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J. Stillwell: The Story of Proof, Princeton University Press, 2022. ISBN:978-0691234366, USD 45.00, 456 pp.

REVIEWED BY TOMMY MURPHY

The physicist Freeman Dyson opined that all mathematicians are by temperament either eagles or frogs. The eagle soars over the mathematical landscape, observing connections between apparently disparate fields and generalising theorems whilst staying light on details, whereas the frog stays in a small locality of the world of mathematics, delving into the intricacies and focused on a deep understanding of specific questions. Dyson's thesis is that the academy needs both viewpoints, and collaborations are enriched when these two tribes work together. Poincaré is widely believed to be the last person who could soar high enough to see the entire mathematical landscape; due to the specialised nature and enormous volume of mathematical research nowadays even our eagles tend to roam within one or two well-trodden research areas.

In "The Story of Proof", John Stillwell has written a well-crafted, thought-provoking meditation on the concept of proof in mathematics, which is used as an organising principle to explain, in broad brushstrokes, how disparate fields of mathematics emerged and lay bare the origins of some of major problems in mathematics. If you are interested in pure mathematics, you should buy and/or read this book. In the spirit of Dyson's dichotomy, it is enlightening and satisfying to learn more about how different areas of research are connected. To give one such example, I had never known how knot theory arose from the study of singularities of algebraic curves. Before reading this book I also did not fully appreciate that Dedekind and Kronecker developed field extensions in an attempt to abstract of the concept of dimension for vector spaces, and used this to answer the ancient Greek problem of duplicating a cube. The book is full of such insights. I learnt field extensions and applications as an undergraduate, and have never really thought about them since or why they were important as it is not relevant to my research. The point is that in today's competitive race to the coal-face of research, we tend to focus on specific topics and questions and much context is lost without an appreciation of how and why these questions arose. As undergraduates we learn about the insolvability of the quintic: we perhaps do not appreciate the reason that surds are of interest in this context is because of the connection with compass-and-straightedge constructions arising from ancient Greek mathematics.

A fascinating theme of the book is the connection between logic and computation. As Stillwell points out, in Greek times logic was comparatively strong but the theory of computation was weak. Thus was born the great glories of Greek mathematics, as opposed to other civilisations of the time who could compute but did not truly understand. Computational techniques advanced tremendously with the advent of calculus, and logic and proof had to catch up. This of course leads us to the natural development of analysis in the eighteenth century. As such, the book begins by focusing on a (somewhat perfunctory) survey of Euclid, before jumping to Hilbert's axiomisation of geometry. At times the pace is too fast and material is not fully explained: we find the

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question of proving consistency of an axiomatic system by constructing a model being mentioned, without motivation, on page 48. Nevertheless, there is much to glean from the text even if some statements wash over the reader initially. The gradual realisation that logic and computation were closely related mathematical concepts is explained well in this book, culminating in a very satisfactory survey of the work of Gödel and Turing. By this stage questions of consistency make more sense to the reader.

Stillwell has many interesting examples explaining how various proofs of a theorem evolved which will enrich the reader's appreciation and understanding. A striking example of this is in the discussion of various failed proofs of the Fundamental Theorem of Algebra. Trying to battle through the maze of details here is what led Bolzano to realise the importance of trying to establish the Intermediate Value Theorem via the least upper bound property: a crucial component in the development of real analysis and closely connected to Dedekind cuts. Another instance, though certainly not novel to this book, is the emphasis placed on explaining the origins of algebraic geometry and placing them in the context of Greek work on conics.

I know of no comparable book on the market. It is not suitable as a textbook for an undergraduate or postgraduate course on the history of mathematics, owing to its dizzying pace and the deliberate choice to omit details and proofs in many places and to emphasise connections. It is not a comprehensive overview of every aspect of the history of mathematics, rather a discourse on what we are really doing in mathematics and how our understanding of proof, computation, and logic have evolved and intertwined over two millennia of human thought, and how remarkably interconnected mathematics is. It is a substantive book that deserves to be read and reflected upon.

Tommy Murphy is Associate Professor at Cal State Fullerton, having completed his Ph.D. under Jürgen Berndt at UCC and postdoctoral fellowships at the Université Libre de Bruxelles (Belgium) and McMaster University (Canada). His research interests span Riemannian and Kähler geometry, focused specifically on Einstein manifolds and symmetric spaces. He also maintains an active interest in the history of mathematics and collaborating with undergraduate students.
Website: https://www.fullerton.edu/math/faculty/tmurphy/

Dept. of Mathematics, California State University Fullerton, 800 N. State College Blvd., Fullerton 92831, CA, USA.
E-mail address: tmurphy@fullerton.edu