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The Golden section in the hypercube

QUANG HUNG TRAN

Abstract. We shall present a way to establish the Golden section in n-dimensional Euclidean space. We use a hypercube covered by a hypersphere and divide the diameters of two opposing facets in a way that depends on the dimension of the space. The Golden ratio will be obtained from the ray connecting these two dividing points intersecting the hypersphere.

1. Introduction

The Golden ratio $\varphi = \frac{\sqrt{5}+1}{2}$ is one of the most beautiful numbers. It has a long history in many different areas of life such as Art, Nature, and Science; see [8, 9]. In Mathematics, the Golden Ratio is mentioned early on, already appearing in Euclid's Elements; see [7], and it has been much studied throughout history; see [11]. In modern Mathematical research, the Golden Ratio remains relevant to some problems, see [3, 10, 12]. In this paper, we introduce and prove our discovery about the occurrence of the Golden ratio in n-dimensional Euclidean space associated with the hypercube [2, 6] and the hypersphere [4, 5, 6].

Theorem 1.1 (Main theorem)**.** *Let $\mathcal{N}$ be a hypercube contained in n-dimensional Euclidean space $\mathbb{E}^n$ ($n \geq 2$). Let $\mathcal{F}_0$ be a facet of $\mathcal{N}$ with center K. Let $\mathcal{F}_0^*$ be the facet opposite to $\mathcal{F}_0$. Let $\mathcal{S}$ be a hypersphere centered at K and passing through all vertices of $\mathcal{F}_0^*$. Let $\mathcal{F}_1$ be a facet of $\mathcal{N}$ that is perpendicular to $\mathcal{F}_0$. Let XY be a diameter of $\mathcal{F}_1$. Let X^* and Y^* be the reflections of X and Y through the center of $\mathcal{N}$. Let Z and Z^* divide the segments XY and Y^*X^*, respectively, in the ratio $n-2$ to 1 i.e.*

$$Z = \frac{(n-2)X+Y}{n-1} \quad \textit{and} \quad Z^* = \frac{(n-2)Y^*+X^*}{n-1}. \tag{1}$$

Let the ray Z^*Z meet the hypersphere $\mathcal{S}$ at Z_0. Then,

$$\frac{Z^*Z}{ZZ_0} = \varphi. \tag{2}$$

Where $n = 2$, we have a configuration with square and circle; see Figure 1.
Where $n = 3$ we have a configuration with cube and sphere; see Figure 2.

2. Proof of main theorem

In this section, we give a proof of Theorem 1.1.

Proof. Let $\mathcal{N} = [-1, 1]^n$ in the Cartesian coordinates of n-dimensional Euclidean space $\mathbb{E}^n$. Let $K = (0, 0, \ldots, 0, -1)$. Thus $\mathcal{S}$ is the hypersphere centered at K and goes through vertex $A_0 = (1, 1, \ldots, 1)$, so $\mathcal{S}$ has equation

$$x_1^2 + x_2^2 + \ldots + (x_n+1)^2 = (1-0)^2 + (1-0)^2 + \ldots + (1-0)^2 + (1-(-1))^2 \tag{3}$$

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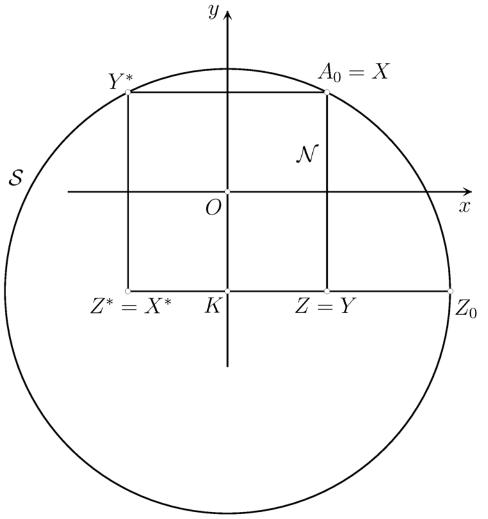

FIGURE 1. Illustrations in two dimensions $n = 2$, $\frac{Z^*Z}{ZZ_0} = \varphi$.

