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Cumann Matamaitice na hÉireann

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Irish Mathematical Society Bulletin

The aim of the *Bulletin* is to inform Society members, and the mathematical community at large, about the activities of the Society and about items of general mathematical interest. It appears twice each year. The *Bulletin* is published online free of charge.

The *Bulletin* seeks articles written in an expository style and likely to be of interest to the members of the Society and the wider mathematical community. We encourage informative surveys, biographical and historical articles, short research articles, classroom notes, book reviews and letters. All areas of mathematics will be considered, pure and applied, old and new.

Correspondence concerning books for review in the *Bulletin* should be directed to

`mailto://reviews.ims@gmail.com`

All other correspondence concerning the *Bulletin* should be sent, in the first instance, by e-mail to the Editor at

`mailto://ims.bulletin@gmail.com`

and only if not possible in electronic form to the address

The Editor
Irish Mathematical Society Bulletin
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Submission instructions for authors, back issues of the *Bulletin,* and further information about the Irish Mathematical Society are available on the IMS website

`http://www.irishmathsoc.org/`

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Irish Mathematical Society Bulletin 90, Winter 2022

CONTENTS

EDITORIAL

The changes in status involved in the creation of the new technological universities has naturally coincided with a wave of what is now known as *rebranding* (an activity one hitherto associated with cattle-rustlers in the Wild West). This affects not just the TU's, but is also observable in venerable constituent universities of the NUI. (In particular, NUIG has become the University of Galway.) Postal addresses and email addresses are affected. I have tried to determine and use the correct forms of these various addresses, but there are bound to be lapses, and for this I apologise. Please let me know of any necessary corrections, for future use.

In this issue, apart from IMS business and meeting reports, we have an article about mathematical education, and classroom notes, book reviews and the problem page. The book reviews are looked after by Eleanor Lingham, and the problem page by Ian Short, and I want to acknowledge their invaluable help.

The book reviews include a dual-language review by Pádraig Ó Catháin. This is a novelty, proposed by the reviewer, who holds a position in DCU related to support for the use of the language. We accept material written in Irish or English. Almost all submissions are in English. We are happy to publish suitable material in Irish. This review is given in both languages, two texts that are intended to have the same meaning. We would not propose to make a habit of this, but I feel that this item will be useful to prospective authors who are inclined to try their hand at writing Irish. Too much that is written in Irish is about the language and related politics, and its survival and development are better served by using it to write about matters of ordinary concern.

The classroom note about the golden ratio earned inclusion for two reasons, apart from the charming coincidence it reports. First, the use of elementary inner-product-space methods makes it suitable as ancillary reading material for undergraduates taking a first course in linear algebra. Second, the use of coordinate methods to prove propositions about synthetic Euclidean geometry has become known in IMO circles as *the Irish method*, and though the note's author (who has a distinguished record in IMO-related education) has no visible Irish connections, his approach to this particular matter can be classed as Irish-in-spirit. To be sure, the Irish method is really the Cartesian method. One has mixed feelings about the fact that Irish IMO candidates have preferred it. On the one hand, Descartes' innovation has proven its worth in many applications over the past four centuries. One the other hand, the lamentable facility with synthetic methods found in our students, and consequent on the catastrophic 'reforms' of the Junior Cycle school geometry programme that began in the sixties, is a matter for regret.

We have just one report from an IMS-sponsored meeting this year, but with the abating of the pandemic the Society has received a reasonable number of applications, and we expect to get back to normal next year. Organisers should report by 15 December, for publication in the Winter issue.

We no longer carry a separate page of news items or of theses defended (due to lack of consistent reporting from the institutions), but I am happy to pass on the news that a doctorate in mathematics has been awarded by the Munster Technological University to Guillermo Cobos-Zara for a thesis entitled *The Ultrahyperbolic Equation.*

For a limited time, beginning as soon as possible after the online publication of this Bulletin, a printed and bound copy may be ordered online on a print-on-demand basis at a minimal price[1]

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[1]Go to www.lulu.com and search for *Irish Mathematical Society Bulletin*.

LINKS FOR POSTGRADUATE STUDY

The following are the links provided by Irish Schools for prospective research students in Mathematics:

DCU: mailto://maths@dcu.ie
TUD: mailto://chris.hills@tudublin.ie
ATU: mailto://creedon.leo@atu.ie
MTU:
http://www.ittralee.ie/en/CareersOffice/StudentsandGraduates/PostgraduateStudy/
UG: mailto://james.cruickshank@nuigalway.ie
MU: mailto://mathsstatspg@mu.ie
QUB:
http://web.am.qub.ac.uk/wp/msrc/msrc-home-page/postgrad_opportunities/
TCD: http://www.maths.tcd.ie/postgraduate/
UCC: http://www.ucc.ie/en/matsci/postgraduate/
UCD: mailto://nuria.garcia@ucd.ie
UL: mailto://sarah.mitchell@ul.ie

The remaining schools with Ph.D. programmes in Mathematics are invited to send their preferred link to the editor.

E-mail address: ims.bulletin@gmail.com

NOTICES FROM THE SOCIETY

Officers and Committee Members 2022

President	Dr Tom Carroll	UCC
Vice-President	Dr Leo Creedon	ATU
Secretary	Dr Derek Kitson	MIC
Treasurer	Dr Cónall Kelly	UCC

Dr R. Flatley, Dr R. Gaburro, Dr D. Mackey, Prof. M. Mathieu, Prof. A. O'Shea, Dr R. Ryan, Assoc. Prof. H. Šmigoc, Dr N. Snigireva.

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Local Representatives

Belfast	QUB	Prof M. Mathieu
Carlow	SETU	Dr D. Ó Sé
Cork	MTU	Dr J. P. McCarthy
	UCC	Dr S. Wills
Dublin	DIAS	Prof T. Dorlas
	TUD, City	Dr D. Mackey
	TUD, Tallaght	Dr C. Stack
	DCU	Prof B. Nolan
	TCD	Prof K. Soodhalter
	UCD	Dr R. Levene
Dundalk	DKIT	Mr Seamus Bellew
Galway	UG	Dr J. Cruickshank
Limerick	MIC	Dr B. Kreussler
	UL	Mr G. Lessells
Maynooth	MU	Prof S. Buckley
Sligo	ATU	Dr L. Creedon
Tralee	MTU	Prof B. Guilfoyle
Waterford	SETU	Dr P. Kirwan

Applying for I.M.S. Membership

(1) The Irish Mathematical Society has reciprocity agreements with the American Mathematical Society, the Deutsche Mathematiker Vereinigung, the Irish Mathematics Teachers Association, the London Mathematical Society, the Moscow Mathematical Society, the New Zealand Mathematical Society and the Real Sociedad Matemática Española.

(2) The current subscription fees are given below:

Institutional member	€200
Ordinary member	€30
Student member	€15
DMV, IMTA, NZMS or RSME reciprocity member	€15
AMS reciprocity member	$20
LMS reciprocity member (paying in Euro)	€15
LMS reciprocity member (paying in Sterling)	£15

The subscription fees listed above should be paid in euro by means of electronic transfer, a cheque drawn on a bank in the Irish Republic, or an international money-order.

(3) The subscription fee for ordinary membership can also be paid in a currency other than euro using a cheque drawn on a foreign bank according to the following schedule:

If paid in United States currency then the subscription fee is US$ 40.
If paid in sterling then the subscription is £30.
If paid in any other currency then the subscription fee is the amount in that currency equivalent to US$ 40.00.

The amounts given in the table above have been set for the current year to allow for bank charges and possible changes in exchange rates.

(4) Any member with a bank account in the Irish Republic may pay his or her subscription by a bank standing order using the form supplied by the Society.

(5) Any ordinary member who has reached the age of 65 years and has been a fully paid up member for the previous five years may pay at the student membership rate of subscription.

(6) Subscriptions normally fall due on 1 February each year.

(7) Cheques should be made payable to the Irish Mathematical Society.

(8) Any application for membership must be presented to the Committee of the I.M.S. before it can be accepted. This Committee meets twice each year.

(9) Please send the completed application form, available at

`http://www.irishmathsoc.org/links/apply.pdf`

with one year's subscription to:

Dr Cónall Kelly
School of Mathematical Sciences
Western Gateway Building, Western Road
University College Cork
Cork, T12 XF62
Ireland

Deceased Members

It is with regret that we report the deaths of members:

Noel White, of TUD, who died on 2 September 2021.

David Tipple, of UCD, who died on 5 June 2022.

Rex Dark, of the University of Galway, who died on 5 December 2022.

E-mail address: subscriptions.ims@gmail.com

PRESIDENT'S REPORT 2022

Committee changes: From January 2023, the President of the Society will be Leo Creedon (ATU, Sligo) and the Vice-President will be Rachel Quinlan (Galway). Derek Kitson (MIC/UL) and Cónall Kelly (UCC) will continue to serve as Secretary and Treasurer, respectively. Thanks to Dana Mackey for her service to the Society during three full terms on the Committee. I will join the Committee as outgoing President in the interests of continuity and institutional memory. With those eligible for re-election having been re-elected for a further term at the AGM, there were no further changes to the Committee.

IMS Bulletin: I renew the Society's appreciation of Tony O'Farrell's work as Editor of the Bulletin and that of the Editorial team. The Bulletin is freely available online from the Society's homepage and, in printed form, from lulu.com. Institutional members now receive a complimentary printed copy of each issue.

As the Society's flagship publication, it is important that the Bulletin continue to develop and to expand its readership, this being best achieved by further improving the breadth and depth of its published articles. To this end, speakers at the September Meeting are encouraged to write an account of their talks for inclusion in the Bulletin.

IMS meeting: The Society's annual 'September Meeting', was held in TU Dublin on 1^{st} and 2^{nd} September at their impressive new Grangegorman campus. The meeting was wonderfully organised by Dana Mackey, Milena Venkova and Mercedes Jordan-Santana who put together an interesting and varied programme of talks. Of particular note was the session on Athena SWAN and the Mathematical Sciences Athena SWAN Network led by Sarah Fink (Athena Swan Ireland) and Niall Madden (Galway).

As is customary, time was set aside at TU Dublin for a meeting of the IMS Committee and for the Society's AGM. The Irish Committee for Mathematics Education (ICME) continues to do excellent work under its Chair, Ann O'Shea (Maynooth), particularly in the area of textbook quality assurance. Its Annual Report 2022 is available on the IMS website. At the AGM, the membership unanimously supported the establishment of a subcommittee to be known as the Irish Committee for Equality, Diversity and Inclusion in Mathematics (ICEDIM) and approved its terms of reference. With the agreement of the IMS Committee, Romina Gaburro (UL) was appointed Chair of ICEDIM, and expressions of interest in membership of the committee have since been sought. The establishment of this committee is a most welcome and timely initiative.

International Mathematical Union: The International Congress of Mathematicians (ICM) was to be held, under the auspices of the International Mathematical Union, in Saint Petersburg from 6^{th} to 14^{th} July 2022. The participation of early career researchers at the ICM was to be encouraged by generous 'Kovalevskaya Grants' covering the registration fee and all local expenses including accommodation and meals. The Committee put much work into administering these grants locally and selecting recipients. Everything changed with Russia's invasion of the Ukraine on 24^{th} February. Following representations from several members, the IMS issued a statement on its website that included the sentence 'The IMS can no longer support the holding of the ICM in Russia.' We also communicated the position of the IMS directly to Prof. Helge Holden, Secretary General of the IMU. The Executive Committee of the IMU reached the decision that ICM2022 would take place as a fully virtual event, hosted outside

Russia but following the original time schedule planned for Saint Petersburg. This is what happened.

