Irish Math. Soc. Bulletin
Number 91, Summer 2023, 83–86
ISSN 0791-5578

PROBLEMS

IAN SHORT

Problems

The first problem this issue was proposed by Toyesh Prakash Sharma of Agra College, India.

Problem 91.1. Prove that the angles α, β and γ of a triangle satisfy

$$\big(1+\sec^2\alpha+\sec^2\beta+\sec^2\gamma\big)(1-\cos\alpha\cos\beta\cos\gamma)\geqslant 8$$

The second problem is from Des MacHale of University College Cork.

Problem 91.2. Prove that the perimeter P and area A of a cyclic quadrilateral satisfy

$$P^2\geqslant 16A,$$

with equality if and only if the cyclic quadrilateral is a square.

The third problem was proposed by Tran Quang Hung of the Vietnam National University at Hanoi, Vietnam.

Problem 91.3. Let $A_0,A_1,\dots,A_n$ be the vertices of a simplex in n-dimensional Euclidean space for which the edges $A_0A_1,A_0A_2,\dots,A_0A_n$ are mutually perpendicular. Let B_i be the centroid of the set of points $\{A_0,A_1,\dots,A_n\}\setminus\{A_i\}$, for $i=0,1,\dots,n$. Consider any point C other than A_0 for which the line through A_0 and C is perpendicular to the hyperplane spanned by $A_1,A_2,\dots,A_n$, and let P be the midpoint of the segment B_0C. Prove that all distances PB_i are equal, for $i=1,2,\dots,n$.

Solutions

Here are solutions to the problems from *Bulletin* Number 89.

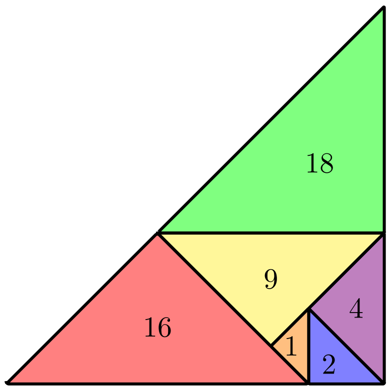
The first problem was solved by Ryan Quinn, Seán Stewart of the King Abdullah University of Science and Technology, Saudi Arabia, the North Kildare Mathematics Problem Club, and the proposer Des MacHale. We present the solution and figures of Seán Stewart.

*Problem 89.1***.** It is well known that it is possible to dissect a square into a finite number of different squares, but that it is not possible to dissect an equilateral triangle into a finite number of different equilateral triangles. Determine whether it is possible to dissect an isosceles right-angled triangle into a finite number of different isosceles right-angled triangles.

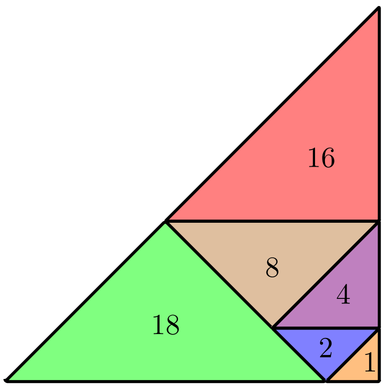
Solution 89.1. One example is shown below. Here an isosceles right-angled triangle with common side length of 10 units is dissected into 6 different isosceles right-angled triangles. The numbers in the triangles record areas.

Received 17-6-2023
DOI:10.33232/BIMS.0091.83.86.

©2023 Irish Mathematical Society



We also give a second example. Here an isosceles right-angled triangle with common side length of $7\sqrt{2}$ units is again dissected into 6 different isosceles right-angled triangles.



$\square$

Stewart references a paper by Skinner, Smith, and Tutte (*Journal of Combinatorial Theory*, Series B **80**, 2000) for related results, and there are further publications and internet resources available. He also asks what is the minimum number of triangles in a dissection that answers Problem 89.1.

The second problem was solved by Daniel Văcaru of Pitești, Romania, Brian Bradie of Christopher Newport University, USA, Seán Stewart, Henry Ricardo of the Westchester Area Math Circle, USA, the North Kildare Mathematics Problem Club, and the proposer, Toyesh Prakash. We present the solution of Brian Bradie (other solutions were similar).

