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PROBLEMS 89

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Problems

The first problem in this issue was posed by Des MacHale of University College Cork.

Problem 89.1. It is well known that it is possible to dissect a square into a finite number of different squares, but that it is not possible to dissect an equilateral triangle into a finite number of different equilateral triangles. Determine whether it is possible to dissect an isosceles right-angled triangle into a finite number of different isosceles right-angled triangles.

In this problem, 'dissect' means 'partition into two or more pieces', and 'different' means that no two of the shapes considered are congruent.

The second problem was suggested by Toyesh Prakash Sharma, Agra College, India.

Problem 89.2. Prove that

$$\int_{-\pi/2}^{\pi/2} \cos^2(\tan x)\, dx = \frac{\pi}{2}(1+e^{-2}).$$

The third problem comes from Finbarr Holland of University College Cork.

Problem 89.3. Let a_k and b_k be real numbers with $a_k < b_k$, for $k = 1, 2, \ldots, n$, and let

$$r_n(z) = \prod_{k=1}^{n} \frac{b_k + z}{a_k + z}.$$

Prove that

$$\int_{-\infty}^{\infty} \log|r_n(ix)|\, dx = \pi \sum_{k=1}^{n} (b_k - a_k).$$

Solutions

Here are solutions to the problems from *Bulletin* Number 87.

I learned the first problem from a paper by Boris Springborn (*Enseign. Math.* 63, 2017, 333–373). It was solved by Riccardo Della Martera and the North Kildare Mathematics Problem Club, who offered two solutions. I present one of the solutions from the Problem Club.

Problem 87.1. Determine the maximum distance between a straight line intersecting a triangle and the vertices of that triangle.

Solution 87.1. Let our triangle be ABC. Imagine drawing circles of radius r about each of A, B and C. Our question is, how big can r be so that a line L can still be drawn between (and possibly tangential to) these circles.

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If the line does not touch any of the circles, then we have some wiggle room, and r could be enlarged. Similarly, if the line touched only one of the circles, then the line could be moved slightly away from the corresponding vertex and all the circles could be enlarged. Thus, the best line must touch at least two circles. Without loss of generality, suppose they are centred at A and B.

Case 1: If L does not cross the side AB, then as L touches circles of the same radius centered at A and B, the line will be parallel to AB on the same side as C. It is obvious that the radius will be half the perpendicular height of C from AB, in which case L touches all three circles. The largest such possible radius would arise when L is parallel to the shortest side of the triangle.

Case 2: Alternatively, if the line crosses the side AB, then it must also cross another side, say AC. If the angle CAB is obtuse, then we can expand the radius until it is half of $|AB|$. Likewise if CBA is obtuse. If both CAB and CBA are acute, it is possible that the circle at C can get in the way of expanding r to half of $|AB|$. In this case, as we have assumed that L touches the circles centered A and B, it will also touch the circle centered at C, and so we are back in Case 1.

So, the answer is half of the tallest perpendicular height, if the triangle's angles are all acute. If any angle is obtuse, then half the length of a longest adjacent side is the answer. □

The second problem was solved by the proposer, Des MacHale, and by the North Kildare Mathematics Problem Club. We present the solution of the Problem Club.

Des provided a second challenge, to prove that if each pair of elements x and y of a ring satisfies $(x^4 - x)y = y(x^4 - x)$ then the ring is commutative. There is a prize of Des's recent book *The Poetry of George Boole* for the first correct, elementary solution, which has yet to be claimed.

*Problem 86.2***.** Prove that if each element x of a ring satisfies $x^4 + x = 2x^3$ then the ring is commutative.

Solution 87.2. First, replace x by $-x$ to get a second identity. If we add and subtract the second identity from the original, then we obtain the pair of identities

$$2x^4 = 0 \quad \text{and} \quad 2x = 4x^3.$$

Hence

$$2x^3 = 4x^5 = 2x \times 2x^4 = 0,$$

so the original identity (with $-x$ in place of x) becomes $x^4 = x$.

Observe that if $a^2 = 0$, then $a = a^4 = (a^2)^2 = 0$.

Next, let e be an idempotent element of the ring (that is, $e^2 = e$). Observe that

$$(ex - exe)^2 = 0 \quad \text{and} \quad (xe - exe)^2 = 0.$$

By the observation just mentioned, we have $ex = exe = xe$. Hence idempotents belong to the centre C of the ring.

Next, notice that $x + x^2$ is an idempotent element, because

$$(x + x^2)^2 = x^2 + 2x^3 + x^4 = x^2 + 0 + x.$$

Hence $x + x^2 \in C$.

Choose any elements x and y of the ring. Expand $x+y+(x+y)^2$ and subtract $x+x^2$ and $y + y^2$ to see that $xy + yx \in C$. In particular, this element commutes with x, so we deduce that $x^2 \in C$. Since $x + x^2 \in C$, we deduce that $x \in C$, as required. □

The third problem was posed by Finbarr Holland of University College Cork. It was solved by Henry Ricardo of the Westchester Area Math Circle, NY, USA, Seán Stewart of the King Abdullah University of Science and Technology, Saudi Arabia, Riccardo Della Martera, Eugene Gath of the University of Limerick, the North Kildare Mathematics Problem Club, and the proposer. We are spoilt for choice with solutions, all excellent. We opt for that of Henry Ricardo, which was similar to others.

Problem 87.3. Determine the sums of the series

$$\sum_{m,n=1}^{\infty} \frac{1}{mn(m+n+1)} \quad \text{and} \quad \sum_{m,n=1}^{\infty} \frac{(-1)^{m+n}}{mn(m+n+1)}.$$

Solution 87.3. Let S be the sum of the first series. Then

$$\begin{aligned} S &= \sum_{m,n=1}^{\infty} \frac{1}{mn} \int_0^1 x^{m+n}\,dx \\ &= \int_0^1 \left(\sum_{m=1}^{\infty} \frac{x^m}{m}\right)\left(\sum_{n=1}^{\infty} \frac{x^n}{n}\right) dx \\ &= \int_0^1 (\ln(1-x))^2\,dx, \end{aligned}$$

where we have applied Fubini's theorem to interchange sums and integrals. The integral can be evaluated by substituting $x = 1 - e^y$ and then integrating by parts twice. We obtain $S = 2$.

Let T be the sum of the second series. Then

$$\begin{aligned} T &= \sum_{m,n=1}^{\infty} \frac{(-1)^{m+n}}{mn} \int_0^1 x^{m+n}\,dx \\ &= \int_0^1 \left(\sum_{m=1}^{\infty} \frac{(-x)^m}{m}\right)\left(\sum_{n=1}^{\infty} \frac{(-x)^n}{n}\right) dx \\ &= \int_0^1 (\ln(1+x))^2\,dx. \end{aligned}$$

Applying the substitution $x = e^y - 1$ and integrating by parts twice gives $T = 2(\ln 2)^2 - 4\ln 2 + 2$. □

We invite readers to submit problems and solutions. Please email submissions to `imsproblems@gmail.com` in any format (we prefer Latex). Submissions for the summer Bulletin should arrive before the end of April, and submissions for the winter Bulletin should arrive by October. The solution to a problem is published two issues after the issue in which the problem first appeared. Please include solutions to any problems you submit, if you have them.

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