The General Assembly of the IMU took place as an in person event in Helsinki on 3rd and 4th July. As a Group II member of the IMU, Ireland has two votes. We felt it was important that Ireland be represented in person and so Derek Kitson and myself travelled to Helsinki as the Ireland delegation. A full account of the General Assembly (report, resolutions, group photo) is available on the IMU website. One of the important outcomes of the meeting was the establishment of a 'reserve fund' to help Adhering Organizations whose dues are in arrears due to temporary extreme, adverse circumstances, this to be funded by earmarked contributions. A related resolution, in response to a request from the Ukrainian Mathematics Union, was that their IMU membership fees for the years 2022 and 2023 were waived, unless these fees will be covered by a third party or parties. There were several such offers from the floor of the GA.

The incoming President of the IMU is Hiraku Nakajima, Japan, and the incoming Secretary General is Christoph Sorger, France, for the term 2023–2026.

The ICM 2026 will be held in Philadelphia, U.S.A., with the General Assembly in New York City.

A bonus of attending the General Assembly in person was the opportunity to attend the presentation of awards, including the Fields medals, on the following day: a most memorable event.

The role of the IMS as IMU Adhering Organisation for Ireland definitely involves extra work for the Committee. Nevertheless, I personally believe that the IMS was the best organisation to take over this role from the Royal Irish Academy as the Society represents those most directly affected, and interested in, the work of the IMU. It is disappointing that the Government, despite many efforts, has not been persuaded of the importance of supporting the IMS in meeting Ireland's annual IMU membership fee (€2,860). I believe it is important that Ireland maintains its status as a Group II member. It would be my hope that outside support to meet this annual charge will soon be found.

European Mathematical Society: Leo Creedon (Vice-President) represented the Society at the meeting of the EMS' governing body, the Council, on 25th and 26th June 2022 in Bled (Slovenia).

Following on from the Council meeting and the decisions made at that meeting, a Meeting of Presidents (of European Mathematical Societies) was held online on 10th September. The in-coming President of the EMS is Jan Philip Solovej (Copenhagen) and Vice-President Beatrice Pelloni (Heriot-Watt). Particular attention at the meeting was drawn to the EMS Topical Activity Groups (EMS TAG) initiative, and to the EMS Young Academy (EMYA) for which the IMS has put forward two nominations. The initial composition of EMYA is expected to be known sometime in January. The President, Volker Mehrmann, expressed a wish to strengthen links between the EMS and 'under-represented regions', for example by increasing membership in EMS committees from such regions. Following his nomination by the IMS, Cónall Kelly has been elected to membership of the EMS Standing Committee for Developing Countries. We expect that similar appointments to EMS committees will be made in future years as vacancies on these committees arise, thereby reinforcing links between the IMS and the EMS.

Finally, it has been an honour to serve as President of the IMS for the past two years. I thank all of you, the members of the IMS, for your support during this time as well as the Committee and especially my fellow officers – Leo, Cónall, David and Derek – for their constant good humour and patience.

Tom Carroll, December 2022.
E-mail address: t.carroll@ucc.ie

Draft minutes of the Irish Mathematical Society Annual General Meeting held on 2nd September 2022 at TU Dublin, Grangegorman

Present: C. Boyd, T. Carroll, L. Creedon, R. Flatley, R. Gaburro, B. Goldsmith, P. Greaney, C. Kelly, D. Kitson, S. Krishnaraj, R. Levene, E. Lingham, P. Lynch, D. Mackey, M. Mackey, N. Madden, D. Malone, P. Mellon, P. Ó Catháin, M. O'Reilly, A. O'Shea, R. Ryan, H. Šmigoc, R. Smith, N. Snigireva

Apologies: J.P. McCarthy, R. Quinlan

1 Agenda / Conflicts of interest

The agenda was accepted and no conflicts of interest were declared.

2 Minutes

The minutes of the AGM held on 3rd September 2021 and published in issue 88 of the Bulletin were accepted with one correction; the 2021 Fergus Gaines cup recipient (recorded under AOB) was Ellen Li (and not Ellen Lee).

3 Matters Arising

E. Lingham (Sheffield-Hallam University) was welcomed as a new editorial board member for the Bulletin.

4 Correspondence

- Fáilte Ireland have offered to provide logistical and financial support for hosting conferences with over 100 international delegates. Financial support would typically range from €10-€30 per international delegate.
- An outreach initiative called *I'm a Scientist* is seeking mathematicians for an activity in October 2022. Further details are available on the IMS twitter account.

5 Membership Applications

18 new membership applications were approved since the last AGM. The new members are: Guillem Cobos, Mary Cunneen, Riccardo Della Martera, Fearghus Downes, Fionn Downes, Ronan Egan, Martin Friesen, Ramen Ghosh, Eva Leahy, Andrew Leader, Denise Lord, Jan Manschot, Dylan McGrath, Timothy McNicholl, Christopher Noonan, Thomas Quinlan, Morgan Robson and Gaston Vergara Hermosilla. Additionally, a lifetime membership application was approved for Eleanor Lingham.

New institutional membership applications were approved for TU Dublin and MTU.

6 President's Report

The president gave a summary of the report (available in full in the Bulletin), covering recent EMS activities (nominations for the EMS Young Academy, the establishment of EMS Topical Activity Groups, the EMS council meeting in Bled), the IMU General Assembly in Helsinki and the Society's response to events in Ukraine. The president also reported on continuing efforts to obtain financial support for IMU fees and members were encouraged to assist with this where possible. The value of IMU activities to mathematics in Ireland was noted and a discussion followed on other possible sources of funding. Suggestions included making a joint bid for financial support with other scientific societies and contacting government backbenchers, opposition party members and journalists. Members were encouraged to follow the IMS twitter account.

7 Treasurer's Report

A report on finances was presented. Printing costs of the Bulletin are expected to increase. There is a downward trend in the closing balance due to the IMU membership fee.

8 Conference Support Fund

The Irish Numerical Analysis Forum was awarded €125 to support the continuation of the online seminar series. Members were encouraged to submit applications for the Nov/Dec deadline.

9 International Mathematical Union

The activities of the IMU and issues around IMU membership had been dealt with as part of the President's Report and the Treasurer's Report.

10 Bulletin

Members were encouraged to submit items to the Bulletin which are of general interest to readers. D. Malone was commended for his work in creating DOI's for the Bulletin. Several suggestions were made on suitable items including articles on mathematics history, mathematics education, and Athena Swan. It was also suggested that the Bulletin could record the abstracts from IMS Conferences and that speakers could submit accompanying articles to the Bulletin. Members were reminded to submit recurring items for the Bulletin in a timely manner. The application to Scopus was discussed.

11 Irish Committee for Mathematics Education (ICME)

A. O'Shea reported that two meetings had been held in the academic year 2021/22. The committee is working on the following topics:

Textbook quality assurance: A subgroup consisting of M. Hanley, R. Flatley and J. Grannell has made an initial report and work is ongoing.

Links with the IAMTA: The committee has contacted the IAMTA with a view to developing resources for the new Applied Mathematics Leaving Certificate course.

Supporting exceptional talent: Work is ongoing.

Developing links between mathematicians and researchers in mathematics education: The first meeting of the online reading seminar, organised by J. Crowley, was held in November 2021. Suggestions for the next topic of discussion are welcome.

National Research Classification System: The committee made a submission to the recent consultation on the National Research Classification System in support of including Mathematics Education as a named research area.

12 Irish Committee for Equality, Diversity and Inclusion in Mathematics (ICEDIM)

Terms of reference for the committee had been prepared and Romina Gaburro was approved as chair of the committee. The importance of addressing EDI issues in mathematics was noted. The committee will advise and report to the IMS committee on EDI matters and aim to give prominence and visibility to women and underrepresented groups in mathematics.

13 Elections

R. Gaburro and A. O'Shea have reached the end of two year terms as committee members, and both are eligible for re-election. T. Carroll has reached the end of a two year term as president, and is consequently not eligible for re-election as president. D. Mackey has reached the end of a six year term as committee member, and is consequently not eligible for re-election as committee member.

L. Creedon was nominated for president by T. Carroll and seconded by M. Mathieu. R. Quinlan was nominated for vice president by T. Carroll and seconded by M. Mathieu. R. Gaburro and A. O'Shea were nominated for committee membership by L. Creedon and seconded by T. Carroll. T. Carroll was nominated for committee membership by

D. Kitson and seconded by L. Creedon. Election to these positions was approved by the meeting.

14 AOB T. Carroll and D. Mackey were thanked for their excellent service as president and committee member respectively.

Derek Kitson
derek.kitson@mic.ul.ie

IMS Annual Scientific Meeting 2022
Technological University Dublin, Grangegorman

September 1st–2nd

The 35th Annual Meeting of the IMS was organised by the School of Mathematical Sciences, TU Dublin at the new Grangegorman campus. The meeting was very well attended by more than 70 participants, including many new members. The organising committee was led by Dana Mackey and consisted of: Nicole Beisiegel, John Butler, Alberto Caimo, Laura Cooke, Chris Hills and Milena Venkova. There were opening remarks from Chris Hills (Head of School) and Dana Mackey, and a welcome address from the Dean of the Faculty of Digital and Data, Prof Pramod Pathak.

The meeting consisted of 12 invited talks on a range of topics in Pure and Applied Mathematics and Statistics, as well as an Athena Swan information session. A key outcome of the AGM was the formal establishment of the Irish Committee for Equality, Diversity and Inclusion in Mathematics, a subcommittee of the IMS, chaired by Romina Gaburro.

The list of all invited talks is below:

- Nicole Beisiegel (TU Dublin) – *Numerical Testcases to Study Proudman Resonance Using Shallow Water Models*
- Leo Creedon (ATU Sligo) – *New algebraic structures which are almost semigroups and rings*
- Ray Ryan (NUI Galway) – *Analysis on vector lattices*
- Rachel Quinlan (NUI Galway) – *How to make up questions about matrices*
- Catalina García (Granada University) – *An overview of multicollinearity: contributions and alternatives to ridge regression*
- Roisin Hill (UL) – *Generating layer-adapted meshes using MPDEs*
- Peter Lynch (UCD) – *Parity and Partition of the Rational Numbers*
- David Malone (Maynooth University) – *Ranking and Rankability*
- Sarah Fink (Athena Swan Ireland) and Niall Madden (NUI Galway) – *Introducing Athena SWAN, and the Mathematical Sciences Athena SWAN Network*
- Eabhnat Ní Fhloinn (DCU)– *Is Maths Different? Experiences and opinions of mathematics lecturers teaching online during the COVID-19 pandemic.*
- Lennon Ó Náraigh (UCD) – *Mathematical and physical modelling of industrial drying*

There were also 11 posters displayed by undergraduate and postgraduate students and early-stage researchers - see list below. The prize for the best poster was awarded to Niall Donlon (UL).

- Priyanka Joshi (UCD) – *Powers of Karpelevič Arcs and their Sparsest Realising Matrices*
- Sowmiya Krishnaraj and Michael Vynnicky (UL) – *Analysing the Effects of Aerosol Particle Deposition on the Pulmonary Tree using Computational Fluid Dynamics (CFD)*
- Thomas Quinlan (UCC) – *Investigating the Effects of Delayed Feedback on the Shutdown of the Atlantic Meridional Overturning Circulation*
- Eva Leahy (UCC) – *Overview of Minimal Surfaces*
- Dylan McGrath (UCC) – *Bump-and-Revalue: Estimating the Greeks*
- Jack Lyons (TU Dublin) – *A Mathematical Model for Holographic Recording in Photopolymer Media with Zeolite Nanoparticles*
- Niall Donlon (UL) – *Stable Reconstruction of Anisotropic Conductivity*
- Edward Donlon (TU Dublin) – *Dynamics of Generalized Random Sequential Adsorption*

- Jason Curran (UL) – *Stability estimates in the inverse problem of diffuse optical tomography via singular solutions*
- Ole Cañadas (DCU) – *Affine Volterra Processes with jumps*
- Yen Thuan Trinh (UCC) – *Pricing multi-asset options using the MONTE CARLO-TREE (MC-TREE) method*
- Fearghus Downes (ATU Sligo) – *Mathematical model of bovine hormonal dynamics*

The webpage for the meeting, which includes abstracts for all talks and posters is archived at

`https://www.tudublin.ie/mathematics/ims-2022/`

The organisers would like to thank the speakers, students who presented posters, and all participants for contributing to a successful and enjoyable meeting.

