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Steven T. Dougherty: Combinatorics and Finite Geometry, Springer, 2020. ISBN:978-3-030-56394-3, GBP 29.99, 369+xv pp.

REVIEWED BY PADRAIG Ó CATHÁIN

This book is an advanced undergraduate introduction to combinatorics, with a particular emphasis on finite geometry and related topics. As such, finite projective planes appear before graphs and connections to codes and designs are discussed in later chapters. The more algebraic and enumerative aspects of combinatorics (e.g. generating functions, group actions) are treated, but less heavily.

Who is the ideal reader?

In the opinion of the reviewer, the author's style is direct and clear. The text is illustrated with many examples, proofs are written in sufficient (but not exhaustive) detail and the reviewer suspects that an advanced undergraduate could tackle large sections of the text independently. While rapid overviews of all required material are contained in the text, it seems implausible that a reader unacquainted with linear algebra will learn enough from eleven pages in Chapter 6 to fully appreciate projective geometries. This section would serve as a review, however. So the ideal reader would have taken courses in linear and abstract algebra, as well as a basic course in discrete mathematics. They would also have a strong grasp of proof writing, and be able to handle a moderate degree of abstraction. Seasoned mathematicians will find something new and interesting there also.

What does the book cover?

The opening chapter could be a free-standing combinatorics course! It is a breezy account of (finite) set theory and binomial coefficients, a rapid overview of permutations and the symmetric group (with a nice graphical interpretation, new to the reviewer, as *Japanese ladders*) and concludes with some examples of generating functions. The style is engaging, the examples are helpful and the proofs are short. Chapter 2 is further background on rings and fields, concluding with some counting problems.

The core text really begins in Chapter 3, which is devoted to the historically important and rather non-trivial problem of the 36 officers. Mutually Orthogonal Latin Squares (MOLS) are motivated and introduced, complete sets are constructed from finite fields at prime power orders, and the problem of determining the number of MOLS in dimension 10 is revealed to be open. In Chapter 4, affine and projective planes are introduced (without linear algebra!) and the equivalence with complete sets of MOLS is given. Chapter 5 is a digression on graph theory, before returning to higher dimensional combinatorial geometries in Chapter 6. This is the heart of the text: the author considers the Desargues configuration and proves Segre's celebrated theorem that an oval in a finite projective plane of odd order is a conic section. He also proves the Bruck-Ryser-Chowla theorem which rules out the existence of certain projective planes. These are **not** easy theorems: they were proved in the middle decades of the twentieth

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century and are rightly recognised as ingenious and important. The author's achievement in presenting them to undergraduates is substantial. Later chapters are devoted to designs, Hadamard matrices, association schemes, coding theory, cryptography and discrete probability. These are well written and largely free-standing, though many rely on results from finite geometry.

So far, so good. Are there any issues with the book?

This book covers an astonishing amount of material. It is perhaps unsurprising then that there were omissions which struck the reviewer as odd and unfortunate. The Veblen-Young axioms for a projective geometry should have been mentioned. The importance of Desargues' theorem (proved only for the Desarguesian planes $\mathrm{PG}_2(\mathbb{F}_q)$) is skirted over, as is any mention of perspective which is arguably the crucial concept for an understanding of the theorem. The proof of Segre's theorem in the published text contains some rather large leaps of logic and a few typographical errors, which together render it cryptic. The reviewer contacted the author about this, and is pleased to report that this resulted in an improved proof now available from the author's website tinyurl.com/poc-fix.

So, what courses could I teach from this book at an Irish university?

An undergraduate course based on this text would require quite a lot of background from the students. Certainly familiarity with linear algebra and discrete mathematics (i.e. covering sets, proofs, modular arithmetic, etc.) would be essential. Groups, rings and fields would be helpful but a course could be plotted around them rather easily (replacing $\mathbb{F}_q$ by $\mathbb{F}_p$ where necessary). With these prerequisites, a third or fourth year class in *combinatorics* or *discrete mathematics* from this text seems perfectly reasonable. Another possibility would be to teach a slightly non-conventional course on *geometry*, supplemented by a little of the classical theory of conic sections over $\mathbb{R}$ and $\mathbb{C}$, perhaps. The reviewer would also be very comfortable teaching a course at beginning graduate level from this text, and it could serve well as a starting point for final year projects or other undergraduate research requirements.

Padraig Ó Catháin received a BA in Mathematics and History from the University of Galway in 2007. Under the direction of Dane Flannery he was awarded the degrees of MLitt in 2008 and PhD in 2012, by the same institution. After a decade at universities in Australia, Finland and the United States, he took a position as Ollamh Cúnta in Fiontar agus Scoil na Gaeilge at Dublin City University in 2022. His research is predominantly in combinatorics, for which he has been awarded the Kirkman Medal of the Institute of Combinatorics and its Applications.

