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Some simple proofs of Lima's two-term dilogarithm identity

SEÁN M. STEWART

Abstract. Recently, Lima found a remarkable two-term dilogarithm identity whose proof was based on a hyperbolic form of a proof for the Basel problem given by Beukers, Kolk, and Calabi. A number of simple proofs for this identity that make use of known functional relations for the dilogarithm function are given and an application of Lima's identity to another two-term dilogarithm evaluation is presented.

1. Introduction

The dilogarithm function defined by $\operatorname{Li}_2(x) := \sum_{n=1}^{\infty} x^n/n^2$ and valid for $|x| \leqslant 1$ is a classical function of mathematical physics. Introduced by Leibniz in 1696 [8, p. 351] and thoroughly discussed by Euler some seventy years later [5, pp. 124–126], it has subsequently been well studied in the literature (for further historical details concerning the function see, for example, [12]). The canonical integral representation for the dilogarithm is

$$\operatorname{Li}_2(x) = -\int_0^x \frac{\log(1-t)}{t}\,dt, \quad x \leqslant 1, \tag{1}$$

an integral that cannot be expressed in terms of elementary functions. Only at a handful of values is the dilogarithm known to reduce to simpler constants. These occur for the eight arguments: $0, \frac{1}{2}, \pm 1, -\varphi, \pm\frac{1}{\varphi}$, and $\frac{1}{\varphi^2}$ [9, pp. 4, 6–7]. Here $\varphi := (1+\sqrt{5})/2$ denotes the golden ratio.

Despite the paucity of special values found for the dilogarithm function it satisfies a multitude of functional relations. Some of these functional relations which we will have a need for are [9, p. 6, Eq. (1.15); p. 5, Eq. (1.11); p. 5, Eq. (1.12); p. 4, Eq. (1.7)]:

$$\operatorname{Li}_2(x) + \operatorname{Li}_2(-x) = \frac{1}{2}\operatorname{Li}_2(x^2), \quad -1 \leqslant x \leqslant 1 \quad \text{(duplication formula)} \tag{2}$$

$$\operatorname{Li}_2(x) + \operatorname{Li}_2(1-x) = \frac{\pi^2}{6} - \log(x)\log(1-x), \quad 0 < x < 1 \quad \text{(Euler's reflexion formula)} \tag{3}$$

$$\operatorname{Li}_2(1-x) + \operatorname{Li}_2\left(1 - \frac{1}{x}\right) = -\frac{1}{2}\log^2(x), \quad x > 0 \quad \text{(Landen's identity)} \tag{4}$$

$$\operatorname{Li}_2(-x) + \operatorname{Li}_2\left(-\frac{1}{x}\right) = -\frac{\pi^2}{6} - \frac{1}{2}\log^2(x), \quad 0 < x \leqslant 1 \quad \text{(inversion formula)} \tag{5}$$

Euler's reflexion formula, Landen's identity, and the inversion formula are examples of two-term dilogarithm identities. Replacing x with $1-x$ in Landen's identity results in

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the following alternative form

$$\operatorname{Li}_2(x) + \operatorname{Li}_2\left(\frac{x}{x-1}\right) = -\frac{1}{2}\log^2(1-x), \quad x < 1. \tag{6}$$

while substituting $x = \frac{1}{2}$ into Euler's reflexion formula leads to the special value

$$\operatorname{Li}_2\left(\frac{1}{2}\right) = \frac{\pi^2}{12} - \frac{1}{2}\log^2(2). \tag{7}$$

Many other functional relations for the dilogarithm can be found. Here the reader is encouraged to consult the works of Kirillov [7] and Gordon and McIntosh [6].

The dilogarithm function today can be found in a wide variety of applications ranging from algebraic K-theory [13], Euler sums [16, 14], to conformal field theory [3]. For those unacquainted with the function it is best summed up in the words of Don Zagier who writes [17, p. 6]:

> ... the dilogarithm is one of the simplest non-elementary functions one can imagine. It is also one of the strangest. It occurs not quite often enough, and in not quite an important enough way, to be included in the Valhalla of the great transcendental functions ... [A]nd yet it occurs too often, and in far too varied contexts, to be dismissed as a mere curiosity. ... Almost all of its appearances in mathematics, and almost all the formulas relating to it, have something of the fantastical in them, as if this function alone among all others possessed a sense of humor.

