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PROBLEMS

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Problems

The first problem this issue was suggested by Des MacHale of University College Cork.

Problem 92.1. Show that the infinite cyclic group is not the full automorphism group of any group.

Des comments that readers may wish to consider the more diffficult (and only partially solved) problem of determining when the finite cyclic group of order n is the full automorphism group of (a) a finite group and (b) an infinite group.

The second problem is from Andrei Zabolotskii of the Open University.

Problem 92.2. Let A be a symmetric square matrix of even order over the ring of integers modulo 2. Suppose that all entries on the leading diagonal of A are 0. Let B be the square matrix obtained from A by replacing each 0 entry with 1 and replacing each 1 entry with 0. Prove that $\det A = \det B$.

Readers who solve Problem 92.2 might care to consider the more challenging question of whether or not the characteristic polynomials of A and B are equal.

The third problem was proposed by Finbarr Holland of University College Cork.

Problem 92.3. For $x > 0$, let $\mu(x)$ denote the ℓ_∞-norm of the sequence

$$u_n(x) = \frac{x^n}{n^n}, \quad n = 1, 2, \dots .$$

Determine

$$\lim_{x\to\infty} \frac{\log \mu(x)}{x}.$$

Solutions

Here are solutions to the problems from *Bulletin* Number 90.

The first problem was solved by Brian Bradie of Christopher Newport University, USA, Elshan Huseynov of ADA University, Azerbaijan, Seán Stewart of the King Abdullah University of Science and Technology, Saudi Arabia, and the North Kildare Mathematics Problem Club. All solutions gave the same correct formula; we provide commentary derived from Seán Stewart's contribution.

Problem 90.1. Let M be any 3-by-3 matrix

$$M = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

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over a field, where $e \neq 0$, and let A, B, C, D be the four submatrices of M given by

$$A = \begin{pmatrix} a & b \\ d & e \end{pmatrix}, \quad B = \begin{pmatrix} b & c \\ e & f \end{pmatrix}, \quad C = \begin{pmatrix} d & e \\ g & h \end{pmatrix}, \quad D = \begin{pmatrix} e & f \\ h & i \end{pmatrix}.$$

Find an expression for $\det M$ in terms of $\det A$, $\det B$, $\det C$, $\det D$, and e.

Solution 90.1. A straightforward computation shows that

$$\det M = \frac{1}{e}(\det A \det D - \det B \det C). \qquad \square$$

This is a special case of a more general identity for n-by-n matrices known as the *Desnanot–Jacobi identity* or sometimes *Dodgson's identity* (see Matrix Analysis (2nd edition), by Horn and Johnson, Section 0.8.11). The method for computing determinants based on these identies is known as the *Dodgson condensation method*.

The second problem was solved by Brian Bradie, Seán Stewart, the North Kildare Mathematics Problem Club, and the proposer Finbarr Holland. Each solution was different; we provide that of Brian Bradie.

*Problem 90.2***.** Prove that

$$\sum_{n=0}^{\infty} a_n \sum_{k=0}^{n} \frac{a_k a_{n-k}}{(2k+1)(2(n-k)+1)} = \frac{2}{\pi} \int_0^{\pi/2} \frac{x^2}{\sin^2 x}\, dx = \log 4,$$

where

$$a_n = \frac{\Gamma(n+\frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(n+1)}, \quad n = 0, 1, 2, \dots .$$

Solution 90.2. Let $f(x) = \log(\sin x)$. Then $f'(x) = \cot x$ and $f''(x) = -\mathrm{cosec}^2 x$. Observe that

$$\int_0^{\pi/2} \frac{x^2}{\sin^2 x}\, dx = -\int_0^{\pi/2} x^2 f''(x)\, dx.$$

Integrating by parts twice gives

$$\int_0^{\pi/2} \frac{x^2}{\sin^2 x}\, dx = 2\big[x f(x)\big]_0^{\pi/2} - \big[x^2 f'(x)\big]_0^{\pi/2} - 2\int_0^{\pi/2} f(x)\, dx = -2\int_0^{\pi/2} f(x)\, dx.$$

Now,

$$\int_0^{\pi/2} \log(\sin x)\, dx = \int_0^{\pi/2} \log(\cos x)\, dx,$$

so

$$\int_0^{\pi/2} f(x)\, dx = \tfrac{1}{2}\int_0^{\pi/2} (\log(\sin x) + \log(\cos x))\, dx = -\tfrac{1}{4}\pi \log 2 + \tfrac{1}{2}\int_0^{\pi/2} \log(\sin 2x)\, dx.$$

After making the substitution $y = 2x$ and rearranging we obtain

$$\int_0^{\pi/2} f(x)\, dx = -\frac{\pi}{2}\log 2.$$

Hence

$$\frac{2}{\pi}\int_0^{\pi/2} \frac{x^2}{\sin^2 x}\, dx = \frac{2}{\pi} \times \pi \log 2 = \log 4.$$

