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Finding the Digits of the Roots of a Polynomial with Horner's Help

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Abstract. We describe a method for extracting the digits of a root of a polynomial by repeatedly transforming the polynomial. The technique uses Horner's synthetic division, and could also use Horner's scheme for polynomial evaluation. While the technique is not competitive with modern computer techniques, it is quite attractive for use by hand in textbook or examination settings.

1. Introduction

In the early 1990s, the first author learned a technique from the second author for finding the roots of a polynomial to a number of decimal places. The authors are both uncertain about the origin of this technique, and we have not found it described in the usual texts (e.g. [1, 2, 3]). The first author picked up the name as *Horner's Method*, which though corroborated by Bráthair Mac Craith's description of the method in Irish as *Modh Horner* [4], does not seem to commonly be used to describe the technique. While Horner's synthetic division is a useful component of the technique, it is not the whole story. The technique also shares some similarities with the common numerical *Bisection Method*, where a root is first bracketed and then the bracket narrowed [1, 2]. However, it has its own distinct flavour and uses methods for manipulation of the roots of polynomials (e.g. via Viéta's formulas [5]) that have been covered on the Leaving Certificate. The technique repeatedly transforms the polynomial using these methods to reveal more decimal places of the root.

In this note, the technique will be described and an example will be given. Similarities to the Bisection Method will be discussed, and another of Horner's Methods will be shown to be useful in making the technique efficient, for example, in exams where simple calculators are permitted. We would welcome any information on the history of this technique.

2. Method

In this technique, we begin with a polynomial $f(x)$, and we are asked to extract the value of a particular root of this polynomial to a number of decimal places. The strategy, which repeatedly extracts part of the value of a specific root, is as follows.

(1) Bracket the root between two consecutive integers. This could be achieved either by using a given value, or by putting $x = 0, 1, 2, \ldots$ and checking for a change in sign of the value of the polynomial. Negative values of x might also be explored.

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(2) If the root is negative, form a new polynomial by changing the sign of all the roots (i.e. multiply the coefficients by powers of (-1), starting with (-1) to the power of 0).
(3) By using the information from step 1, this polynomial has a root between n and $n+1$. This reveals the integer part of the root.
(4) To get a root in the range $(0,1)$, a new polynomial is created by reducing the roots by the value n. This is achieved using synthetic division.
(5) To generate a root from this new polynomial in the range $(0,10)$, all the roots of this polynomial are scaled by a factor of 10 (i.e., multiply the coefficients by powers of ten, starting with 10 to the power of 0).

    By checking this resulting polynomial at $x = 1, 2, \ldots 9$, we find that the sign changes between n and $n+1$. This value of n is the next decimal place value of the original root. This is because the previous step effectively removed the integer part of the root, and this step scaled the result by ten.
(6) While more digits (decimal places) are required, repeat the previous steps 4–5, which reduce the roots by n and then increase all the roots by a factor of 10. Otherwise stop.

Example. *Find, to 2 decimal places, the root of the polynomial*

$$f(x) = x^3 + 6x^2 + 9x + 17$$

near $x = -4$.

The technique works by repeatedly transforming the polynomial so that information can be found about the original root by evaluating the resulting polynomial at integer values of x.

The first step of the technique is to bracket the root between two integers. In this case, we can check $f(-4) > 0$ and $f(-5) < 0$, so, there is a root in $(-5,-4)$. As this root is negative, transform the polynomial by changing the sign of its roots (see step 2 above). Of course, this amounts to changing the sign of every second coefficient, and we are now working with

$$x^3 - 6x^2 + 9x - 17.$$

By construction, this polynomial has a root in $(4,5)$. To extract the next decimal place, reduce the roots of this polynomial by 4. This can be done by repeatedly using Horner's synthetic division.

$$\begin{array}{c|rrr|r}
 & 1 & -6 & 9 & -17 \\
\mathbf{4} & & 4 & -8 & 4 \\
\cline{2-5}
 & 1 & -2 & 1 & -13 \\
 & & 4 & 8 & \\
\cline{2-4}
 & 1 & 2 & 9 & \\
 & & 4 & & \\
\cline{2-3}
 & 1 & 6 & &
\end{array}$$

By reading the coefficients of this new polynomial from left to right along the bottom and up the 'remainders', the new polynomial

$$x^3 + 6x^2 + 9x - 13$$

is found, which has a root in $(0,1)$.

Now scale up the root by a factor of 10 (see step 5 above) and produce another new polynomial

$$f_1(x) = x^3 + 60x^2 + 900x - 13000,$$

which has a root in $(0,10)$. The integer part of the root corresponds to the next digit (decimal place) of our original root. Check this new polynomial $f_1(x)$ at $x = 1, \ldots 9$, to identify a change of sign between $f_1(8)$ and $f_1(9)$. This means that the root of the original polynomial is between -4.8 and -4.9.