New results found for the function therefore remain important. One such result was recently given by Lima who gave the remarkable two-term dilogarithm identity [10, Eq. (11)]

$$\operatorname{Li}_2\left(\sqrt{2}-1\right) - \operatorname{Li}_2\left(1-\sqrt{2}\right) = \frac{\pi^2}{8} - \frac{1}{2}\log^2\left(\sqrt{2}+1\right). \tag{8}$$

It was obtained by evaluating an integral that stemmed from a double integral used in a proof for the Basel problem given by Beukers, Kolk, and Calabi [2] where a non-trivial trigonometric change of variables is used, except with the trigonometric change of variables changed to its analogous hyperbolic form. What makes Lima's identity so interesting is that it is thought to not follow trivially from any previously known two-term dilogarithm identities [4].

Recently Campbell gave a new proof for Lima's identity using a series transformation obtained via Legendre polynomial expansions [4]. In this note we give three separate simple proofs for this same result. The first follows from the three functional relations (2) to (4), the second from a four-term dilogarithm functional relation, while the third from the evaluation of a definite integral in two different ways. As one application of Lima's identity, we will use it to show that

$$\operatorname{Li}_2\left(-\sqrt{2}\right) + \operatorname{Li}_2\left(-1-\sqrt{2}\right) = -\frac{5\pi^2}{24} - \frac{1}{2}\log\left(1+\sqrt{2}\right)\log\left(2+2\sqrt{2}\right). \tag{9}$$

Other non-trivial two-term dilogarithm identities due to Ramanujan can be found listed in [1, pp. 324–325] and still others are given by Loxton in [11]. Here by non-trivial we mean those two-term dilogarithm identities that do not directly follow on substituting for some value of x into one of the two-term functional relations for the dilogarithm function.

2. Simple proofs of Lima's identity using functional relations

The two proofs we give here for Lima's identity make use of various functional relations for the dilogarithm function.

2.1. **Using Landen's identity, Euler's reflexion formula, and the duplication formula.** For the first of the proofs we give for Lima's identity we proceed by employing Landen's identity. For the first dilogarithm term appearing in (8) we have

$$\begin{aligned}\mathrm{Li}_2\left(\sqrt{2}-1\right) &= \mathrm{Li}_2\left(1-(2-\sqrt{2})\right) = -\mathrm{Li}_2\left(1-\frac{1}{2-\sqrt{2}}\right) - \frac{1}{2}\log^2\left(2-\sqrt{2}\right)\\ &= -\mathrm{Li}_2\left(-\frac{1}{\sqrt{2}}\right) - \frac{1}{2}\log^2\left(\frac{2}{2+\sqrt{2}}\right)\\ &= -\mathrm{Li}_2\left(-\frac{1}{\sqrt{2}}\right) - \frac{1}{2}\left(\log(2)-\log(2+\sqrt{2})\right)^2.\end{aligned}$$

Noting that $\log(2+\sqrt{2}) = \frac{1}{2}\log(2)+\log(1+\sqrt{2})$, then

$$\begin{aligned}\mathrm{Li}_2\left(\sqrt{2}-1\right) &= -\mathrm{Li}_2\left(-\frac{1}{\sqrt{2}}\right) - \frac{1}{2}\left(\frac{1}{2}\log(2)-\log(1+\sqrt{2})\right)^2\\ &= -\mathrm{Li}_2\left(-\frac{1}{\sqrt{2}}\right) - \frac{1}{8}\log^2(2) + \frac{1}{2}\log(2)\log\left(1+\sqrt{2}\right) - \frac{1}{2}\log^2\left(1+\sqrt{2}\right).\end{aligned} \tag{10}$$

And for the second dilogarithm term appearing in (8), applying Landen's identity followed by Euler's reflexion formula one obtains

$$\begin{aligned}\mathrm{Li}_2\left(1-\sqrt{2}\right) &= -\mathrm{Li}_2\left(1-\frac{1}{\sqrt{2}}\right) - \frac{1}{2}\log^2\left(\sqrt{2}\right)\\ &= -\left[\frac{\pi^2}{6} - \log\left(\frac{1}{\sqrt{2}}\right)\log\left(1-\frac{1}{\sqrt{2}}\right) - \mathrm{Li}_2\left(\frac{1}{\sqrt{2}}\right)\right] - \frac{1}{8}\log^2(2)\\ &= \mathrm{Li}_2\left(\frac{1}{\sqrt{2}}\right) - \frac{\pi^2}{6} + \frac{1}{2}\log(2)\log(2+\sqrt{2}) - \frac{1}{8}\log^2(2)\\ &= \mathrm{Li}_2\left(\frac{1}{\sqrt{2}}\right) - \frac{\pi^2}{6} + \frac{1}{2}\log(2)\log(1+\sqrt{2}) + \frac{1}{8}\log^2(2),\end{aligned} \tag{11}$$