Report by Dana Mackey, TU Dublin.
dana.mackey@tudublin.ie

Reports of Sponsored Meetings

Report received of a sponsored meeting held in 2022:

Groups in Galway meets the Irish Geometry Conference 2022
May 18–20, 2022, NUI Galway

The first joint meeting of *Groups in Galway* and the *Irish Geometry Conference* took place at NUI Galway from 18–20 May 2022. The meeting was organised by John Burns, Angela Carnevale, Martin Kerin, and Tobias Rossmann (all from NUI Galway). There were 14 talks (11 in person, 3 given remotely) spread over three days. A poster session took place on Thursday afternoon. Over 50 people attended the meeting.

Speakers and talks:

- Sahana Balasubramanya (University of Münster): *Actions of solvable groups on hyperbolic spaces*
- Peter Brooksbank (Bucknell University): *Computing intersections of classical groups*
- Marston Conder (University of Auckland): *Distinguishing triangle groups by their finite quotients*
- James Cruickshank (NUI Galway): *Symmetric pseudo-triangulations and contact graphs*
- Viveka Erlandsson (University of Bristol): *Dihedral subgroups of lattices in* $\mathrm{PSL}(2,\mathbb{R})$
- Brendan Guilfoyle (Munster Technological University): *The failure of geometry in dimension* 4
- Lee Kennard (Syracuse University): *Graph systoles and torus representations*
- Ilaria Mondello (Paris-East Créteil University): *Gromov-Hausdorff limits of manifolds with a Kato bound on the Ricci curvature*
- Lucia Morotti (Leibniz University Hannover): *Homogeneous reductions of spin representations*
- Marco Radeschi (University of Notre Dame): *Invariant theory without groups*
- Krishnan Shankar (University of Oklahoma / NSF): *Growth competitions in non-positive curvature*
- Mima Stanojkovski (RWTH Aachen University / MPI for Mathematics in the Sciences Leipzig): *Automorphisms and isomorphism testing of elliptic groups*
- James B. Wilson (Colorado State University): *Counting in Groups by Coordinatizing Geometries*
- David Wraith (Maynooth University): *Intermediate curvatures: what are they and why should you care?*

The conference website `https://torossmann.github.io/ggg22/` contains abstracts of the talks and further information.

Report by Tobias Rossmann, NUI Galway
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David A. Tipple, 1942-2022

MICHAEL MACKEY AND MARIA MEEHAN

David Tipple was born on 7 December 1942 in Manchester, the only child of John and Mary Tipple. His father was a highly skilled craftsman who worked as a tool-room turner. David was always proud of his working-class roots. For his secondary education he attended Xaverian College, a Catholic grammar school in Manchester. A keen musician, with a special interest in classical music, he played the oboe in Manchester Youth Orchestra. He went on to the University of Manchester, where he obtained a BSc and an MSc in Mathematics.

On completion of his master's degree, David secured a position as lecturer at University College Dublin (UCD) and moved there in 1967. As a young lecturer, he worked on his PhD, which was awarded in 1972 from the University of Manchester under the supervision of Professor Michael Barratt. (cf. *A note on the metastable homotopy groups of torsion spheres.* Bull. London Math. Soc. 3 (1971), 303–306.)

David joined UCD at a time of expansion and growth. The university was moving from the city centre to the Belfield campus which was still a sprawling woodland. David and a close-knit group of colleagues kept logs of the birds they spotted on their lunchtime walks and this catalogue is now archived in UCD library. David's office reflected his own neatness and organisation. He would never leave in the evening without first filing or disposing of each piece of paper and applying a dust cover to each piece of equipment. Computers and computing were a great interest of his and he was an enthusiastic early adopter of Gnu/Linux and an advocate for the value of programming skills to mathematics students. David played a prominent role in the introduction of the Honours BA in Mathematical Studies to UCD. David felt a three year honours offering was required and he was instrumental in the design and delivery of the new programme, which included components in symbolic computation (Maple) as well as traditional programming skills (Pascal).

David meticulously prepared all his lectures and his courses were highly organised. Students held him in high regard and he generously gave of his time to any student who appeared at his office door, which was always open. He also contributed to student life through his long term role as senior trasurer to UCD Dramsoc.

David was an active member of the Irish Mathematical Society and an officer of the society for four years from 1990 to 1993 when he served as treasurer. His penchant for detail and organisation were a great advantage in the role. The physical records from that period are systematically sorted in manilla folders, the titles carefully inked in David's neat hand. (He refused to write on paper with anything but a fountain pen.) An indicator of his methods is that, if there were no invoices or membership applications for a given period, David would nevertheless create the appropriate file with no contents. A significant contribution by David from this period was the redrafting of the IMS constitution, undertaken in collaboration with his UCD colleague Mícheál Ó Searcóid. The original IMS constitution from 1976 consisted of just nine articles and nine rules,

Key words and phrases. Tipple, Obituary.
Received on 15-12-2022.
DOI:10.33232/BIMS.0090.13.15.

FIGURE 1. David Tipple (Photograph: M. Mackey)

but as the society grew, it was apparent that extension and refinement were required. For example, the term limits on membership of the committee were introduced with this revision. Mícheál credits David for the succintness and precision of language of the resulting articles and the constitution has not changed substantially since. The society's current system of conference funding, with application made via the treasurer, also seems to have started during David's term in the role.

David was a popular, affable and much appreciated colleague in the UCD Department of Mathematics. He was particularly helpful to new and younger members of staff (as the authors can attest) and was always happy to share his knowledge of many topics outside mathematics, the state of the Lancashire cricket team being a particular favourite.

After his retirement in 2006, David pursued his interests with his usual vigour. An avid reader, with an interest in the history of typography and printing, he volunteered at Project Gutenberg. He was particularly proud of a collaboration with his friend, and professor of linguistics, Kevin Cathcart. They worked together on The Book of the Twelve Prophets, vol. II, by George Adam Smith which had more than 1,500 footnotes, many in Greek and Hebrew, with David ensuring the fonts were faithfully reproduced.

In 1975 David had met Fíona, a librarian newly arrived to UCD. They married in 1976 and enjoyed a further 46 happy years together. They both loved the Irish countryside and regularly holidayed there. David's fondness for Ireland was apparent when he became an Irish citizen in 2005.

David passed away following a short illness on June 5th 2022 and is survived by Fíona.

(A version of this article has appeared in the London Mathematical Society Newsletter.)

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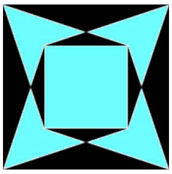

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SPIRIT Maths: Giving students what they want

VIOLETA MORARI AND SHANE O ROURKE

Abstract. We give an overview of a teaching and learning project in Munster Technological University, called SPIRIT Maths. The project's aim was to find out students' perceptions and attitudes towards mathematics, establish student preferences regarding learning resources, and develop and pilot resources consistent with these preferences.

1. Introduction

Mathematics educators have long observed deep-seated difficulties that many students encounter, and have grappled with the thorny question of how to address these difficulties. How do students find the experience of learning mathematics and how does it compare to their expectations? How can we establish what the key conceptual problems are and focus our pedagogical energies on these? What exactly would help students to overcome these problems?

In Munster Technological University (MTU) we have done our share of observing and grappling, leading to several teaching and learning initiatives in MTU. Most recently, we have surveyed students to ask about their perceptions of mathematics, and what resources they think would be most helpful to them in learning it. Moreover, we have developed learning resources in response to the answers to this survey. This project goes by the name **SPIRIT Maths** – Student Perceptions Informing and Redefining Teaching in Mathematics. This article presents an overview of the project. More details on the resources developed in the course of this project as well as the survey analysis can be found in [3] and [2].

We have no illusions that we've found the ultimate answers to the questions above. Other students in other institutions may have different backgrounds and expectations from ours. Moreover, students are not necessarily best placed to assess what are the impediments to their learning, and what resources would help them. Nevertheless, we believe that our findings (which contained a few surprises for the experienced lecturers involved in the project) and the nature of the resources developed may be useful elsewhere.

2. Background

MTU came into existence on 1 January 2021 as Ireland's second Technological University, arising from the former Cork Institutes of Technology (CIT) and the Institute of Technology Tralee (ITT). For convenience we will refer exclusively, to MTU throughout this overview of our project, even though all references prior to 2021 should more

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correctly be to CIT. Mathematics is, and has long been, an important subject in MTU. Mathematics features in 58 out of 75 CAO programmes offered in MTU; in 2020-2021 it was taken by 2260 of 3169 (71.3%) of all first year students. While the mathematics department offers programmes in mathematics, notably a Higher Diploma and an MSc in Data Science and Analytics, the vast majority of the mathematics teaching in MTU is service teaching, with students in the areas of business and engineering accounting for most of staff contact hours.

Anecdotally, mathematics has been a rather unpopular subject with a reputation – somewhat deserved, alas – as having a high failure rate. In 2019-2020 as many as 50% of students failed certain mathematics modules at their first attempt. Happily, the pass rate rises significantly when one considers in addition further attempts, but it is clear that a significant number of students have problems with mathematics. This raises the following questions:

(1) What exactly are the mathematical problems that students face?
(2) How can we lecturers help them?

Even students who attend lectures and tutorials, and who have difficulties with their mathematics modules are remarkably slow to ask questions or to seek help from their lecturer, so the answers to these questions are not self-evident. The SPIRIT Maths project sought to address these questions head-on by surveying MTU students who have taken a mathematics module.

Having learned the answer to the second question (some answers at any rate), we developed interactive digital learning resources that these students were looking for. We also developed a diagnostic test to help identify topics that students find difficult and direct the students to those resources.

The project was carried out by 12 lecturers in the Department of Mathematics in MTU with the support of several colleagues in client departments.

3. The surveys

Two surveys were carried out, one in June 2020, and one in February 2021.[1] The two surveys were sent to all students who had already completed at least one mathematics or statistics module in MTU. The first survey captured the perceptions of students that were on site for traditional in-class delivery (pre-pandemic). Once staff in MTU – along with institutions throughout the world – were forced to pivot to remote teaching, the team was keen to also capture this aspect of the student experience. This led to additional questions on the second survey. It is worth noting that the comparisons of the responses from the two surveys revealed no statistically significant differences, so the data from the two surveys were combined for analysis. [2]

Below we summarise some of the key points arising from the surveys. Further details can be found in [2].

(1) 16.1% of respondents did not realise a mandatory mathematics module was going to be part of their chosen programme.
(2) 52.3% of respondents find mathematics difficult.
(3) Presented with several types of learning resources such as internet resources, lecturer-provided notes and videos, textbooks, and MTU's Academic Learning Support Centre, it is notable that an overwhelming majority favoured resources

[1] It should be pointed out that while this places them in the midst of the COVID-19 pandemic, the project was planned before such a situation could have been envisaged.

[2] In total, 1633 students were invited to participate, from which 310 responses were obtained.

provided by the lecturer. This suggests that students don't make use of external resources such as the Khan Academy nearly as much as might have been suspected.