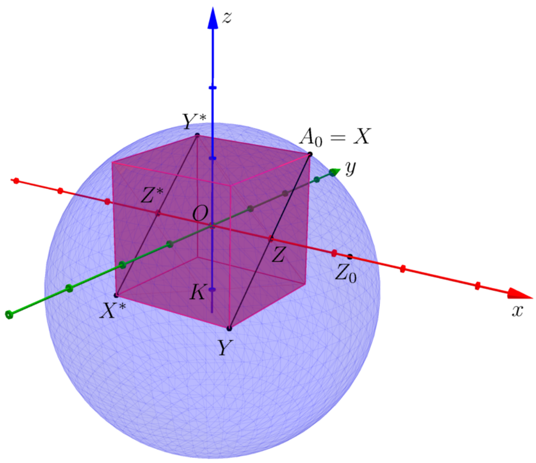

FIGURE 2. Illustrations in three dimensions $n = 3$, $\frac{Z^*Z}{ZZ_0} = \varphi$.

which is

$$x_1^2 + x_2^2 + \ldots + (x_n + 1)^2 = n + 3. \tag{4}$$

Since XY is a diameter of $\mathcal{F}_1$, which is perpendicular to $\mathcal{F}_0$ (centered at K), we may choose $X = A_0 = (1, 1, \ldots, 1)$ and then Y is the reflection of X in the center $K_1 = (1, 0, \ldots, 0)$. Therefore $Y = 2K_1 - X = (1, -1, -1, \ldots, -1)$. Now X^* and Y^* are the reflections of X and Y, respectively, in the center $O = (0, 0, \ldots, 0)$ of $\mathcal{N}$, so we obtain the coordinates

$$X^* = (-1, -1, \ldots, -1)$$

and

$$Y^* = (-1, 1, \ldots 1).$$

Thus,

$$Z = \frac{(n-2)X + Y}{n-1} = \left(1, \frac{n-3}{n-1}, \frac{n-3}{n-1}, \ldots, \frac{n-3}{n-1}\right)$$

and

$$Z^* = \frac{(n-2)Y^* + X^*}{n-1} = \left(-1, \frac{n-3}{n-1}, \frac{n-3}{n-1}, \dots, \frac{n-3}{n-1}\right).$$

From these, the line ZZ^* has parametric equation

$$X = Z^* + t \cdot \overrightarrow{ZZ^*} = \left(-1-2t, \frac{n-3}{n-1}, \frac{n-3}{n-1}, \dots, \frac{n-3}{n-1}\right). \tag{5}$$

The intersection of the ray Z^*Z (equation (5)) and the hypersphere $\mathcal{S}$ (equation (4)) is the point $Z_0 = Z^* + t_0 \cdot \overrightarrow{ZZ^*}$ $(t_0 > 0)$, where t_0 satisfies the equation

$$(-1-2t_0)^2 + (n-2)\left(\frac{n-3}{n-1}\right)^2 + \left(\frac{2n-4}{n-1}\right)^2 = n+3, \tag{6}$$

which is equivalent to

$$(1+2t_0)^2 = n+3 - \frac{(n-2)(n-3)^2 + 4(n-2)^2}{(n-1)^2}. \tag{7}$$

Therefore

$$(1+2t_0)^2 = 5 \tag{8}$$

or

$$t_0 = \frac{\sqrt{5}-1}{2} = \frac{1}{\varphi}. \tag{9}$$

Since $Z_0 = Z^* + t_0 \cdot \overrightarrow{ZZ^*}$,

$$\frac{ZZ^*}{Z^*Z_0} = \frac{1}{t_0}.$$

Hence equation (2) holds. This completes the proof. □

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Quang Hung Tran graduated with a First-class BS (2008), and a Master's (2011) in pure mathematics from the Vietnam National University at Hanoi. He is a geometry teacher at the High School for Gifted Students of Vietnam National University at Hanoi. Many of his students have won Gold medals in the International Mathematical Olympiad. His main interests include many aspects of geometry such as multidimensional Euclidean and non-Euclidean geometry, geometric algebra, computational geometry, and algebraic geometry. He also enjoys reading and spending time with his family.

High school for Gifted Students, Hanoi University of Science, Vietnam National University at Hanoi, Hanoi, Vietnam.
E-mail address: tranquanghung@hus.edu.vn