*Problem 89.2***.** Prove that

$$\int_{-\pi/2}^{\pi/2} \cos^2(\tan x)\, dx = \frac{\pi}{2}(1+e^{-2}).$$

Solution 89.2. With the substitution $u = \tan x$ and the half-angle identity

$$\cos^2 u = \frac{1}{2} + \frac{1}{2}\cos 2u,$$

we have

$$\begin{aligned}
\int_{-\pi/2}^{\pi/2} \cos^2(\tan x)\, dx &= \int_{-\infty}^{\infty} \frac{\cos^2 u}{1+u^2}\, du \\
&= \frac{1}{2}\int_{-\infty}^{\infty} \frac{1}{1+u^2}\, du + \frac{1}{2}\int_{-\infty}^{\infty} \frac{\cos 2u}{1+u^2}\, du \\
&= \frac{\pi}{2} + \frac{1}{2}\int_{-\infty}^{\infty} \frac{\cos 2u}{1+u^2}\, du.
\end{aligned}$$

Now,

$$\int_{-\infty}^{\infty} \frac{\cos 2u}{1+u^2}\,du = \mathrm{Re}\int_{-\infty}^{\infty} \frac{e^{2ui}}{1+u^2}\,du,$$

and

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{e^{2ui}}{1+u^2}\,du &= \lim_{R\to\infty}\int_{C_R} \frac{e^{2ui}}{1+u^2}\,du \\ &= 2\pi i\ \mathrm{Res}\left(\frac{e^{2ui}}{1+u^2}; u=i\right) \\ &= 2\pi i \times \frac{e^{-2}}{2i} = \frac{\pi}{e^2},\end{aligned}$$

where C_R is the contour $(-R,R)\cup\{Re^{i\theta} : 0 \leqslant \theta \leqslant \pi\}$. Thus,

$$\int_{-\pi/2}^{\pi/2} \cos^2(\tan x)\,dx = \frac{\pi}{2} + \frac{\pi}{2e^2} = \frac{\pi}{2}(1+e^{-2}). \qquad \square$$

The third problem comes from Finbarr Holland of University College Cork. It was solved by Seán Stewart, Daniel Văcaru, the North Kildare Mathematics Problem Club, and the proposer. We present the solution of the problem club.

Problem 89.3. Let a_k and b_k be positive real numbers with $a_k < b_k$, for $k = 1, 2, \ldots, n$, and let

$$r_n(z) = \prod_{k=1}^{n} \frac{b_k + z}{a_k + z}.$$

Prove that

$$\int_{-\infty}^{\infty} \log|r_n(ix)|\,dx = \pi \sum_{k=1}^{n} (b_k - a_k).$$

Solution 89.3. We will prove the result for $n = 1$; the general result then follows because $\log|x|$ converts products into sums. Let $r(x) = r_1(x)$, $a = a_1$, and $b = b_1$.

Notice that

$$\log|r(ix)| = \log\left|\frac{b+ix}{a+ix}\right| = \log\sqrt{\frac{b^2+x^2}{a^2+x^2}} = \frac{1}{2}\left(\log(b^2+x^2) - \log(a^2+x^2)\right).$$

Integration by parts shows that the antiderivitive of $\log(a^2+x^2)$ is

$$x\log(a^2+x^2) + 2a\tan^{-1}(x/a) - 2x.$$

Hence

$$\int_{-\infty}^{\infty} \log|r(ix)|\,dx = \frac{1}{2}\lim_{R\to\infty}\left[x\log\left(\frac{b^2+x^2}{a^2+x^2}\right) + 2b\tan^{-1}(x/b) - 2a\tan^{-1}(x/a)\right]_{-R}^{R}.$$

The term involving the logarithm tends to zero at both limits, by L'Hôpital's rule, so we are left with

$$\int_{-\infty}^{\infty} \log|r(ix)|\,dx = \frac{1}{2}\Big((2b-2a)\times\frac{\pi}{2} - (2b-2a)\times\Big(-\frac{\pi}{2}\Big)\Big) = \pi(b-a). \qquad \square$$

Editor's remark: I added the assumption that a_k and b_k are positive to the problem, which is necessary if the solution of the integral is to retain its present form. The requirement that $a_k < b_k$ is redundant. Finbarr points out that for real numbers a_k and b_k (not necessarily positive), we have

$$\int_{-\infty}^{\infty} \log|r_n(ix)|\,dx = \pi \sum_{k=1}^{n} (|b_k| - |a_k|).$$

We invite readers to submit problems and solutions. Please email submissions to `imsproblems@gmail.com` in any format (we prefer LaTeX). Submissions for the summer Bulletin should arrive before the end of April, and submissions for the winter Bulletin should arrive by October. The solution to a problem is published two issues after the issue in which the problem first appeared. Please include solutions to any problems you submit, if you have them.

School of Mathematics and Statistics, The Open University, Milton Keynes MK7 6AA, United Kingdom