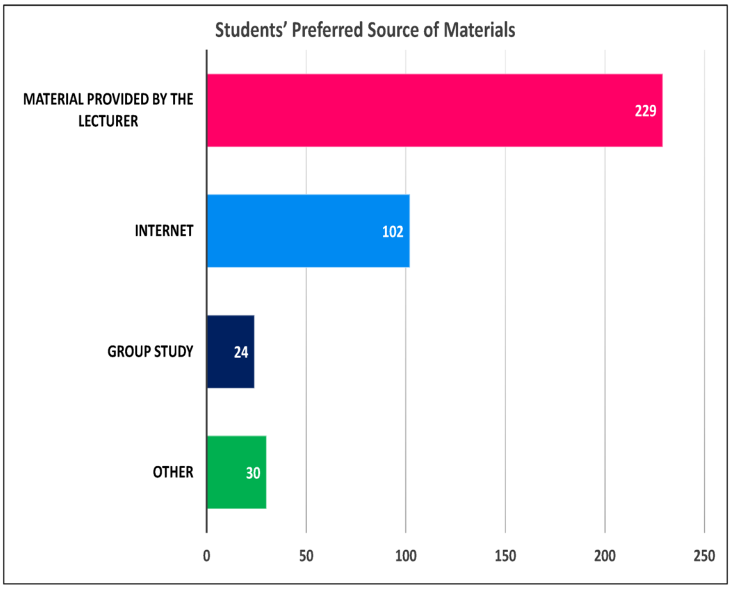

FIGURE 1. Students' preferred source

(4) The 'need to practise to master methods' and the 'need to think and understand rather than learn off' are the two main things students dislike about learning mathematics.

Somewhat encouragingly, 28.5% of 309 students said there is nothing they dislike about learning mathematics. Of the remaining 221 respondents, 43.4% disliked the need to practise to master methods, and 18.6% said they dislike 'the need to think and understand rather than learn off'[3].

(5) There is no consensus among learners on the most desirable teaching mode (online or in-person) for mathematics, with roughly equal numbers of students thinking mathematics is more suited to online delivery, less suited to online delivery and equally suited to online delivery compared to other disciplines. This suggests that a fully remote delivery of mathematics modules would not be welcomed by a significant proportion of learners.

(6) Surprisingly, while initially students expected mathematics to be significantly harder at third-level than at second-level, by the time they had completed their mathematics module, 81.9% of all respondents found first year mathematics in MTU similar or only 'a bit harder' than at second-level.

(7) While 55.5% of students say they do two or more hours per week of independent work on mathematics, 34.1% of students said they do one hour, and 10.4% said they do no independent work on mathematics.

[3]Respondents could choose more than one response here, so there may be an overlap between these cases.

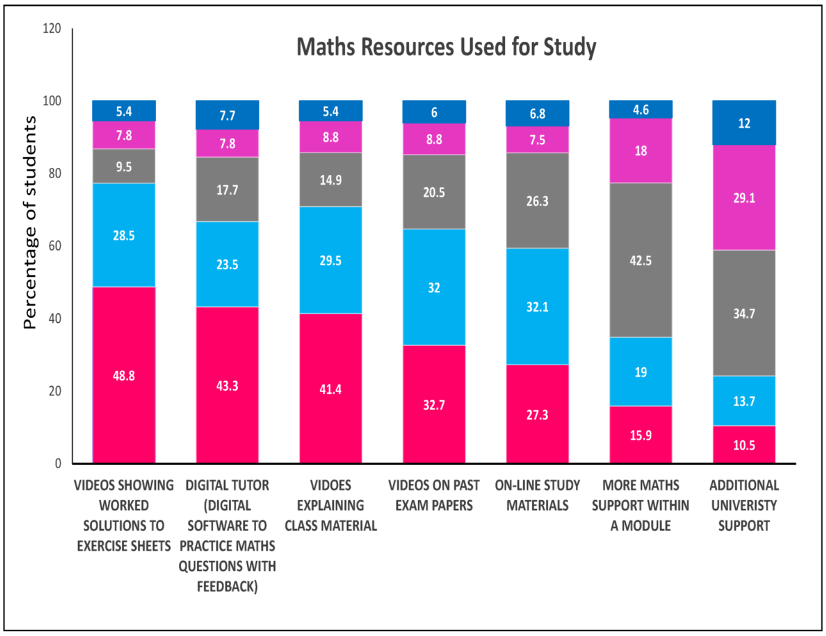

FIGURE 2. Students' preferences of resources

4. The Diagnostic Test

A comprehensive review of the literature of diagnostic testing at third level, particularly in Ireland, revealed that a lot of diagnostic testing is concerned with testing students on various topics that they are supposed to know about going into third level. Those tests typically establish a cut-off score such that students scoring below that score would be identified as 'at risk' of struggling with maths at third level.

The team took a slightly different approach and developed a diagnostic test that would allow immediate feedback to both students and lecturers. The purpose of the test was to identify topics that students find problematic, rather than 'at risk' students. The final diagnostic test is quite short, consisting of 15 short questions that test core knowledge, such as the following:

- Without the use of a calculator, solve for p:
$$\frac{492^{84} \times 492^{7}}{492^{13}} = 492^{p}$$
- Solve for v:
$$T = \frac{2v}{g} + 5$$
with the possible choices listed below:
  (1) $v = Tg - 7$
  (2) $v = \dfrac{g(T-5)}{2}$
  (3) $v = \dfrac{Tg-5}{2}$
  (4) $v = \dfrac{2(T-5)}{g}$

(5) $v = \frac{Tg}{7}$
(6) $v = \frac{g(T+5)}{2}$
(7) $v = \frac{Tg}{2} - 5$

Each question answered incorrectly is highlighted and students are invited to either:

(1) Explore a recommended SPIRIT Maths Digital Resource.
(2) Watch a video of the solution.
(3) Book a one to one meeting in the Academic Learning Centre.

The Diagnostic Test is available via Maths Online, a specialised module developed by the Academic Learning Centre in MTU. This module is automatically accessible to every student that takes a maths or stats module through Canvas, the Virtual Learning Environment (VLE) used in the University. It can also be imported by lecturers into their own Canvas modules if they wish to do so. Lecturers can see the results of the diagnostic test and take appropriate action as needed.

5. The Digital Resources

As seen above, students are very focused on their own lecturer's materials. While there are many excellent resources on mathematical topics on the internet, students are much more likely to engage with materials provided by their lecturer.

With this in mind, the SPIRIT Maths team set about developing digital resources easily accessible from Canvas. We picked two of the biggest first year maths modules, MATH6051 (First Year Business) and MATH6014 (First Year Engineering) in MTU Cork. There are over 500 students taking these modules which is why the team decided to focus on them first.

The team developed three interlinked and complementary resources:

(1) H5P interactive self-test questions.
(2) Corresponding videos showing worked solutions.
(3) Bank of practice questions developed using Numbas.

H5P is a tool widely used to create interactive content. H5P content is responsive and mobile-friendly, which means that users will experience the same rich content on computers, smartphones and tablets alike. All the user needs is a web browser. It also can be easily integrated into Canvas, as well as other VLEs.

The H5P questions created by the team enable students to input their answer in the provided box and verify it by clicking the 'check' button. If the answer is incorrect, they can choose to either watch the video solution to that particular question or practise a similar question in Numbas.

Another useful feature of H5P is that reports of submitted answers are available to the lecturer which will help the lecturer to understand where most of the difficulties lie.

Aligned then to each H5P video are worked out solutions the team created as well as a bank of Numbas questions. The videos with the solutions were created using packages such as Explain Everything and Educreations.

Numbas is an e-assessment tool supported by an open source platform which is particularly suitable for maths-related topics as it allows easy randomisation of variables and names. It also can provide instant feedback and hints. So, after watching a video, the student can choose to attempt a similar question in Numbas. If the student answered a question correctly, a green tick will appear to confirm this. If, on the other hand, the student is unsure of how to proceed, they can choose to open a hint. After each hint they can continue with the question, or reveal the full solution. The student then can choose to try a similar question of the same type.

Numbas has been extensively used in MTU Cork in mathematics modules and has been proven to be a very effective tool [1]. Figure 3 illustrates the three interlinked resources developed by the team for a typical engineering question.

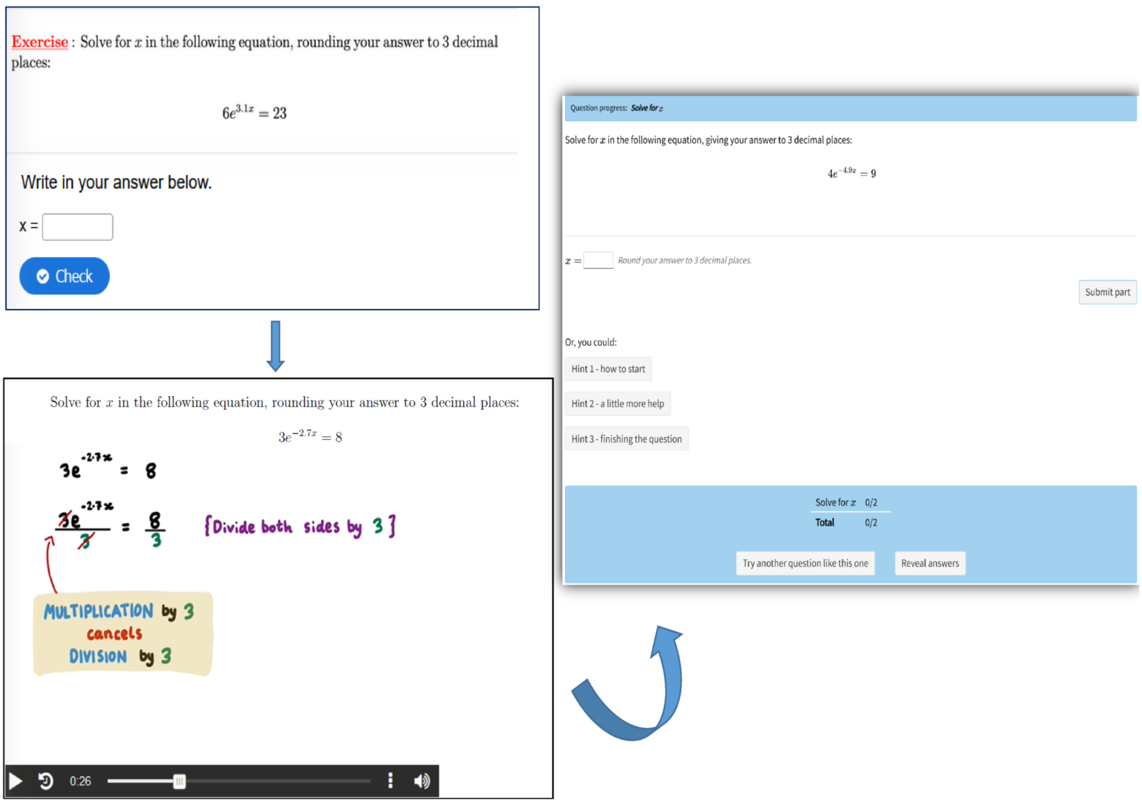

FIGURE 3. Interlinked resources for a problem involving the exponential function.

In summary, SPIRIT Maths sought to address the challenges that students encounter in learning mathematics in MTU, noting that this is predominantly service teaching. The type of resources described above were chosen in response to primary data obtained through a survey of MTU students. Preliminary feedback from a focus group, as well as initial analysis of Canvas data for one of the piloted modules (MATH6051) suggests that the use of the resources correlates with a higher module grade and that the resources are effective for self-directed learning. However, the results of our survey underline the importance of lecturers promoting the resources to their students. It is also important to obtain ongoing feedback and to incorporate this when revising the resources.