The process can be repeated to find more decimal places of the original root. Repeated synthetic division is used to move the root from $(8, 9)$ to $(0, 1)$,

$$\begin{array}{c|rrr|r}
 & 1 & 60 & 900 & -13000 \\
\mathbf{8} & & 8 & 544 & 11552 \\
\hline
 & 1 & 68 & 1444 & -1448 \\
 & & 8 & 608 & \\
\cline{2-4}
 & 1 & 76 & 2052 & \\
 & & 8 & & \\
\cline{2-3}
 & 1 & 84 & &
\end{array}$$

giving $x^3 + 84x^2 + 2050x - 1448$. Subsequently scaling up the root by a factor of 10 gives $f_2(x) = x^3 + 840x^2 + 205000x - 1448000$. Checking this polynomial at $x = 1, \ldots 9$, we find a sign change between $x = 6$ and $x = 7$, so the original root is now known to be between -4.86 and -4.87.

3. Discussion

This technique is quite attractive for use by hand. If the initial polynomial has integer coefficients, all the quantities remain integers. It is also clearly designed as a decimal-friendly technique — it sandwiches the roots between round decimal values, allowing the extraction of a particular number of digits.

It also has clear similarities with the bracketing and bisection technique. Both begin by bracketing the root to some interval and then iterate to refine the root. However, with this technique the polynomial changes over each iteration, while in bisection the interval changes at each step. The bisection technique is more general and can work with any continuous function $f(x)$, while for this technique $f(x)$ must be a polynomial. On the other hand, this technique only needs to evaluate its functions at integers, while the bisection technique typically winds up with increasingly long decimals.

One could also compare bisection and this technique in terms of efficiency. At each step, bisection evaluates $f(x)$ once and then halves the size of the interval. Consequently, it uses roughly $\log_2 10$ function evaluations to refine the root by a factor of 10. The technique presented above refines the root by a factor of 10 on each step, but uses more function evaluations — with intuition or binary search this can be done in 3–4 function evaluations per step, which is actually similar to bisection. Of course, the technique also needs a synthetic division step and a scale-by-ten step, which the bisection technique does not require.

Note that both techniques need to evaluate the polynomial, which gives us a chance to use another method of Horner's for polynomial evaluation. Using this method, a polynomial of degree m can be evaluated with just m multiplies and m additions/subtractions. Suppose you wish to find:

$$p(x) = a_m x^m + a_{n-1}x^{n-1} + \ldots + a_2x^2 + a_1x + a_0,$$

you can write this as

$$p(x) = ((\ldots((xa_m + a_{m-1})x + a_{m-2})x + \ldots + a_2)x + a_1)x + a_0.$$

So, you can start with a_m, multiply by x, add a_{m-1}, multiply by x, ..., add a_1, multiply by x, and finally add a_0. Note that with judicious use of the "=" key, this method can be used to evaluate polynomials on a non-programmable calculator without the use of any intermediate values that need to be stored or written down. For example, suppose we are working with

$$f_1(x) = 1x^3 + 60x^2 + 900x - 13000$$

and we want to find $f_1(8)$, then we can do the following:

$\boxed{1}$ $\boxed{\times}$ $\boxed{8}$ $\boxed{+}$ $\boxed{60}$ $\boxed{=}$ $\boxed{\times}$ $\boxed{8}$ $\boxed{+}$ $\boxed{900}$ $\boxed{=}$ $\boxed{\times}$ $\boxed{8}$ $\boxed{-}$ $\boxed{13000}$,

giving the answer -1448. While this trick is particularly handy if the x value is a single digit, it can also be used at more messy x values by storing x in the calculator's memory.

$\boxed{1}$ $\boxed{\times}$ $\boxed{\text{RCL}}$ $\boxed{+}$ $\boxed{60}$ $\boxed{=}$ $\boxed{\times}$ $\boxed{\text{RCL}}$ $\boxed{+}$ $\boxed{900}$ $\boxed{=}$ $\boxed{\times}$ $\boxed{\text{RCL}}$ $\boxed{-}$ $\boxed{13000}$.

It seems that this trick might be useful in exams where polynomial evaluations are common, such as the current Leaving Certificate.

Of course, more powerful techniques could be used, such as Newton's Method [1, 2, 3] or extracting the roots using Sturm's Theorem [6]. These methods need more technical tools and also usually require division, which is curiously absent from the method we describe!

4. Conclusion

We have described a technique for extracting digits of roots of a polynomial that is quite suited to manual use. We noted its similarities and differences to the closely related and better-known bisection technique. We would be interested to know more about the history of this method, its naming and its use, and welcome feedback if it is familiar to any readers.

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