Leagan gaeilge den léarmheas céanna:

Is buntreoir an leabhar seo don fhochéimí aibí ar an matamaitic theaglamach, le béim ar leith ar chéimseata thar uimhirchoirp críochta agus ar ábhair ghaolta leis seo. Ní cúis ionaidh é mar sin go bhfuil cur síos ar na plánaí teilgeacha críochta sula bhfeictear grafanna. Cuirtear na plánaí teilgeacha i bhfeidhm agus cur síos á dhéanamh ag an údar ar chódaigh agus dearaí theaglamach níos déanaí sa théacs. Déantar cur síos freisin ar na gnéithe ailgéabracha agus áireamha den mhatamaitic theaglamach (m.sh. teoiric ghníomhú na ngrúpaí, feidhmeanna giniúna), ach gan róbhéim orthu.

Cé do an leabhar seo?

Dar leis an léirmheastóir, tá stíl dheas sholéite ag an údar. Baintear úsáid as neart samplaí, déantar mionphlé ar na cruthúnais (gan dul thar fóir). Dá bhrí sin, ba chóir go mbeadh fochéimí láidir in ann dul i ngleic le roinnt mhaith den leabhar faoina stuaim fhéin. Cé go dtugtar léirmheas tapaidh ar chuile ghné atá ag teastáil sa théacs, is ar ndóigh go mbeadh duine ar bith gan an ailgéabar líneach ar a dtoil acu in ann dóthain a bhaint ó aon leathanach déag i gCaibidil 6 ionas go mbeadh léirthuiscint cheart aige ar na gcéimseataí theilgeacha. D'fheilfeadh sé go maith mar achoimre ar an ábhar, áfach. Mar sin, bheadh cúrsaí sna ailgéabair líneach agus teibí araon, chomh maith le bunchúrsa sa mhatamatic scoite de dhíth ar an léitheoir. Ba chóir go mbeadh taithí mhaith acu le cruthúnais, agus go mbeadh roinnt mhaith teibiú feicthe acu. Cé gur ar fhochéimithe atá an leabhair dírithe, tá neart ann ar mhaitheas an mhatamataiceora lánfhásta freisin.

Céard atá sa leabhar?

Tá neart sa chéad chaibidil le haghaidh cúrsa iomlán sa mhatamataic theaglamach a mhúineadh! Insint bhreá thapaidh atá ann ar theoiric na tacair críochta, na chomhéifeachtaí dhéthéarmacha, iomalartuithe agus an ghrúpa siméadrach (le léiriú deas grafaiceach i dtéarmaí *Japanese ladders*, nach bhfuil feicthe cheana ag an léirmheastóir). Tá críoch leis an chaibidil le roinnt samplaí de na feidhmeanna giniúna. Tá stíl shoiléir sholéite ag an údar, tá mianach sna samplaí a thugtar agus tá na cruthúnais gearr agus dírithe. Sa dhara caibidil tugtar tuilleadh buneolais maidir le faileanna agus uimhirchoirp, le beagán ar áireamh ar deireadh.

Cuirtear tús le príomhábhar an téacs sa thríú caibidil, ina ndéantar cur síos ar fhadhb na 36 Oifigigh, a raibh tábhacht ar leith ag baint leis i stair na matamataic. Mar seo, tugtar spreagadh agus intreoir ar na *Cearnógaí Laidneach Comhortagánach* (MOLS, ag baint úsáid as an ngiorrúcháin a úsáidtear go hiondúil). Tugtar déantús ar tacair iomláin MOLS ag baint úsáid as uimhirchoirp ag ord ar bith ar cumhacht d'uimhir phríomha é, agus cuirtear in iúl don léitheoir gur fadhb oscailte í uaslíon na MOLS in ord a deich. Sa cheathrú caibidil, tugtar cur síos ar na plánaí teilgeacha agus fineacha[1] (gan chaint ar an ailgéabar líneach!) agus cuirtear an comhionnanais le tacar iomlán MOLS in iúl. Is aistear ar leataobh an cúigiú caibidil ar bhealach, i dtreo na ghrafanna, sula bhfilltear ar na céimseataí teilgeacha de dhiminsean níos mó ná dó i gCaibidil 6. Is í seo croílár an leabhair: déantar cíoradh ar chumraíocht Desargues, agus tugtar cruthúnais iomlán ar theoraim thábhachtach de chuid Segre a dhéanann cur síos ar na hubhcruthanna i bplána teilgeach d'ord chorr i dtéarmaí chónghearrtha. Tugtar cruthúnas iomlán freisin ar an teoraim de chuid Bruck-Ryser-Chowla, a chuireann cosc ar uimhreacha áirithe mar ord ar phlána teilgeach. Ní torthaí éasca iad seo! Aimsíodh iad i lár an fichiú aois, agus meastar go forleathan go bhfuil tábhacht agus sárbhua ar leith ag baint leo. Tá gaisce déanta ag an údar anseo agus iad ar fáil ar leibhéal soléite don fhochéimí. Tá caibidil níos déanaí sa leabhar dírithe ar dhearaí teaglamacha, maitrisí Hadamard, scéimeanna comhthiomsaitheacha, teoraic an chódaigh, cripteagrafaíocht agus an dóchúlacht scoite.