where again the result $\log(2+\sqrt{2}) = \frac{1}{2}\log(2)+\log(1+\sqrt{2})$ has been used. Taking the difference between (10) and (11) we see that

$$\begin{aligned}\mathrm{Li}_2\left(\sqrt{2}-1\right) - \mathrm{Li}_2\left(1-\sqrt{2}\right) &= \frac{\pi^2}{6} - \frac{1}{2}\log^2(1+\sqrt{2}) - \frac{1}{4}\log^2(2)\\ &\quad - \left[\mathrm{Li}_2\left(\frac{1}{\sqrt{2}}\right) + \mathrm{Li}_2\left(-\frac{1}{\sqrt{2}}\right)\right].\end{aligned} \tag{12}$$

A value for the dilogarithm term appearing within the square brackets on the right of the equality in (12) can be found from the duplication formula. Setting $x = \frac{1}{\sqrt{2}}$ in (2) we see that

$$\mathrm{Li}_2\left(\frac{1}{\sqrt{2}}\right) + \mathrm{Li}_2\left(-\frac{1}{\sqrt{2}}\right) = \frac{1}{2}\mathrm{Li}_2\left(\frac{1}{2}\right) = \frac{\pi^2}{24} - \frac{1}{4}\log^2(2),$$

where the result given in (7) has been used. Thus (12) reduces to (8) and completes the first of our proofs for Lima's identity.

2.2. **Using a four-term dilogarithm functional relation.** We first give a four-term functional relation involving dilogarithms.

Theorem 2.1. *For $-1 \leqslant x \leqslant 1$ the following four-term functional relation involving dilogarithms holds:*

$$\mathrm{Li}_2\left(\frac{1-x}{1+x}\right) - \mathrm{Li}_2\left(-\frac{1-x}{1+x}\right) = \frac{\pi^2}{4} + \mathrm{Li}_2(-x) - \mathrm{Li}_2(x) + \log(x)\log\left(\frac{1+x}{1-x}\right). \tag{13}$$

Proof. In view of (1) it is immediate that $\frac{d}{dx}\operatorname{Li}_2(x) = -\log(1-x)/x$. Consider

$$\begin{aligned}\frac{d}{dx}\left[\operatorname{Li}_2\left(\frac{1-x}{1+x}\right) - \operatorname{Li}_2\left(-\frac{1-x}{1+x}\right)\right] &= \frac{2}{1-x^2}\log\left(\frac{2x}{1+x}\right) - \frac{2}{1-x^2}\log\left(\frac{2}{1+x}\right)\\ &= \frac{2}{1-x^2}\log(x).\end{aligned}$$

Integrating the above expression with respect to x gives

$$\operatorname{Li}_2\left(\frac{1-x}{1+x}\right) - \operatorname{Li}_2\left(-\frac{1-x}{1+x}\right) = 2\int\frac{\log(x)}{1-x^2}\,dx = \int\frac{\log(x)}{1-x}\,dx + \int\frac{\log(x)}{1+x}\,dx + C, \quad (14)$$

after a partial fraction decomposition has been employed. Here C is an arbitrary constant of integration. Making the change of variable of $t \mapsto 1-t$ in (1) we see that the first integral appearing in (14) is

$$\int\frac{\log(t)}{1-t}\,dt = \operatorname{Li}_2(1-t), \quad (15)$$

where, for convenience, we have dropped the arbitrary constant of integration. For the second integral appearing in (14), integrating by parts followed by a change of variable of $t \mapsto -t$ leads to

$$\int\frac{\log(t)}{1+t}\,dt = \log(t)\log(1+t) + \operatorname{Li}_2(-t), \quad (16)$$

where once more for convenience the arbitrary constant of integration has been dropped. Thus (14) becomes

$$\operatorname{Li}_2\left(\frac{1-x}{1+x}\right) - \operatorname{Li}_2\left(-\frac{1-x}{1+x}\right) = \operatorname{Li}_2(1-x) + \operatorname{Li}_2(-x) + \log(x)\log(1+x) + C. \quad (17)$$

To find the constant C, we set $x = 0$. Doing so we find

$$C = -\operatorname{Li}_2(-1) = \frac{\pi^2}{12}.$$

Here the value for $\operatorname{Li}_2(-1)$ is found on setting $x = 1$ in the inversion formula of (5). Substituting the value found for C into (17), after applying Euler's reflexion formula to the term $\operatorname{Li}_2(1-x)$ the desired result then follows. □

Remark 2.2. The identity given by (13) is not new. It is listed, for example, online at The Wolfram Functions Site [15].