It is of course difficult to draw firm conclusions on what are the determining factors on a student's module grade. Nevertheless, the initial analysis encourages us to roll out similar resources throughout the university in future.

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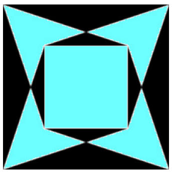

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Finding the Digits of the Roots of a Polynomial with Horner's Help

DAVID MALONE AND MAURICE MAXWELL

ABSTRACT. We describe a method for extracting the digits of a root of a polynomial by repeatedly transforming the polynomial. The technique uses Horner's synthetic division, and could also use Horner's scheme for polynomial evaluation. While the technique is not competitive with modern computer techniques, it is quite attractive for use by hand in textbook or examination settings.

1. Introduction

In the early 1990s, the first author learned a technique from the second author for finding the roots of a polynomial to a number of decimal places. The authors are both uncertain about the origin of this technique, and we have not found it described in the usual texts (e.g. [1, 2, 3]). The first author picked up the name as *Horner's Method*, which though corroborated by Bráthair Mac Craith's description of the method in Irish as *Modh Horner* [4], does not seem to commonly be used to describe the technique. While Horner's synthetic division is a useful component of the technique, it is not the whole story. The technique also shares some similarities with the common numerical *Bisection Method*, where a root is first bracketed and then the bracket narrowed [1, 2]. However, it has its own distinct flavour and uses methods for manipulation of the roots of polynomials (e.g. via Viéta's formulas [5]) that have been covered on the Leaving Certificate. The technique repeatedly transforms the polynomial using these methods to reveal more decimal places of the root.

In this note, the technique will be described and an example will be given. Similarities to the Bisection Method will be discussed, and another of Horner's Methods will be shown to be useful in making the technique efficient, for example, in exams where simple calculators are permitted. We would welcome any information on the history of this technique.

2. Method

In this technique, we begin with a polynomial $f(x)$, and we are asked to extract the value of a particular root of this polynomial to a number of decimal places. The strategy, which repeatedly extracts part of the value of a specific root, is as follows.

(1) Bracket the root between two consecutive integers. This could be achieved either by using a given value, or by putting $x = 0, 1, 2, \ldots$ and checking for a change in sign of the value of the polynomial. Negative values of x might also be explored.

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Thanks to the editor who provided useful insights and references.

(2) If the root is negative, form a new polynomial by changing the sign of all the roots (i.e. multiply the coefficients by powers of (-1), starting with (-1) to the power of 0).
(3) By using the information from step 1, this polynomial has a root between n and $n+1$. This reveals the integer part of the root.
(4) To get a root in the range $(0,1)$, a new polynomial is created by reducing the roots by the value n. This is achieved using synthetic division.
(5) To generate a root from this new polynomial in the range $(0,10)$, all the roots of this polynomial are scaled by a factor of 10 (i.e., multiply the coefficients by powers of ten, starting with 10 to the power of 0).

    By checking this resulting polynomial at $x = 1, 2, \dots 9$, we find that the sign changes between n and $n+1$. This value of n is the next decimal place value of the original root. This is because the previous step effectively removed the integer part of the root, and this step scaled the result by ten.
(6) While more digits (decimal places) are required, repeat the previous steps 4–5, which reduce the roots by n and then increase all the roots by a factor of 10. Otherwise stop.

Example. *Find, to 2 decimal places, the root of the polynomial*

$$f(x) = x^3 + 6x^2 + 9x + 17$$

near $x = -4$.

The technique works by repeatedly transforming the polynomial so that information can be found about the original root by evaluating the resulting polynomial at integer values of x.

The first step of the technique is to bracket the root between two integers. In this case, we can check $f(-4) > 0$ and $f(-5) < 0$, so, there is a root in $(-5,-4)$. As this root is negative, transform the polynomial by changing the sign of its roots (see step 2 above). Of course, this amounts to changing the sign of every second coefficient, and we are now working with

$$x^3 - 6x^2 + 9x - 17.$$

By construction, this polynomial has a root in $(4,5)$. To extract the next decimal place, reduce the roots of this polynomial by 4. This can be done by repeatedly using Horner's synthetic division.

$$\begin{array}{c|rrr|r}
 & 1 & -6 & 9 & -17 \\
\mathbf{4} & & 4 & -8 & 4 \\
\hline
 & 1 & -2 & 1 & -13 \\
 & & 4 & 8 & \\
\hline
 & 1 & 2 & 9 & \\
 & & 4 & & \\
\hline
 & 1 & 6 & &
\end{array}$$

By reading the coefficients of this new polynomial from left to right along the bottom and up the 'remainders', the new polynomial

$$x^3 + 6x^2 + 9x - 13$$

is found, which has a root in $(0,1)$.

Now scale up the root by a factor of 10 (see step 5 above) and produce another new polynomial

$$f_1(x) = x^3 + 60x^2 + 900x - 13000,$$

which has a root in $(0,10)$. The integer part of the root corresponds to the next digit (decimal place) of our original root. Check this new polynomial $f_1(x)$ at $x = 1, \dots 9$, to identify a change of sign between $f_1(8)$ and $f_1(9)$. This means that the root of the original polynomial is between -4.8 and -4.9.

The process can be repeated to find more decimal places of the original root. Repeated synthetic division is used to move the root from $(8, 9)$ to $(0, 1)$,

	1	60	900	−13000
8		8	544	11552
	1	68	1444	−1448
		8	608	
	1	76	2052	
		8		
	1	84		

giving $x^3 + 84x^2 + 2050x - 1448$. Subsequently scaling up the root by a factor of 10 gives $f_2(x) = x^3 + 840x^2 + 205000x - 1448000$. Checking this polynomial at $x = 1, \dots 9$, we find a sign change between $x = 6$ and $x = 7$, so the original root is now known to be between -4.86 and -4.87.

3. Discussion

This technique is quite attractive for use by hand. If the initial polynomial has integer coefficients, all the quantities remain integers. It is also clearly designed as a decimal-friendly technique — it sandwiches the roots between round decimal values, allowing the extraction of a particular number of digits.

It also has clear similarities with the bracketing and bisection technique. Both begin by bracketing the root to some interval and then iterate to refine the root. However, with this technique the polynomial changes over each iteration, while in bisection the interval changes at each step. The bisection technique is more general and can work with any continuous function $f(x)$, while for this technique $f(x)$ must be a polynomial. On the other hand, this technique only needs to evaluate its functions at integers, while the bisection technique typically winds up with increasingly long decimals.

One could also compare bisection and this technique in terms of efficiency. At each step, bisection evaluates $f(x)$ once and then halves the size of the interval. Consequently, it uses roughly $\log_2 10$ function evaluations to refine the root by a factor of 10. The technique presented above refines the root by a factor of 10 on each step, but uses more function evaluations — with intuition or binary search this can be done in 3–4 function evaluations per step, which is actually similar to bisection. Of course, the technique also needs a synthetic division step and a scale-by-ten step, which the bisection technique does not require.

Note that both techniques need to evaluate the polynomial, which gives us a chance to use another method of Horner's for polynomial evaluation. Using this method, a polynomial of degree m can be evaluated with just m multiplies and m additions/subtractions. Suppose you wish to find:

$$p(x) = a_m x^m + a_{n-1}x^{n-1} + \ldots + a_2x^2 + a_1x + a_0,$$

you can write this as

$$p(x) = ((\ldots((xa_m + a_{m-1})x + a_{m-2})x + \ldots + a_2)x + a_1)x + a_0.$$

So, you can start with a_m, multiply by x, add a_{m-1}, multiply by x, …, add a_1, multiply by x, and finally add a_0. Note that with judicious use of the "=" key, this method can be used to evaluate polynomials on a non-programmable calculator without the use of any intermediate values that need to be stored or written down. For example, suppose we are working with

$$f_1(x) = 1x^3 + 60x^2 + 900x - 13000$$

and we want to find $f_1(8)$, then we can do the following:

1 × 8 + 60 = × 8 + 900 = × 8 - 13000 ,

giving the answer -1448. While this trick is particularly handy if the x value is a single digit, it can also be used at more messy x values by storing x in the calculator's memory.

1 × RCL + 60 = × RCL + 900 = × RCL - 13000 .

It seems that this trick might be useful in exams where polynomial evaluations are common, such as the current Leaving Certificate.

Of course, more powerful techniques could be used, such as Newton's Method [1, 2, 3] or extracting the roots using Sturm's Theorem [6]. These methods need more technical tools and also usually require division, which is curiously absent from the method we describe!

4. Conclusion

We have described a technique for extracting digits of roots of a polynomial that is quite suited to manual use. We noted its similarities and differences to the closely related and better-known bisection technique. We would be interested to know more about the history of this method, its naming and its use, and welcome feedback if it is familiar to any readers.

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The Golden section in the hypercube

QUANG HUNG TRAN

Abstract. We shall present a way to establish the Golden section in n-dimensional Euclidean space. We use a hypercube covered by a hypersphere and divide the diameters of two opposing facets in a way that depends on the dimension of the space. The Golden ratio will be obtained from the ray connecting these two dividing points intersecting the hypersphere.

1. Introduction

The Golden ratio $\varphi = \frac{\sqrt{5}+1}{2}$ is one of the most beautiful numbers. It has a long history in many different areas of life such as Art, Nature, and Science; see [8, 9]. In Mathematics, the Golden Ratio is mentioned early on, already appearing in Euclid's Elements; see [7], and it has been much studied throughout history; see [11]. In modern Mathematical research, the Golden Ratio remains relevant to some problems, see [3, 10, 12]. In this paper, we introduce and prove our discovery about the occurrence of the Golden ratio in n-dimensional Euclidean space associated with the hypercube [2, 6] and the hypersphere [4, 5, 6].

Theorem 1.1 (Main theorem). *Let $\mathcal{N}$ be a hypercube contained in n-dimensional Euclidean space $\mathbb{E}^n$ ($n \geq 2$). Let $\mathcal{F}_0$ be a facet of $\mathcal{N}$ with center K. Let $\mathcal{F}_0^*$ be the facet opposite to $\mathcal{F}_0$. Let $\mathcal{S}$ be a hypersphere centered at K and passing through all vertices of $\mathcal{F}_0^*$. Let $\mathcal{F}_1$ be a facet of $\mathcal{N}$ that is perpendicular to $\mathcal{F}_0$. Let XY be a diameter of $\mathcal{F}_1$. Let X^* and Y^* be the reflections of X and Y through the center of $\mathcal{N}$. Let Z and Z^* divide the segments XY and Y^*X^*, respectively, in the ratio $n-2$ to 1 i.e.*

$$Z = \frac{(n-2)X + Y}{n-1} \quad \text{and} \quad Z^* = \frac{(n-2)Y^* + X^*}{n-1}. \tag{1}$$

Let the ray Z^*Z meet the hypersphere $\mathcal{S}$ at Z_0. Then,

$$\frac{Z^*Z}{ZZ_0} = \varphi. \tag{2}$$

Where $n = 2$, we have a configuration with square and circle; see Figure 1.
Where $n = 3$ we have a configuration with cube and sphere; see Figure 2.

2. Proof of main theorem

In this section, we give a proof of Theorem 1.1.

