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This volume consists of nineteen chapters discussing different branches of geometry from a historical perspective. The intended audience is that of the general mathematical community with an interest in how certain geometrical ideas developed and evolved over time. It is important to note that each chapter is written by practising experts in the field as opposed to historians trying to capture developments in geometry from a layperson's perspective. In many of the essays, it is striking to see the impact that philosophy had on the development of mathematical ideas down through the ages.

As might be expected, the range of writing styles is diverse and this tends to undermine the overall coherency of the text. The chapter lengths vary from just ten pages to a hefty ninety-two pages which contributes to a disjointed effect for the reader. Despite these aesthetical concerns, there is an effort to create a logical progression between the essays in terms of content. The book has two separate sections, the old and the new. The first (spanning seven chapters) examines topics that have roots in Greek antiquity and the second (consisting of the remaining chapters) concerns itself with more modern material.

The first essay looks at Plato's theory of anthyphairesis, a topic from the fourth century BCE, which has its modern counterpart in the theory of continued fractions. Plato believed that mathematics was a means to gaining a better understanding of reality, he was convinced that geometry was the key to unlocking the secrets of the universe. The author contends that Plato's theory of knowledge of Forms is built upon the concept of periodic anthyphairesis. Plato was critical of the axiomatic method, believing that it led to an overreliance on hypotheses divorced from true knowledge. He argued for the sole use of Division and Collection, a philosophical version of the periodic anthyphairesis, to acquire any knowledge in geometry.

The second chapter looks at the work of the topologist Ren Thom's reevaluation of Aristotle's writings on science, in particular biology. Thom identified that much of Aristotle's assertions have a definite topological content, even if they were written over two thousand years before the field of topology was formally born. The author contends that Thom succeeded in providing a link between modern mathematics and science in Greek antiquity.

The ideas of Thomas Kuhn, concerning paradigm shifts, are considered in the third chapter. A paradigm shift is a fundamental change in the basic concepts and experimental practices of a scientific discipline. The author believes that this is not applicable in mathematics, arguing that mathematical thought evolves in a continuous manner rather than an abrupt change. To illustrate this belief, the reader is given an account of the development of one mathematical thought over time. It begins with Ptolemy's dynamical model of the solar system, used to explain the variations in speed and direction of the apparent motion of the Sun, Moon and planets. It then explains how his

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thoughts in this area were considered by the likes of Fourier and Mendeleev before finishing with Schrodinger's work on quantum mechanics, described as 'a complexification of Ptolemy's epicycles'.

Convexity, in the mathematical writings of Greek antiquity, is considered in chapter four. Aristotle, Euclid, Apollonius, Archimedes and more are shown to have explored the area of convexity in their mathematical work before it became recognised formally in the twentieth century. It is possible to trace Aristotle's ideas around the topic of convexity to the more formal work of the likes of Minkowski and Caratheodory. This idea supports the contention of chapter three that mathematical ideas evolve over time rather than being the product of a paradigm shift.

In chapter five, the author considers the relationship between mathematics and art. Modernism in mathematics is described here as the algebraization of spatiality while maintaining geometrical and topological thinking. It looks at the work of Fermat and Descartes, where the notion of algebraization of geometry became necessary. This is developed further using the 'irrationality' of quantum mechanics as an example and how Heisenberg's work led to a 'rationalization'. This is linked to the problematic nature of the incommensurable that was considered in ancient Greek mathematics, before concluding with a consideration of John Tate's principle of 'Think geometrically, prove algebraically'.

The sixth chapter considers the evolution of the ancient Greek theory of curves up to the synthetic differential geometry of today. It looks at the work of several major mathematicians along this journey: Huygens on evolutes, Euclid on curvature of surfaces up to Busemann, Feller and Alexandrov on Carnot groups. The contribution of each mathematician is considered in great detail in an effort to show how the idea was shaped and developed over time.