Next, observe that

$$a_n = \frac{\Gamma(n+\frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(n+1)} = \frac{1}{2^{2n}}\binom{2n}{n}.$$

Using the generating function

$$\arcsin x = \sum_{n=0}^{\infty} \frac{a_n}{2n+1} x^{2n+1},$$

we see that the convolution

$$\sum_{k=0}^{n}\left(\frac{a_k}{2k+1}\right)\left(\frac{a_{n-k}}{2(n-k)+1}\right)$$

is the coefficent of x^{2n+2} in $(\arcsin x)^2$. But

$$(\arcsin x)^2 = \frac{1}{2}\sum_{n=1}^{\infty}\frac{(2x)^{2n}}{n^2}\binom{2n}{n}^{-1},$$

so

$$\sum_{k=0}^{n}\left(\frac{a_k}{2k+1}\right)\left(\frac{a_{n-k}}{2(n-k)+1}\right) = \frac{1}{2}\frac{2^{2n+2}}{(n+1)^2}\binom{2n+2}{n+1}^{-1}$$

and

$$\begin{aligned} a_n\sum_{k=0}^{n}\frac{a_k a_{n-k}}{(2k+1)(2(n-k)+1)} &= \frac{1}{2^{2n}}\binom{2n}{n}\times\frac{1}{2}\frac{2^{2n+2}}{(n+1)^2}\binom{2n+2}{n+1}^{-1} \\ &= \frac{2}{(2n+1)(2n+2)} \\ &= 2\left(\frac{1}{2n+1}-\frac{1}{2n+2}\right). \end{aligned}$$

Finally,

$$\begin{aligned} \sum_{n=0}^{\infty}a_n\sum_{k=0}^{n}\frac{a_k a_{n-k}}{(2k+1)(2(n-k)+1)} &= 2\sum_{n=0}^{\infty}\left(\frac{1}{2n+1}-\frac{1}{2n+2}\right) \\ &= 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n} = \log 4. \end{aligned}$$

□

The third problem was solved by Brian Bradie, Finbarr Holland, Kee-Wai Lau of Hong Kong, China, the North Kildare Mathematics Problem Club, and the proposer, Seán Stewart. We present the solution of Finbarr Holland.

Problem 90.3. Evaluate

$$\sum_{m,n=0}^{\infty}\binom{2m}{m}^2\binom{2n}{n}\frac{1}{2^{4m+2n}(m+n+1)}.$$

Solution 90.3. Let

$$a_n = \binom{2n}{n}\frac{1}{2^{2n}}, \quad n = 0, 1, \dots .$$

It is easy to show by induction that

$$a_n = \frac{\Gamma(\frac{1}{2}+n)}{\Gamma(\frac{1}{2})\Gamma(n+1)}, \quad n = 0, 1, \dots .$$

Let $f(z) = 1/\sqrt{1-z}$, for $|z| < 1$. Then

$$f(z) = \sum_{n=0}^{\infty} a_n z^n.$$

With this notation, the sum S that we are to evaluate satisfies

$$\begin{aligned} S &= \sum_{m,n=0}^{\infty} \frac{a_m^2 a_n}{m+n+1} \\ &= \int_0^1 \sum_{m,n=0}^{\infty} a_m^2 a_n t^{m+n}\, dt \\ &= \int_0^1 \left(\sum_{m=0}^{\infty} a_m^2 t^m\right)\left(\sum_{n=0}^{\infty} a_n t^n\right) dt \\ &= \sum_{m=0}^{\infty} a_m^2 \int_0^1 t^m f(t)\, dt. \end{aligned}$$

Observe that

$$\int_0^1 t^m f(t)\, dt = \int_0^1 t^m (1-t)^{-1/2}\, dt = \frac{\Gamma(\frac{1}{2})\Gamma(m+1)}{\Gamma(m+\frac{3}{2})} = \frac{1}{a_m(m+\frac{1}{2})}.$$

Hence

$$S = \sum_{m=0}^{\infty} \frac{a_m}{m+\frac{1}{2}} = 2\int_0^1 \sum_{m=0}^{\infty} a_m t^{2m}\, dt = 2\int_0^1 \frac{1}{\sqrt{1-t^2}}\, dt = \pi. \qquad \square$$

We invite readers to submit problems and solutions. Please email submissions to `imsproblems@gmail.com` in any format (we prefer LaTeX). Submissions for the summer Bulletin should arrive before the end of April, and submissions for the winter Bulletin should arrive by October. The solution to a problem is published two issues after the issue in which the problem first appeared. Please include solutions to any problems you submit, if you have them.

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