Proof. Let $\mathcal{N} = [-1, 1]^n$ in the Cartesian coordinates of n-dimensional Euclidean space $\mathbb{E}^n$. Let $K = (0, 0, \ldots, 0, -1)$. Thus $\mathcal{S}$ is the hypersphere centered at K and goes through vertex $A_0 = (1, 1, \ldots, 1)$, so $\mathcal{S}$ has equation

$$x_1^2 + x_2^2 + \ldots + (x_n + 1)^2 = (1-0)^2 + (1-0)^2 + \ldots + (1-0)^2 + (1-(-1))^2 \tag{3}$$

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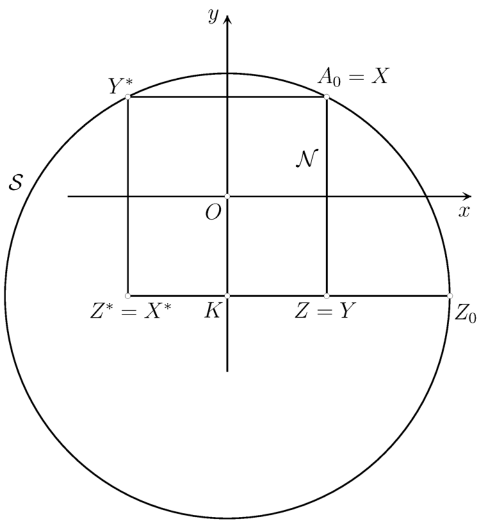

FIGURE 1. Illustrations in two dimensions $n = 2$, $\frac{Z^*Z}{ZZ_0} = \varphi$.

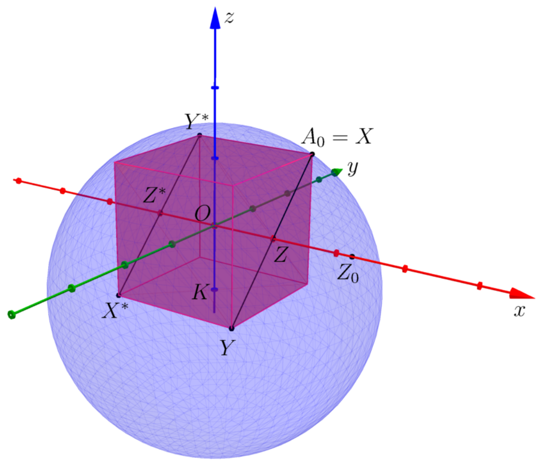

FIGURE 2. Illustrations in three dimensions $n = 3$, $\frac{Z^*Z}{ZZ_0} = \varphi$.

which is

$$x_1^2 + x_2^2 + \ldots + (x_n + 1)^2 = n + 3. \tag{4}$$

Since XY is a diameter of $\mathcal{F}_1$, which is perpendicular to $\mathcal{F}_0$ (centered at K), we may choose $X = A_0 = (1, 1, \ldots, 1)$ and then Y is the reflection of X in the center $K_1 = (1, 0, \ldots, 0)$. Therefore $Y = 2K_1 - X = (1, -1, -1, \ldots, -1)$. Now X^* and Y^* are the reflections of X and Y, respectively, in the center $O = (0, 0, \ldots, 0)$ of $\mathcal{N}$, so we obtain the coordinates

$$X^* = (-1, -1, \ldots, -1)$$

and

$$Y^* = (-1, 1, \ldots 1).$$

Thus,

$$Z = \frac{(n-2)X + Y}{n-1} = \left(1, \frac{n-3}{n-1}, \frac{n-3}{n-1}, \ldots, \frac{n-3}{n-1}\right)$$

and

$$Z^* = \frac{(n-2)Y^* + X^*}{n-1} = \left(-1, \frac{n-3}{n-1}, \frac{n-3}{n-1}, \ldots, \frac{n-3}{n-1}\right).$$

From these, the line ZZ^* has parametric equation

$$X = Z^* + t \cdot \overrightarrow{ZZ^*} = \left(-1 - 2t, \frac{n-3}{n-1}, \frac{n-3}{n-1}, \ldots, \frac{n-3}{n-1}\right). \tag{5}$$

The intersection of the ray Z^*Z (equation (5)) and the hypersphere $\mathcal{S}$ (equation (4)) is the point $Z_0 = Z^* + t_0 \cdot \overrightarrow{ZZ^*}$ ($t_0 > 0$), where t_0 satisfies the equation

$$(-1 - 2t_0)^2 + (n-2)\left(\frac{n-3}{n-1}\right)^2 + \left(\frac{2n-4}{n-1}\right)^2 = n + 3, \tag{6}$$

which is equivalent to

$$(1 + 2t_0)^2 = n + 3 - \frac{(n-2)(n-3)^2 + 4(n-2)^2}{(n-1)^2}. \tag{7}$$

Therefore

$$(1 + 2t_0)^2 = 5 \tag{8}$$

or

$$t_0 = \frac{\sqrt{5} - 1}{2} = \frac{1}{\varphi}. \tag{9}$$

Since $Z_0 = Z^* + t_0 \cdot \overrightarrow{ZZ^*}$,

$$\frac{ZZ^*}{Z^*Z_0} = \frac{1}{t_0}.$$

Hence equation (2) holds. This completes the proof. □

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Peter Lynch: That's Maths III, Logic Press, 2022. ISBN:9781471757525, EUR 13.69*, 282 pp..

REVIEWED BY RICHARD M. ARON

When picking up a new book, the first things that a potential reader may ask are: Is this book for me? Who are the intended readers?

I was sceptical of Peter Lynch's response to these questions which appears in the preface, namely that the book is suitable for any interested person who has studied mathematics at secondary school level, and that it is especially aimed at teachers of mathematics and science at school and university levels. After all, what could I, who have lectured to university students for around half a century, learn from TM III?

I felt that way until I got to page 5, when I read about a certain Dr Muriel Bristol who strongly preferred that milk should precede tea when being served. Although the three-page account left me in the dark about whether and when Dr Bristol took sugar, I did learn that as a consequence of this 1935 episode, the term *statistical significance* became prominent.

Like TM II, this volume has 64 short chapters, each averaging four pages in length. Many of the articles are based on Lynch's fortnightly column in the *Irish Times*, while some come from his blog (thatsmaths.com). References for further study are included at the end of most short chapters; typically, these references are to helpful *Wikipedia* notes, or to accessible texts, or even to *YouTube* videos.

A significant difference in TM III is that one quarter of the chapters (specifically, the last 16 chapters), require more sophistication and knowledge on the part of the reader. Thus, for example, the first of these – Chapter 49 – is about the "elegance of complex analysis". Starting with a picture of Cauchy, in five pages we are led to line integrals, Laurent series, and the Residue Theorem. Of course, nothing is proved, but it is reasonable to think that for many this would be an excellent introduction encouraging further study of complex variables.

Similarly, the three-page sketch about why the sphere and the torus are not homeomorphic, titled "Doughnuts $\neq$ Dumplings", gently leads the reader through an intuitive argument about why the torus cannot be deformed to a sphere, with a strong hint that group theory and algebraic topology hold the answers. (Here, I wished that the heuristic argument had treated the case of space-filling curves on the sphere.)

To me, one of the most fascinating chapters is about the Euler-Borel divergent series

$$\sum_{k=0}^{\infty}(-1)^k k!$$

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to which Euler assigned the value 0.596347562 . . .. There's a nice justification, which roughly goes:

$$\begin{aligned}\sum_{k=0}^{\infty}(-1)^k k! &= \sum_{k=0}^{\infty}(-1)^k \int_0^{\infty} x^k e^{-x}\,dx \\ &= \sum_{k=0}^{\infty}\int_0^{\infty}(-x)^k e^{-x}\,dx \\ &= \int_0^{\infty}\frac{e^{-x}}{1+x}\,dx \\ &\sim 0.596347562\ldots.\end{aligned}$$

(Surely, only an over-zealous person would question any of the three equalities!). In fact, as suggested by Lynch, I went to *Wikipedia* to learn that Émile Borel had explained the above to Mittag-Leffler, who was not impressed.

There are few book authors, mathematical or otherwise, who sincerely suggest that the reader should *ramble* through its pages, more or less randomly. This is indeed the case here, where readers are encouraged to wander through the book, skipping over sections they find too difficult. General topics appear in one chapter, followed by several chapters on disparate things, only to reappear later on. Thus, for example, Chapter 13 in TM III is called "The Rise and Rise of Women in Mathematics", with "The 'Superior Genius' of Sophie Germain" being Chapter 43. (The contribution by women was already noted in TM I with a short chapter about the first female PhD, Sofia Kovalevskaya, who did her work with Karl Weierstrass.)

By way of conclusion, I return to answer my initial questions: I think that, anyone with even a vague interest and slight background in mathematics, will find many interesting topics in TM III. Given the author's firm but gentle hand, the book will find a very appreciative audience.

Richard M. Aron was a Lecturer in Mathematics at Trinity during the 1970's and early 1980's. He has returned as a visitor to both TCD and UCD on several occasions. In 2015 he retired as Professor of Mathematics at Kent State University (Ohio, USA) and is semi-retired as an Editor of the *Journal of Mathematical Analysis and Applications*.

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Riccardo Benedetti: Lectures on Differential Topology, AMS, 2021. ISBN: 978-1-4704-6674-9, USD 85.00, 425 pp.

REVIEWED BY BENJAMIN MCKAY

Both the quantity and quality of textbooks decays, on average, with the level at which they are set, leaving us with so many excellent textbooks in calculus and linear algebra, and so few bridging the gap between basic graduate topics and current research. This book makes a daring attempt to help the student get a big picture of differential topology. The author develops the central pillars of differential topology on compact manifolds: transversality, bordism, cobordism, characteristic numbers, 3-manifolds and 4-manifolds, starting by laying out some basic definitions and theorems in the concrete context of open sets in Euclidean space, and then for embedded submanifolds of Euclidean space. This book is written for graduate students in mathematics without prior study of algebraic topology; there are many excellent books in this area (we mention a few below), but still a wide gap between the books and research. The author does not mention whether he expects the reader to have prior familiarity with differential geometry, but the pace would be too quick for a first encounter with manifolds.

Benedetti's approach is quite concrete, emphasising cut and paste. The classical approach to characteristic classes of Pontryagin looks at the zeroes of generic sections of vector bundles, varying with the choice of section, following the algebraic geometer's picture of a pencil of curves on a surface. This picture reveals that these sections all have the same homology class, which is thus a topological invariant of the vector bundle itself. The author explains that all vector bundles are pulled back from the tautological vector bundles on Grassmannians, hence from the tautological vector bundle on the infinite Grassmannian, and therefore the vector bundles are classified by the homotopy classes of maps to the infinite Grassmannian.

The author provides the bulk of Whitney's proof of his embedding and immersion theorems, including the cancellations of oppositely signed double points using the Whitney trick. His insistence on working only with compact manifolds makes the existence of a Euclidean embedding almost obvious, but he chooses to follow the path of Whitney, which gives the proof for noncompact manifolds and also gives information about embedding dimensions. Again, the approach is through transversality arguments and through elementary cut and paste. He approaches Morse theory, in much the style of Milnor's book, emphasising handle decompositions and discussing handle sliding. Then he works through bordism and cobordism invariants, some theory of 3-manifolds and 4-manifolds including the Arf invariant.

At some points, the book runs very quickly through subtle points. The reader might like to have C.T.C. Wall's book [8] on hand. Compared to the textbooks of Adachi [1], Bröcker and Jänich [2], Hirsch [3], Milnor [4, 5], Munkres [6], Shastri [7], Wall [8], or Wallace [9], this book emphasises low dimensional topology more, and proceeds closer to current research, making more frequent use of references to proofs so as to skip some of the complicated bits of the proofs. The index is unfortunately brief.

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Steven T. Dougherty: Combinatorics and Finite Geometry, Springer, 2020. ISBN:978-3-030-56394-3, GBP 29.99, 369+xv pp.

REVIEWED BY PADRAIG Ó CATHÁIN

This book is an advanced undergraduate introduction to combinatorics, with a particular emphasis on finite geometry and related topics. As such, finite projective planes appear before graphs and connections to codes and designs are discussed in later chapters. The more algebraic and enumerative aspects of combinatorics (e.g. generating functions, group actions) are treated, but less heavily.

