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The Beauty of Simultaneous Equations

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Abstract. The old chestnut of "Two Trains and a Fly" is well known, but what happens in windy conditions? Simultaneous equations provide an attractive solution.

Two trains 90 miles apart are travelling towards each other along the same track. The first train goes 50 miles per hour; the second train trundles at 40 miles per hour. A fly hovers just above the front of the first train. It buzzes from the first train to the second train, turns around immediately, flies back to the first train, and turns around again. It goes on flying back and forth between the two trains until they collide. If the fly's speed is 60 miles per hour how far will it travel?

Nowadays it is generally accepted that there are two ways to solve this problem. The **pedestrian** method is to sum an infinite geometric series, but there is also a **smart** way which simply observes that the trains will collide after one hour and during that time the fly will have flown 60 miles which must therefore be the answer.

The story goes that when this question was addressed to von Neumann he thought for a couple of seconds before answering "60 miles". The questioner complimented him, "Well done. Most people try to sum the infinite series." to which he famously replied, "What do you mean? That's how I did it!"

The purpose of this note is to point out an apparent oversight. There is a third method of solution which is superior to both those above. Indeed it is one of the best-disguised applications of simultaneous equations that I have ever come across.

Consider a practical generalization by introducing a gentle breeze. How far does the fly travel if there is a constant wind blowing at 2 mph from the first train towards the second one?

A naïve attempt to answer this might place an observer on a parallel track travelling at the same velocity as the wind, namely 2 mph. The fly is always travelling at 60 mph relative to the observer and the two trains still meet after one hour. After that time the fly will have flown 60 miles in the observer's frame of reference and the observer himself will have moved 2 miles, so the "absolute" total distance travelled by the fly seems to be 62 miles.

Unfortunately this "solution" can be easily debunked by spotting that the fly is a 2-speed object which manages 62 mph downwind but only 58 mph in the other direction. In order to cover 62 miles within the hour it would have to maintain the higher speed throughout, which it clearly doesn't do. So this idea is flawed and we must try again.

A much better approach is to use simultaneous equations. Let x and y denote the time in hours the fly spends at 62 mph and 58 mph respectively.

Then $\quad x + y = 1 \quad$ (being the length of time until the trains meet)

and $\quad 62x - 58y = 50 \quad$ (the distance between start point and collision point).

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Straightforward computation yields $x = 0{\cdot}9$, $y = 0{\cdot}1$ and since the total distance flown during the hour in question is simply $62x + 58y$ the right answer is $61{\cdot}6$ miles. Anyone who doubts this may confirm it by following von Neumann's thought process and summing the infinite geometric series. Assume the trains are 459 units apart at the start of an iteration. After its out and back trip the fly will have covered 308 units and the trains will be 9 units apart. So each iteration is $\frac{9}{459} = \frac{1}{51}$ of the previous one, and the distance we want is

$$\frac{308 \times 90}{459}\left[1 + \left(\frac{1}{51}\right) + \left(\frac{1}{51}\right)^2 + \left(\frac{1}{51}\right)^3 + \dots\right] = \frac{3080 \times 51}{51 \times 50} = 61\tfrac{3}{5} \text{ miles .}$$

In fact if the wind speed is w miles per hour then $x = (110-w)/120$, $y = (10+w)/120$ and it is easily calculated that the total distance travelled by the fly is $60+w(50-w)/60$ miles. Naturally w must lie somewhere in the range -10 to 20 or else the fly will be unable to keep pace with one of the trains and so will happily miss the collision.

Finally return to the original question, in other words the special case when $w = 0$. The simultaneous equations become $x + y = 1$ and $60x - 60y = 50$ and the total distance flown is $60x + 60y$. For the purpose of deducing the latter the second equation is redundant. From the first equation alone we reach the smart conclusion that the fly travels exactly 60 miles before being squashed.

I have searched the World-Wide-Web in the expectation of finding this more general and (in my view) more satisfactory approach to Two Trains and a Fly. However I've found very few references to possible wind effects, and none that uses simultaneous equations to address the matter.

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