows you to see the reasoning behind the theory you are introduced to, and their importance becomes clear further into the course. As well as useful and well-thought out diagrams to support explanations and understanding, there are also intuitive descriptions. This ensures the book is accessible and largely well-explained. Despite being an introductory course, the author manages to convey the topic in a way that encourages you to pursue the topic further and it makes you want to learn more about knot theory, especially with the final two chapters giving a brief introduction to other topics and the mathematicians that were pivotal in the growing of this topic.

A brief insight into some chapters of the book is as follows.

Chapters $1-3$ provide an introduction into the main elements of knot theory, covering Reidemeister moves, the Conway polynomial and the arithmetic of knots.

Chapters $4-6$ contain interesting theorems and definitions and facts.

Chapter 6 also has information on things that the reader could further explore, for example the Dowker-Thistlewaite code.

Chapter 7 relates theory of braids to where their applications appear.

The final lecture provides a brief history of knot theory and also contains images of the main contributors to the subject.

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At the end of each chapter there is a number of exercises. These exercises are challenging, but allow for a deep understanding of the material. Some of the exercises ask for proofs to theorems introduced in the lectures, and proofs for how some lemmas imply theorems. This shows how the material interacts. This layout provides an interesting flow of the content and shows how much of the course content can be derived through elementary knowledge.

There are some areas of the book which could benefit from some further explanations, such as how to calculate $\nabla(L)$, and how to calculate the link number. A course based on the material in this book would provide a good framework and introduction to knot theory. The historical account in the final chapter is divided in sections, so if the book is used to support a taught course, the history could be interwoven into the relevant sections of the theory. This could be especially useful for an undergraduate audience, showing when and by whom the theory was introduced. It could also be suitable as a self-study guide or textbook, although it may be beneficial to have as a taught course, especially as a number of proofs to theorems, lemmas and remarks are left as exercises for the reader. However, the course is very accessible for the recommended readership and an interested reader has much of the information necessary to build knowledge to understand and learn the content.

In summary, the author has created a very useful and concise book that provides an interesting introduction to some of the main results of knot theory such as the Alexander-Conway knot polynomial, the Jones polynomial and the Vassiliev invariants.

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# PROBLEMS

IAN SHORT

## Problems

A polygon is said to be *inscribed* in a simple closed curve in the plane if all the vertices of the polygon lie on the curve.

**Problem 93.1.** Find a simple closed curve in the plane that does not have an inscribed regular pentagon.

The same problem but with a square instead of a pentagon is unsolved; it is known as the *inscribed square problem*.

**Problem 93.2.** Determine the least positive integer $n$ for which a continued fraction

$$\cfrac{1}{b_1+\cfrac{1}{b_2+\cfrac{1}{b_3+\cdots+\cfrac{1}{b_n}}}}$$

has value $\infty$, where $b_i$ are Gaussian integers each of modulus greater than 1.

The third problem was passed on to me by Andrei Zabolotskii of the Open University, who encountered it at the Moscow Mathematical Olympiad in 2005. There are recent publications on the problem.

**Problem 93.3.** Dissect a disc into a finite number of congruent connected pieces (reflections allowed) in such a way that at least one piece does not contain the centre of the disc inside it or on its boundary.

## Solutions

Here are solutions to the problems from *Bulletin* Number 91.

The first problem was solved by the North Kildare Mathematics Problem Club and the proposer, Toyesh Prakash Sharma of Agra College, India. We present an abridged version of the solution of the proposer, which begins with an attractive equation for the cosines of the angles of a triangle.

*Problem 91.1*. Prove that the angles $\alpha$, $\beta$, and $\gamma$ of a triangle satisfy

$$\big(1+\sec^2\alpha+\sec^2\beta+\sec^2\gamma\big)(1-\cos\alpha\cos\beta\cos\gamma)\geqslant 8$$

We ought to assume here that none of the angles are right-angles, so that none of the cosines of the angles vanish.

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*Solution 91.1.* Observe that

$$\cos^2\alpha+\cos^2\beta+\cos^2\gamma+2\cos\alpha\cos\beta\cos\gamma-1=0.$$

An elementary if tedious way to establish this equation is to use the cosine rule to express $\cos\alpha$, $\cos\beta$, and $\cos\gamma$ in terms of the lengths of the sides of the triangle; we omit the details.

