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Michael Z. Spivey: The Art of Proving Binomial Identities, CRC Press, 2019.
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REVIEWED BY HENRY RICARDO

Probably all readers of this journal have encountered binomial coefficients many times in their teaching and/or research. Even Gilbert and Sullivan's Modern Major General declared "About binomial theorem I'm teeming with a lot o' news." As someone who is not a combinatorialist (or a military officer) but who haunts the problem sections of various mathematical journals, I have encountered many challenging problems involving binomial coefficients. Often my instinct is to check Gould's impressive compendium [1] of over 500 binomial coefficient identities—but, unfortunately, there are no proofs.

In ten chapters, Michael Spivey succeeds in bringing together in a systematic way the many methods used in dealing with binomial coefficients. The techniques covered in this book consist of algebra (including finite difference methods and complex numbers), calculus, linear algebra, and combinatorics/probability. Anyone absorbing the techniques in Spivey's book should be prepared to understand Gould's collection and tackle deeper works such as Riordan's classic monograph [2]. This is not an encyclopedia, although the convenient Index of Identities and Theorems makes it easy to find alternative proofs or other uses of particular identities. Because of the way Spivey's book ties together several key courses in the undergraduate mathematics curriculum, a university maths department could base a senior seminar or capstone course/project on this book.

The author starts by proving the equivalence of four definitions of the binomial coefficient for integers n and k with $n \geq k \geq 0$:

(1) The number of subsets of size k formed from a set of n elements;
(2) $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$ when $n-1 \geq k \geq 1$ with boundary conditions $\binom{n}{0} = \binom{n}{n} = 1$;
(3) the coefficient of x^k in the expansion of $(x+1)^n$ in powers of x;
(4) $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.

Later, Spivey generalises to $n \in \mathbb{R}$ and $k \in \mathbb{Z}$.

Two of my favourite topics are given good introductory treatments: the central binomial coefficient (CBC) and reciprocal binomial coefficients (RBCs). One worked-out example yields this gem:

$$\sum_{n=0}^{\infty} \frac{1}{\binom{2n}{n}} = \frac{4}{3} + \frac{2\pi\sqrt{3}}{27}.$$

The CBC is developed further in other sections of the book, including discussions of Catalan numbers and lattice paths. A number of other sums (finite and infinite) of RBCs are discussed and are listed in the very useful Index to Identities and Theorems, but the term reciprocal binomial coefficient is missing from the main index.

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Some of the over 300 numbered identities appear as examples, while others are used as exercises. Throughout the book there are historical inserts, presenting snippets of information about various mathematicians: Eric Temple Bell, the Bernoulli family, ..., Newton, Pascal, Stirling, ..., Vandermonde, Zeilberger. There are also end-of-chapter notes that provide context for the chapter's results and/or recommendations for further reading. Appendices include an 82-item Bibliography and Hints and Solutions to Exercises—the latter particularly useful for self-study.

In the final chapter ("Mechanical Summation"), art gives way to science. Here the author explains the powerful Gosper-Zeilberger algorithm, both Gosper's original form and Zeilberger's extension. Hypergeometric series are introduced early. The treatment contains "very heavy algebra", in the author's words, but is a tour de force of exposition.

In summary, this is a delightful and useful book: A readable introduction to binomial coefficients and many of their applications for the advanced undergraduate or graduate student, an aid to those mathematically mature individuals who are not combinatorialists, an inspiration for those who attempt to solve problems involving binomial coefficients.

References

[1] H. W. Gould: *Combinatorial Identities* (*Revised Edition*), Morgantown Printing, Morgantown, West Virginia, 1972.

[2] J. Riordan: *Combinatorial Identities*, Robert E. Krieger Publishing Company, New York, 1979.

Henry Ricardo received his Ph.D. from Yeshiva University in New York. Since retiring from The City University of New York as Professor of Mathematics, he has been a volunteer with the Westchester Area Math Circle (Purchase, NY). He is an avid problemist and regularly attempts problems in more than a dozen journals worldwide.

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