Who is the ideal reader?
In the opinion of the reviewer, the author's style is direct and clear. The text is illustrated with many examples, proofs are written in sufficient (but not exhaustive) detail and the reviewer suspects that an advanced undergraduate could tackle large sections of the text independently. While rapid overviews of all required material are contained in the text, it seems implausible that a reader unacquainted with linear algebra will learn enough from eleven pages in Chapter 6 to fully appreciate projective geometries. This section would serve as a review, however. So the ideal reader would have taken courses in linear and abstract algebra, as well as a basic course in discrete mathematics. They would also have a strong grasp of proof writing, and be able to handle a moderate degree of abstraction. Seasoned mathematicians will find something new and interesting there also.

What does the book cover?
The opening chapter could be a free-standing combinatorics course! It is a breezy account of (finite) set theory and binomial coefficients, a rapid overview of permutations and the symmetric group (with a nice graphical interpretation, new to the reviewer, as *Japanese ladders*) and concludes with some examples of generating functions. The style is engaging, the examples are helpful and the proofs are short. Chapter 2 is further background on rings and fields, concluding with some counting problems.

The core text really begins in Chapter 3, which is devoted to the historically important and rather non-trivial problem of the 36 officers. Mutually Orthogonal Latin Squares (MOLS) are motivated and introduced, complete sets are constructed from finite fields at prime power orders, and the problem of determining the number of MOLS in dimension 10 is revealed to be open. In Chapter 4, affine and projective planes are introduced (without linear algebra!) and the equivalence with complete sets of MOLS is given. Chapter 5 is a digression on graph theory, before returning to higher dimensional combinatorial geometries in Chapter 6. This is the heart of the text: the author considers the Desargues configuration and proves Segre's celebrated theorem that an oval in a finite projective plane of odd order is a conic section. He also proves the Bruck-Ryser-Chowla theorem which rules out the existence of certain projective planes. These are **not** easy theorems: they were proved in the middle decades of the twentieth

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century and are rightly recognised as ingenious and important. The author's achievement in presenting them to undergraduates is substantial. Later chapters are devoted to designs, Hadamard matrices, association schemes, coding theory, cryptography and discrete probability. These are well written and largely free-standing, though many rely on results from finite geometry.

So far, so good. Are there any issues with the book?

This book covers an astonishing amount of material. It is perhaps unsurprising then that there were omissions which struck the reviewer as odd and unfortunate. The Veblen-Young axioms for a projective geometry should have been mentioned. The importance of Desargues' theorem (proved only for the Desarguesian planes $\mathrm{PG}_2(\mathbb{F}_q)$) is skirted over, as is any mention of perspective which is arguably the crucial concept for an understanding of the theorem. The proof of Segre's theorem in the published text contains some rather large leaps of logic and a few typographical errors, which together render it cryptic. The reviewer contacted the author about this, and is pleased to report that this resulted in an improved proof now available from the author's website tinyurl.com/poc-fix.

So, what courses could I teach from this book at an Irish university?

An undergraduate course based on this text would require quite a lot of background from the students. Certainly familiarity with linear algebra and discrete mathematics (i.e. covering sets, proofs, modular arithmetic, etc.) would be essential. Groups, rings and fields would be helpful but a course could be plotted around them rather easily (replacing $\mathbb{F}_q$ by $\mathbb{F}_p$ where necessary). With these prerequisites, a third or fourth year class in *combinatorics* or *discrete mathematics* from this text seems perfectly reasonable. Another possibility would be to teach a slightly non-conventional course on *geometry*, supplemented by a little of the classical theory of conic sections over $\mathbb{R}$ and $\mathbb{C}$, perhaps. The reviewer would also be very comfortable teaching a course at beginning graduate level from this text, and it could serve well as a starting point for final year projects or other undergraduate research requirements.

Padraig Ó Catháin received a BA in Mathematics and History from the University of Galway in 2007. Under the direction of Dane Flannery he was awarded the degrees of MLitt in 2008 and PhD in 2012, by the same institution. After a decade at universities in Australia, Finland and the United States, he took a position as Ollamh Cúnta in Fiontar agus Scoil na Gaeilge at Dublin City University in 2022. His research is predominantly in combinatorics, for which he has been awarded the Kirkman Medal of the Institute of Combinatorics and its Applications.

Leagan gaeilge den léarmheas céanna:

Is buntreoir an leabhar seo don fhochéimí aibí ar an matamaitic theaglamach, le béim ar leith ar chéimseata thar uimhirchoirp críochta agus ar ábhair ghaolta leis seo. Ní cúis ionaidh é mar sin go bhfuil cur síos ar na plánaí teilgeacha críochta sula bhfeictear grafanna. Cuirtear na plánaí teilgeacha i bhfeidhm agus cur síos á dhéanamh ag an údar ar chódaigh agus dearaí theaglamach níos déanaí sa théacs. Déantar cur síos freisin ar na gnéithe ailgéabracha agus áireamha den mhatamaitic theaglamach (m.sh. teoiric ghníomhú na ngrúpaí, feidhmeanna giniúna), ach gan róbhéim orthu.

Cé do an leabhar seo?

Dar leis an léirmheastóir, tá stíl dheas sholéite ag an údar. Baintear úsáid as neart samplaí, déantar mionphlé ar na cruthúnais (gan dul thar fóir). Dá bhrí sin, ba chóir go mbeadh fochéimí láidir in ann dul i ngleic le roinnt mhaith den leabhar faoina stuaim fhéin. Cé go dtugtar léirmheas tapaidh ar chuile ghné atá ag teastáil sa théacs, is ar ndóigh go mbeadh duine ar bith gan an ailgéabar líneach ar a dtoil acu in ann dóthain a bhaint ó aon leathanach déag i gCaibidil 6 ionas go mbeadh léirthuiscint cheart aige ar na gcéimseataí theilgeacha. D'fheilfeadh sé go maith mar achoimre ar an ábhar, áfach. Mar sin, bheadh cúrsaí sna ailgéabair líneach agus teibí araon, chomh maith le bunchúrsa sa mhatamatic scoite de dhíth ar an léitheoir. Ba chóir go mbeadh taithí mhaith acu le cruthúnais, agus go mbeadh roinnt mhaith teibiú feicthe acu. Cé gur ar fhochéimithe atá an leabhair dírithe, tá neart ann ar mhaitheas an mhatamataiceora lánfhásta freisin.

Céard atá sa leabhar?

Tá neart sa chéad chaibidil le haghaidh cúrsa iomlán sa mhatamataic theaglamach a mhúineadh! Insint bhreá thapaidh atá ann ar theoiric na tacair críochta, na chomhéifeachtaí dhéthéarmacha, iomalartuithe agus an ghrúpa siméadrach (le léiriú deas grafaiceach i dtéarmaí *Japanese ladders*, nach bhfuil feicthe cheana ag an léirmheastóir). Tá críoch leis an chaibidil le roinnt samplaí de na feidhmeanna giniúna. Tá stíl shoiléir sholéite ag an údar, tá mianach sna samplaí a thugtar agus tá na cruthúnais gearr agus dírithe. Sa dhara caibidil tugtar tuilleadh buneolais maidir le faileanna agus uimhirchoirp, le beagán ar áireamh ar deireadh.

Cuirtear tús le príomhábhar an téacs sa thríú caibidil, ina ndéantar cur síos ar fhadhb na 36 Oifigigh, a raibh tábhacht ar leith ag baint leis i stair na matamataic. Mar seo, tugtar spreagadh agus intreoir ar na *Cearnógaí Laidneach Comhortagánach* (MOLS, ag baint úsáid as an ngiorrúcháin a úsáidtear go hiondúil). Tugtar déantús ar tacair iomláin MOLS ag baint úsáid as uimhirchoirp ag ord ar bith ar cumhacht d'uimhir phríomha é, agus cuirtear in iúl don léitheoir gur fadhb oscailte í uaslíon na MOLS in ord a deich. Sa cheathrú caibidil, tugtar cur síos ar na plánaí teilgeacha agus fineacha[1] (gan chaint ar an ailgéabar líneach!) agus cuirtear an comhionnanais le tacar iomlán MOLS in iúl. Is aistear ar leataobh an cúigiú caibidil ar bhealach, i dtreo na ghrafanna, sula bhfilltear ar na céimseataí teilgeacha de dhiminsean níos mó ná dó i gCaibidil 6. Is í seo croílár an leabhair: déantar cíoradh ar chumraíocht Desargues, agus tugtar cruthúnais iomlán ar theoraim thábhachtach de chuid Segre a dhéanann cur síos ar na hubhcruthanna i bplána teilgeach d'ord chorr i dtéarmaí chónghearrtha. Tugtar cruthúnas iomlán freisin ar an teoraim de chuid Bruck-Ryser-Chowla, a chuireann cosc ar uimhreacha áirithe mar ord ar phlána teilgeach. Ní torthaí éasca iad seo! Aimsíodh iad i lár an fichiú aois, agus meastar go forleathan go bhfuil tábhacht agus sárbhua ar leith ag baint leo. Tá gaisce déanta ag an údar anseo agus iad ar fáil ar leibhéal soléite don fhochéimí. Tá caibidil níos déanaí sa leabhar dírithe ar dhearaí teaglamacha, maitrisí Hadamard, scéimeanna comhthiomsaitheacha, teoraic an chódaigh, cripteagrafaíocht agus an dóchúlacht scoite.

[1] Eitleán cleamhnais a thug Google Translate ar seo!! Moltar tearma.ie do théarmaí teicniúla.

Tá siad seo soléite agus neamhspleách go maith ar a chéile, seachas go bhfuil roinnt mhaith dóibh ag brath ar an gcéimseata críochta.

Maith go leor. An bhfuil lucht ar bith ar an leabhar?

Clúdaítear an t-uafás ábhar sa leabhar. Ní ionadh ar bith é mar sin go raibh roinnt rudaí in easnamh, ar a laghad i súile an léirmheastóra. Is mór an trua, mar shampla, gur fágadh aicsímí Veblen-Young ar an chéimseata theilgeach ar lár sa théacs. Cé go ndéantar cur síos ar chumraíocht Desargues, agus go dtugtar cruthúnas dó dos na plánaí ar a nglaotar *Desarguesian* orthu, níor dhearnadh iarracht ar bith cur síos a dhéanamh ar an tábhacht a bhaineann leis ó thaobh an comhordanáidiú. Níl trácht ar bith ach oiread ar an bpeirspictíocht, ar cheann de na coincheapa is tábhachtaí ar fad sa réimse seo. Sa chruthúnas ar Theoraim Segre, tá roinnt bhotún clóghrafach ann, a fhágann an scéal doiléir go maith. Chuaigh an léirmheastóir i dteagmháil leis an údar faoi seo áfach, agus is deas le rá go bhfuil leagan nua den chruthúnas seo ar fáil anois ó shuíomh idirlín an údair tinyurl.com/poc-fix.

Ar deireadh, cén cineál cúrsa ar fiú múineadh ón leabhar seo in ollscoil Éireannach?