If one sets $x = \sqrt{2}-1$ in (13), as

$$\frac{1-x}{1+x} = x = \sqrt{2}-1,$$

one finds

$$\operatorname{Li}_2\left(\sqrt{2}-1\right) - \operatorname{Li}_2\left(1-\sqrt{2}\right) = \frac{\pi^2}{4} + \operatorname{Li}_2\left(1-\sqrt{2}\right) - \operatorname{Li}_2\left(\sqrt{2}-1\right) - \log^2\left(\sqrt{2}-1\right),$$

or

$$\operatorname{Li}_2\left(\sqrt{2}-1\right) - \operatorname{Li}_2\left(1-\sqrt{2}\right) = \frac{\pi^2}{8} - \frac{1}{2}\log^2\left(\sqrt{2}-1\right) = \frac{\pi^2}{8} - \frac{1}{2}\log^2\left(1+\sqrt{2}\right),$$

since $\log(\sqrt{2}-1) = -\log(1+\sqrt{2})$ and completes the second of our proofs for Lima's identity.

3. An application

As an application we now give a two-term dilogarithm identity that makes use of Lima's identity. This is identity (9). To prove this result, setting $x = -\sqrt{2}$ in identity (6) yields

$$\operatorname{Li}_2(-\sqrt{2}) = -\operatorname{Li}_2(2-\sqrt{2}) - \frac{1}{2}\log^2(1+\sqrt{2}). \tag{18}$$

Applying Euler's reflexion formula to the $\operatorname{Li}_2(2-\sqrt{2})$ term produces

$$\begin{aligned}\operatorname{Li}_2(2-\sqrt{2}) = \operatorname{Li}_2\Big(1-(\sqrt{2}-1)\Big) &= \frac{\pi^2}{6} - \operatorname{Li}_2(\sqrt{2}-1) - \log(\sqrt{2}-1)\log(2-\sqrt{2})\\ &= \frac{\pi^2}{6} - \operatorname{Li}_2(\sqrt{2}-1) + \frac{1}{2}\log(2)\log(1+\sqrt{2}) - \log^2(1+\sqrt{2}),\end{aligned}$$

since $\log(2-\sqrt{2}) = \frac{1}{2}\log(2) + \log(\sqrt{2}-1)$ and $\log(\sqrt{2}-1) = -\log(1+\sqrt{2})$. Thus (18) becomes

$$\operatorname{Li}_2(-\sqrt{2}) = -\frac{\pi^2}{6} - \frac{1}{2}\log(2)\log(1+\sqrt{2}) + \frac{1}{2}\log^2(1+\sqrt{2}) + \operatorname{Li}_2(\sqrt{2}-1). \tag{19}$$

Next, setting $x = 1+\sqrt{2}$ in the inversion formula yields

$$\operatorname{Li}_2(-1-\sqrt{2}) = -\operatorname{Li}_2(1-\sqrt{2}) - \frac{\pi^2}{6} - \frac{1}{2}\log^2(1+\sqrt{2}). \tag{20}$$

Adding (19) and (20) gives

$$\operatorname{Li}_2(-\sqrt{2}) + \operatorname{Li}_2(-1-\sqrt{2}) = \operatorname{Li}_2(\sqrt{2}-1) - \operatorname{Li}_2(1-\sqrt{2}) - \frac{\pi^2}{3} - \frac{1}{2}\log(2)\log(1+\sqrt{2}). \tag{21}$$

On substituting Lima's identity into (21) the two-term dilogarithm identity given in (9) immediately follows.

While the result given in (21) is interesting in its own right, it is important for another reason. If a method that is independent of Lima's identity can be found which gives the value for the dilogarithm sum appearing to the left of the equality in (21), it will give a third proof for Lima's identity. This will now be shown using a definite integral that is evaluated in two different ways.

The definite integral we consider is

$$J = \int_0^1 \frac{\operatorname{arcsinh}(x)}{x\sqrt{1+x^2}}\,dx.$$

Substituting $x = \sinh(t)$ followed by substituting $t = \log(u)$ we find

$$J = \int_0^{\log(1+\sqrt{2})} \frac{t}{\sinh(t)}\,dt = 2\int_1^{1+\sqrt{2}} \frac{\log(u)}{u^2-1}\,du,$$

or

$$J = -\int_1^{1+\sqrt{2}} \frac{\log(u)}{1-u}\,du - \int_1^{1+\sqrt{2}} \frac{\log(u)}{1+u}\,du,$$

after a partial fraction decomposition has been made. The first of the integrals to the right of the equality is (15), the second is (16). Thus

$$\begin{aligned}J &= -\operatorname{Li}_2(1-u)\Big|_1^{1+\sqrt{2}} - \Big[\log(u)\log(1+u) + \operatorname{Li}_2(-u)\Big]_1^{1+\sqrt{2}}\\ &= -\frac{\pi^2}{12} - \frac{1}{2}\log(2)\log(1+\sqrt{2}) - \log^2(1+\sqrt{2}) - \operatorname{Li}_2(-\sqrt{2}) - \operatorname{Li}_2(-1-\sqrt{2}).\end{aligned}$$