It follows immediately that

$$1-\cos\alpha\cos\beta\cos\gamma=\frac{1}{2}(1+\cos^2\alpha+\cos^2\beta+\cos^2\gamma),$$

and hence

$$\begin{aligned}&\left(1+\sec^2\alpha+\sec^2\beta+\sec^2\gamma\right)(1-\cos\alpha\cos\beta\cos\gamma)\\&\qquad=\frac{1}{2}\left(1+\sec^2\alpha+\sec^2\beta+\sec^2\gamma\right)\left(1+\cos^2\alpha+\cos^2\beta+\cos^2\gamma\right).\end{aligned}$$

By applying the arithmetic-geometric mean inequality (twice) we obtain

$$\begin{aligned}&\left(1+\sec^2\alpha+\sec^2\beta+\sec^2\gamma\right)(1-\cos\alpha\cos\beta\cos\gamma)\\&\qquad\geqslant\tfrac{1}{2}\times 4\sqrt[4]{\sec^2\alpha\sec^2\beta\sec^2\gamma}\times 4\sqrt[4]{\cos^2\alpha\cos^2\beta\cos^2\gamma}=8.\end{aligned}$$ □

The second problem was solved by Henry Ricardo of the Westchester Area Math Circle, USA, Seán Stewart of the King Abdullah University of Science and Technology, Saudi Arabia, the North Kildare Mathematics Problem Club, and the proposer Des MacHale of University College Cork. We provide the solution of Henry Ricardo which, like some (but not all) of the other solutions, uses Brahmagupta's elegant formula for the area of a cyclic quadrilateral.

*Problem 91.2.* Prove that the perimeter $P$ and area $A$ of a cyclic quadrilateral satisfy

$$P^2\geqslant 16A,$$

with equality if and only if the cyclic quadrilateral is a square.

*Solution 91.2.* The area $A$ of a cyclic quadrilateral with sides $a,b,c$, and $d$ is given by Brahmagupta's formula

$$A=\sqrt{(s-a)(s-b)(s-c)(s-d)},$$

where $s=P/2=(a+b+c+d)/2$. By applying the arithmetic-geometric mean (AGM) inequality we obtain

$$\begin{aligned}A&=\left(\sqrt[4]{(s-a)(s-b)(s-c)(s-d)}\right)^2\\&\leqslant\left(\frac{4s-P}{4}\right)^2\\&=\frac{P^2}{16}.\end{aligned}$$

Equality holds in the AGM inequality if and only if $s-a=s-b=s-c=s-d$, that is, if and only if the quadrilateral is a square. □

The third problem was solved by the North Kildare Mathematics Problem Club and the proposer, Tran Quang Hung of the Vietnam National University at Hanoi, Vietnam. We present the solution of the problem club.

*Problem 91.3.* Let $A_0, A_1, \dots, A_n$ be the vertices of a simplex in $n$-dimensional Euclidean space for which the edges $A_0A_1, A_0A_2, \dots, A_0A_n$ are mutually perpendicular. Let $B_i$ be the centroid of the set of points $\{A_0, A_1, \dots, A_n\} \setminus \{A_i\}$, for $i = 0, 1, \dots, n$. Consider any point $C$ other than $A_0$ for which the line through $A_0$ and $C$ is perpendicular to the hyperplane spanned by $A_1, A_2, \dots, A_n$, and let $P$ be the midpoint of the segment $B_0C$. Prove that all distances $PB_i$ are equal, for $i = 1, 2, \dots, n$.

*Solution 91.3.* We may assume that $A_0$ is the origin and $A_j = a_je_j$, for $j = 1, 2, \dots, n$, where $(e_1, e_2, \dots, e_n)$ is an orthonormal basis for $\mathbb{R}^n$, and $a_j > 0$. Then

$$B_i = \frac{1}{n}\sum_{j\neq i} a_je_j.$$

The plane spanned by $A_1, A_2, \dots, A_n$ has equation

$$\frac{x_1}{a_1} + \frac{x_2}{a_2} + \cdots \frac{x_n}{a_n} = 1,$$

so the perpendicular vector $C$ takes the form

$$C = \lambda\left(\frac{1}{a_1}, \frac{1}{a_2}, \dots, \frac{1}{a_n}\right),$$

for some nonzero real number $\lambda$. Then

$$B_0 = \frac{1}{n}\sum_{j=1}^{n} a_je_j \quad \text{and} \quad P = \frac{1}{2}\left(\frac{1}{n}\sum_{j=1}^{n} a_je_j + \lambda\sum_{j=1}^{n}\frac{1}{a_j}e_j\right),$$

so

$$\begin{aligned}|PB_i|^2 &= \frac{1}{4}\left(\left(\frac{a_i}{n} + \frac{\lambda}{a_i}\right)^2 + \sum_{j\neq i}\left(\frac{a_j}{n} - \frac{\lambda}{a_j}\right)^2\right)\\ &= \frac{1}{4}\sum_{j=1}^{n}\left(\frac{a_j^2}{n^2} + \frac{\lambda^2}{a_j^2}\right) + \frac{\lambda(2-n)}{2n},\end{aligned}$$

which is independent of the index $i$, as required. □

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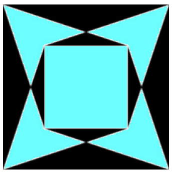

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