Bheadh bunús sách láidir sa mhatamataic ag teastáil ó fhochéimithe agus iad ag cur tús le cúrsa bunaithe ar an leabhar seo. Cinnte ba chóir go mbeadh an ailgéabair líneach agus an mhatamataic scoite (sé sin tacair, cruthúnais, uimhríocht mhodúlach, srl.) ar a dtoil acu. Ba mhaith an rud é freisin go mbeadh tuiscint acu ar ghrúpaí, faileanna agus uimhirchoirp, ach bheadh sé sodhéanta iad seo a sheachaint sa théacs (ag déanamh ionadú ar $\mathbb{F}_q$ le $\mathbb{F}_p$ agus a leithid). Leis na réamhriachtanais seo, d'fheadfaí chúrsa don tríú nó don cheathrú bhliain sa *mhatamataic scoite* nó *theaglamach* a theagaisc ón leabhar seo. Bheadh sé indéanta freisin cúrsa beagán neamhghnách a thabhairt ar an *gcéimseata*, le breis cuirthe isteach ar theoiric na chóngearradh clasaiceach thar na huimhirchoirp $\mathbb{R}$ agus $\mathbb{C}$. Bheadh an léirmheastóir lán-sásta freisin rang ar léibhéal na hiarchéime tosaíocha a mhúineadh as an leabhar. Is foinse iontach maith é freisin ar ábhar do thionscnaimh na bliana deireanaí nó mar sheoladh ar an taighde ar leibhéal na buncéimithe.

Padraig Ó Catháin Bhain śe amach BA sa mhatamataic agus sa stair ó Ollscoil na Gaillimhe sa bhlian 2007. Bronnadh céim MLitt air i 2008 agus céim PhD i 2012, agus é ag déanamh taighde faoi stiúr Dane Flannery san áit céanna. Théis deich mbliana ag obair thar lear san Astráil, san Fhionlainn agus sna Stáit Aontaithe, d'fhill sé ar Éirinn i 2022 chun post a ghlacadh mar Ollamh Cúnta le Fiontar agus Scoil na Gaeilge le hOllscoil Chathair Bhaile Átha Cliath. Tá an chuid is mó den taighde aige sa mhatamataic teaglamach, agus bronnadh Bonn Kirkman de chuid an *Institute of Combinatorics and its Applications* air in aitheantas ar seo.

Stór Focail

Céimseata	Geometry
Cónghearradh	Conic section
Críochta	Finite
Cód, Códaigh	Code, Codes
Cumraíocht	Configuration
Dearadh theaglamach	Combinatorial design
Feidhmeanna ghiniúna	Generating functions
Gníomhú na ngrúpaí	Group actions
Matamaitic theaglamach	Combinatorics
Scéim comhthiomsaitheach	Association Scheme
Scoite	Discrete
Teibí, Teibiú	Abstract, Abstraction
Teilgeach	Projective
Ubhcruth	Oval
Uimhirchoirp (críochta)	(finite) Field

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PROBLEMS

IAN SHORT

Problems

The first problem this issue is an elementary observation I stumbled upon recently.

Problem 90.1. Let M be any 3-by-3 matrix

$$M = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

over a field, where $e \neq 0$, and let A, B, C, D be the four submatrices of M given by

$$A = \begin{pmatrix} a & b \\ d & e \end{pmatrix}, \quad B = \begin{pmatrix} b & c \\ e & f \end{pmatrix}, \quad C = \begin{pmatrix} d & e \\ g & h \end{pmatrix}, \quad D = \begin{pmatrix} e & f \\ h & i \end{pmatrix}.$$

Find an expression for $\det M$ in terms of $\det A$, $\det B$, $\det C$, $\det D$, and e.

The second problem is from Finbarr Holland of University College Cork.

Problem 90.2. Prove that

$$\sum_{n=0}^{\infty} a_n \sum_{k=0}^{n} \frac{a_k a_{n-k}}{(2k+1)(2(n-k)+1)} = \frac{2}{\pi} \int_0^{\pi/2} \frac{x^2}{\sin^2 x}\, dx = \log 4,$$

where

$$a_n = \frac{\Gamma(n+\frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(n+1)}, \quad n = 0, 1, 2, \dots .$$

The third problem was proposed by Seán Stewart of the King Abdullah University of Science and Technology, Saudi Arabia.

Problem 90.3. Evaluate

$$\sum_{m,n=0}^{\infty} \binom{2m}{m}^2 \binom{2n}{n} \frac{1}{2^{4m+2n}(m+n+1)}.$$

Solutions

Here are solutions to the problems from *Bulletin* Number 88.

The first problem was solved by Kee-Wai Lau of Hong Kong, China, Seán Stewart, the North Kildare Mathematics Problem Club, and the proposer Anthony O'Farrell, editor of this Bulletin. We present the solution of Kee-Wai Lau.

Problem 88.1. Consider the sequence $x_0, x_1, \dots$ defined by $x_0 = \sqrt{5}$ and $x_n = \sqrt{2 + x_{n-1}}$, for $n = 1, 2, \dots$. Prove that

$$\prod_{n=1}^{\infty} \frac{2}{x_n} = 2 \log\left(\frac{1+\sqrt{5}}{2}\right).$$

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Solution 88.1. Let $c = \log((1+\sqrt{5})/2)$. Using the identity $1 + \cosh(2x) = 2\cosh^2 x$, it can be proven by induction that

$$x_n = 2\cosh\left(\frac{c}{2^n}\right), \quad \text{for } n = 0, 1, \ldots.$$

Then, using the identity $\sinh(2x) = 2\sinh x \cosh x$, it can be proven by induction that

$$\prod_{k=1}^{n} \frac{2}{x_k} = 2^{n+1} \sinh\left(\frac{c}{2^n}\right), \quad \text{for } n = 1, 2, \ldots.$$

Hence

$$\prod_{k=1}^{\infty} \frac{2}{x_k} = 2c \times \lim_{n\to\infty} \frac{\sinh(c/2^n)}{c/2^n} = 2c,$$

as required. □

The second problem was solved by the North Kildare Mathematics Problem Club and the proposer, J.P. McCarthy of Munster Technological University. Here is the solution of the Problem Club.

*Problem 88.2***.** Let P be a 3-by-3 matrix each entry of which is an n-by-n complex Hermitian matrix; that is, each entry P_{ij} is an n-by-n complex matrix equal to its own conjugate transpose P_{ij}^*. Suppose that the sum along any row or column of P is the n-by-n identity matrix I_n:

$$\sum_{k=1}^{3} P_{kj} = \sum_{k=1}^{3} P_{ik} = I_n.$$

Suppose also that the entries of P along rows and columns satisfy

$$P_{ik}P_{il} = \delta_{kl}P_{ik} \quad \text{and} \quad P_{kj}P_{lj} = \delta_{kl}P_{kj},$$

where δ_{kl} is 1 if k and l are equal and otherwise it is 0 (and no summation convention should be applied). Prove that the entries of P commute with one another.

Solution 88.2. The given data imply that, regarded as linear operators on $\mathbb{C}^n$, the matrices P_{ij} are idempotents, orthogonal projections, and each row or column gives an orthogonal decomposition of $\mathbb{C}^n$ as a sum of three corresponding subspaces $V_{ij} = \mathrm{im}(P_{ij})$. We are also given that P_{ij} commutes with every P_{ik} and P_{kj}, that is, P_{ij} commutes with every entry in its row and every entry in its column. So it remains to show that each entry commutes with the entries outside its row and column.

The given conditions remain valid under permutation of rows or columns, so it suffices to prove that P_{11} commutes with P_{22}.

If $v \in V_{22}$, then $P_{12}v = 0$ so $P_{11}v + P_{13}v = v$, and also $P_{23}v = 0$ so $P_{13}v + P_{33}v = v$. Hence $P_{11}v = P_{33}v$. Thus for any $v \in \mathbb{C}^n$, $P_{11}P_{22}v = P_{33}P_{22}v$. Thus $P_{11}P_{22} = P_{33}P_{22}$. Taking the conjugate transpose, we also have $P_{22}P_{11} = P_{22}P_{33}$. Permuting indices, we deduce that

$$P_{ii}P_{jj} = P_{ii}P_{kk} \quad \text{and} \quad P_{ii}P_{jj} = P_{kk}P_{jj}$$

whenever i, j and k are mutually-distinct indices. Thus

$$P_{11}P_{22} = P_{33}P_{22} = P_{33}P_{11} = P_{22}P_{11}.$$

So P_{11} commutes with P_{22}, as required. □

The third problem was posed by Seán Stewart of the King Abdullah University of Science and Technology, Saudi Arabia. It was solved by Ankush Kumar Parcha, a student from Indira Gandhi National Open University, New Delhi, India, Finbarr Holland, the North Kildare Mathematics Problem Club, and the proposer. We present the solution of the problem club.

*Problem 88.3***.** Prove that

$$\sum_{n=2}^{\infty} \frac{(-1)^n H_{\lfloor n/2 \rfloor}}{n} = (\log 2)^2,$$

where H_n denotes the nth harmonic number

$$H_n = \sum_{k=1}^{n} \frac{1}{k}$$

and $\lfloor \cdot \rfloor$ denotes the floor function.

Solution 88.3. The sum of the series is

$$\begin{aligned}
&\int_0^1 \left(H_1 y - H_1 y^2 + H_2 y^3 - H_2 y^4 + H_3 y^5 - H_3 y^6 + \cdots\right) dy \\
&= \int_0^1 \left(y + \frac{y^3}{2} + \frac{y^5}{3} + \cdots\right)\left(1 - y + y^2 - y^3 + \cdots\right) dy \\
&= -\int_0^1 \frac{\log(1-y^2)}{y(1+y)}\, dy \\
&= -\int_0^1 \left(\log(1-y) + \log(1+y)\right)\left(\frac{1}{y} - \frac{1}{y+1}\right) dy.
\end{aligned}$$

This gives us four integrals, and we proceed to evaluate them in terms of the dilogarithm

$$\operatorname{Li}_2(x) = -\int_0^x \frac{\log(1-t)}{t}\, dt, \quad x \leqslant 1$$

(cf. Seán Stewart's article in BIMS89). We have

$$\begin{aligned}
&\int_0^1 \frac{\log(1-y)}{y}\, dy = -\operatorname{Li}_2(1), \\
&\int_0^1 \frac{\log(1+y)}{y}\, dy = \int_0^{-1} \frac{\log(1-t)}{t}\, dt = -\operatorname{Li}_2(-1), \\
&\int_0^1 \frac{\log(1+y)}{1+y}\, dy = \int_0^1 \log(1+y)\, d\log(1+y) = \tfrac{1}{2}(\log 2)^2,
\end{aligned}$$

and

$$\begin{aligned}
\int_0^1 \frac{\log(1-y)}{1+y}\, dy &= \int_1^2 \frac{\log(2-x)}{x}\, dx \\
&= \int_1^2 \frac{\log 2 + \log(1 - \frac{x}{2})}{x}\, dx \\
&= (\log 2)^2 + \int_{\frac{1}{2}}^1 \frac{\log(1-t)}{t}\, dt \\
&= (\log 2)^2 - \operatorname{Li}_2(1) + \operatorname{Li}_2(\tfrac{1}{2}).
\end{aligned}$$

Combining these values, we see that the sum of the series is

$$\operatorname{Li}_2(1) + \operatorname{Li}_2(-1) + \tfrac{1}{2}(\log 2)^2 + (\log 2)^2 - \operatorname{Li}_2(1) + \operatorname{Li}_2(\tfrac{1}{2}).$$

Putting in the known values

$$\operatorname{Li}_2(1) = \frac{\pi^2}{6}, \quad \operatorname{Li}_2(-1) = -\frac{\pi^2}{12}, \quad \operatorname{Li}_2(\tfrac{1}{2}) = \frac{\pi^2}{12} - \tfrac{1}{2}(\log 2)^2,$$

we obtain the solution $(\log 2)^2$. □

We invite readers to submit problems and solutions. Please email submissions to `imsproblems@gmail.com` in any format (we prefer LaTeX). Submissions for the summer Bulletin should arrive before the end of April, and submissions for the winter Bulletin should arrive by October. The solution to a problem is published two issues after the issue in which the problem first appeared. Please include solutions to any problems you submit, if you have them.

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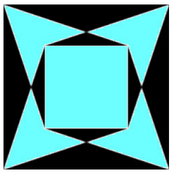

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