Evaluating the definite integral for J a second time but in a different way, noting that for $x > 0$

$$\int \frac{dx}{x\sqrt{1+x^2}} = -\operatorname{arccoth}(\sqrt{1+x^2}) = -\operatorname{arctanh}\left(\frac{1}{\sqrt{1+x^2}}\right) = -\operatorname{arcsinh}\left(\frac{1}{x}\right),$$

where, for convenience, the various arbitrary constants of integration have been dropped, integrating by parts we have

$$J = -\operatorname{arcsinh}(x)\operatorname{arcsinh}\left(\frac{1}{x}\right)\Big|_0^1 + \int_0^1 \frac{\operatorname{arcsinh}\left(\frac{1}{x}\right)}{\sqrt{1+x^2}}\,dx = -\log^2(1+\sqrt{2}) + \int_0^1 \frac{\operatorname{arcsinh}\left(\frac{1}{x}\right)}{\sqrt{1+x^2}}\,dx.$$

Here the result $\operatorname{arcsinh}(1) = \log(1+\sqrt{2})$ has been used. Enforcing a substitution of $x \mapsto \frac{1}{x}$ produces

$$J = -\log^2(1+\sqrt{2}) + \int_1^\infty \frac{\operatorname{arcsinh}(x)}{x\sqrt{1+x^2}}\,dx = -\log^2(1+\sqrt{2}) + \int_0^\infty \frac{\operatorname{arcsinh}(x)}{x\sqrt{1+x^2}}\,dx - J,$$

or

$$J = -\frac{1}{2}\log^2(1+\sqrt{2}) + \frac{1}{2}\int_0^\infty \frac{\operatorname{arcsinh}(x)}{x\sqrt{1+x^2}}\,dx = -\frac{1}{2}\log^2(1+\sqrt{2}) + I.$$

A value for the remaining integral I can be readily found. Substituting $x = \sinh(u)$ gives

$$I = \frac{1}{2}\int_0^\infty \frac{u}{\sinh(u)}\,du = \int_0^\infty \frac{ue^{-u}}{1-e^{-2u}}\,du,$$

where the definition for the hyperbolic sine function in terms of exponentials has been used. Expanding the denominator as an infinite geometric series one has

$$I = \sum_{n=0}^\infty \int_0^\infty ue^{-(2n+1)u}\,du.$$

The interchange that has been made here between the integration sign and the summation is permissible due to the posititivity of all terms involved. Integrating by parts we find

$$I = \sum_{n=0}^\infty \frac{1}{(2n+1)^2} = \sum_{n=1}^\infty \frac{1}{n^2} - \sum_{n=1}^\infty \frac{1}{(2n)^2} = \left(1-\frac{1}{4}\right)\sum_{n=1}^\infty \frac{1}{n^2} = \frac{3}{4}\cdot\frac{\pi^2}{6} = \frac{\pi^2}{8}.$$

Here the well-known result for the Basel problem of $\sum_{n=1}^\infty \frac{1}{n^2} = \frac{\pi^2}{6}$ has been used. The absolute convergence of the series allows for the rearrangement of its terms.

Returning to the integral J, we find

$$J = \frac{\pi^2}{8} - \frac{1}{2}\log^2(1+\sqrt{2}).$$

Equating the two values found for J leads to the result given in (9), which when substituted into (21) leads to Lima's identity, thereby providing our third proof for this remarkable result.

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Seán M. Stewart comes from Australia. For many years he taught mathematics to engineers in Kazakhstan and the United Arab Emirates. He now lectures at King Abdullah University of Science and Technology in Saudi Arabia. He has always found it hard to resist the challenge of a definite integral and is the author of *How to integrate it: A practical guide to finding elementary integrals* published by Cambridge University Press.

Physical Sciences and Engineering Division, King Abdullah University of Science and Technology, Thuwal 23955-6900, Saudi Arabia.
E-mail address: `sean.stewart@physics.org`