[1]Eitleán cleamhnais a thug Google Translate ar seo!! Moltar tearma.ie do théarmaí teicniúla.

Tá siad seo soléite agus neamhspleách go maith ar a chéile, seachas go bhfuil roinnt mhaith dóibh ag brath ar an gcéimseata críochta.

Maith go leor. An bhfuil lucht ar bith ar an leabhar?

Clúdaítear an t-uafás ábhar sa leabhar. Ní ionadh ar bith é mar sin go raibh roinnt rudaí in easnamh, ar a laghad i súile an léirmheastóra. Is mór an trua, mar shampla, gur fágadh aicsímí Veblen-Young ar an chéimseata theilgeach ar lár sa théacs. Cé go ndéantar cur síos ar chumraíocht Desargues, agus go dtugtar cruthúnas dó dos na plánaí ar a nglaotar *Desarguesian* orthu, níor dhearnadh iarracht ar bith cur síos a dhéanamh ar an tábhacht a bhaineann leis ó thaobh an comhordanáidiú. Níl trácht ar bith ach oiread ar an bpeirspictíocht, ar cheann de na coincheapa is tábhachtaí ar fad sa réimse seo. Sa chruthúnas ar Theoraim Segre, tá roinnt bhotún clóghrafach ann, a fhágann an scéal doiléir go maith. Chuaigh an léirmheastóir i dteagmháil leis an údar faoi seo áfach, agus is deas le rá go bhfuil leagan nua den chruthúnas seo ar fáil anois ó shuíomh idirlín an údair tinyurl.com/poc-fix.

Ar deireadh, cén cineál cúrsa ar fiú múineadh ón leabhar seo in ollscoil Éireannach?

Bheadh bunús sách láidir sa mhatamataic ag teastáil ó fhochéimithe agus iad ag cur tús le cúrsa bunaithe ar an leabhar seo. Cinnte ba chóir go mbeadh an ailgéabair líneach agus an mhatamataic scoite (sé sin tacair, cruthúnais, uimhríocht mhodúlach, srl.) ar a dtoil acu. Ba mhaith an rud é freisin go mbeadh tuiscint acu ar ghrúpaí, faileanna agus uimhirchoirp, ach bheadh sé sodhéanta iad seo a sheachaint sa théacs (ag déanamh ionadú ar $\mathbb{F}_q$ le $\mathbb{F}_p$ agus a leithid). Leis na réamhriachtanais seo, d'fheadfaí chúrsa don tríú nó don cheathrú bhliain sa *mhatamataic scoite* nó *theaglamach* a theagaisc ón leabhar seo. Bheadh sé indéanta freisin cúrsa beagán neamhghnách a thabhairt ar an *gcéimseata*, le breis cuirthe isteach ar theoiric na chóngearradh clasaiceach thar na huimhirchoirp $\mathbb{R}$ agus $\mathbb{C}$. Bheadh an léirmheastóir lán-sásta freisin rang ar léibhéal na hiarchéime tosaíocha a mhúineadh as an leabhar. Is foinse iontach maith é freisin ar ábhar do thionscnaimh na bliana deireanaí nó mar sheoladh ar an taighde ar leibhéal na buncéimithe.

Padraig Ó Catháin Bhain sé amach BA sa mhatamataic agus sa stair ó Ollscoil na Gaillimhe sa bhlian 2007. Bronnadh céim MLitt air i 2008 agus céim PhD i 2012, agus é ag déanamh taighde faoi stiúr Dane Flannery san áit céanna. Théis deich mbliana ag obair thar lear san Astráil, san Fhionlainn agus sna Stáit Aontaithe, d'fhill sé ar Éirinn i 2022 chun post a ghlacadh mar Ollamh Cúnta le Fiontar agus Scoil na Gaeilge le hOllscoil Chathair Bhaile Átha Cliath. Tá an chuid is mó den taighde aige sa mhatamataic teaglamach, agus bronnadh Bonn Kirkman de chuid an *Institute of Combinatorics and its Applications* air in aitheantas ar seo.

Stór Focail

Céimseata	Geometry
Cónghearradh	Conic section
Críochta	Finite
Cód, Códaigh	Code, Codes
Cumraíocht	Configuration
Dearadh theaglamach	Combinatorial design
Feidhmeanna ghiniúna	Generating functions
Gníomhú na ngrúpaí	Group actions
Matamaitic theaglamach	Combinatorics
Scéim comhthiomsaitheach	Association Scheme
Scoite	Discrete
Teibí, Teibiú	Abstract, Abstraction
Teilgeach	Projective
Ubhcruth	Oval
Uimhirchoirp (críochta)	(finite) Field

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