In chapter seven, geometry is considered a means for describing the shape of the universe. The author explores the development of this idea from ancient times to the modern day. It looks at the areas of cosmology and the philosophy of space and time. Topology, set theory, differential and projective geometry are all employed to explore concepts like infinity, infinitesimal and curvature. This marks the end of the first section that considered geometry's origins in ancient times.

Chapters seven and eight focus on configuration theorems, these are theorems within projective geometry whose statements involve finite sets of points and arrangements of lines. Chapter eight looks at the importance of Pappus' and Desargues' configuration theorems, dating back to the fourth and seventeenth centuries respectively. The authors view the theorems as a bridge between geometry and algebra. These theorems did not gain recognition until the twentieth century when the area received increased consideration. Chapter nine explores the impact that configuration theorems have had and the many results that they have yielded in modern dynamics. The chapter closes with an examination of Richard Schwartz's work on the pentagram map and the theory of skewers.

The essay in chapter ten looks at the work of Henri Poincar in the area of topology. Poincar's philosophy is very much that of a scientist originating in his own daily practice of science and the scientific debates of his time. The author shows that he was also strongly influenced by contemporary philosophical doctrines, such as Kant and Althusser. Consideration is given to the influence of several philosophers such as Frege, Husserl and Russell in the area of geometry.

Chapter eleven looks at the applications of the study of the dynamics of the iterates of a map found by perturbing the germ at the origin of a planar rotation. The author explains how such work led to the development of the Andronov-Hopf-Neimark-Sacker bifurcation theory, Kolmogorov-Arnold-Moser theory and Poincar's theory of normal forms.

The next chapter looks at Gromov's h-principle, which gives conditions where a manifold carrying a geometric structure in a weak sense carries a genuine geometric structure. Other work relating to this h-principle completed by Thom, Smale, Eliashberg, Mishachev and Thurston is also discussed.

Chapter thirteen considers flexibility and rigidity phenomena in symplectic geometry. The chapter begins with an outline of Poincar's desire for the development of this branch of geometry. This came to pass years after his death when symplectic topology emerged through the work of Arnold and Gromov. The chapter concludes with an account of the most recent findings in this area.

A historical survey on the theory of locally homogenous geometric structures is presented in chapter fourteen. The theory is traced back to Charles Ehresmann in the early twentieth century. Further work led to the Ehresmann-Weil-Thurston holonomy principle, which identifies a relation between the classification of geometric structures on a manifold and the representation of its fundamental group into a Lie group. The rest of the chapter looks at discrete subgroups of Lie groups.

Chapter fifteen looks again at the work of Ehresmann, but this time in relation to the development of differential geometry. An Ehresmann connection and its importance for fibre bundles is explained. His work on jet bundles is also accounted for.

Finsler and Riemannian geometries are compared in chapter sixteen. The emphasis is on the asymmetry of the distance function associated with a Finsler manifold. References to several mathematicians working in this area are made.

The next two chapters look at the topology of 3- and 4- manifolds. Chapter seventeen looks at the work completed around the three-dimensional Poincar conjecture. It took over a hundred years before the first valid proof emerged. A lot of mathematics was generated in the search for such a proof. Chapter eighteen is much broader, describing the important problems surrounding four-dimensional manifolds and the work of many mathematicians. It also accounts for the proof of the Poincar conjecture in higher dimensions.

The final chapter provides an interesting contrast to the preceding essays. It is an autobiographical account by Valentin Ponaru concentrating on the period where he decided to become a mathematician. He describes the problems that he worked on and those that he corresponded with at the time. It gives a first-hand account of what life was like for a mathematician behind the iron curtain.

This book allows the reader to grasp just how large the area of geometry is. It is apparent that not all topics within geometry were covered, most likely due to a lack of suitable authors. The major contributors like Poincar, Ehresmann, Thom, Thurston and Gromov are referenced repeatedly. Most authors succeed in linking the mathematical work of today with its origins in the past. Due to the contrasting writing styles and varying chapter lengths, this book is not an easy read. This volume would be a useful reference text in a library, where readers could use it to research the historical origins of certain geometrical topics and see how they developed over time.

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