Irish Math. Soc. Bulletin
Number 87, Summer 2021, 35–43
ISSN 0791-5578

Weighted Sylvester sums on the Frobenius set

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Abstract. Let a and b be relatively prime positive integers. In this paper the weighted sum $\sum_{n\in\mathrm{NR}(a,b)}\lambda^{n-1}n^m$ is given explicitly or in terms of the Apostol-Bernoulli numbers, where m is a nonnegative integer, and $\mathrm{NR}(a,b)$ denotes the set of positive integers nonrepresentable in terms of a and b.

1. Introduction

The *Frobenius Problem* is to determine the largest positive integer that is NOT representable as a nonnegative integer combination of given positive integers that are coprime (see [13] for general references).

Given positive integers $a_1,\dots,a_m$ with $\gcd(a_1,\dots,a_m)=1$, it is well-known that for all sufficiently large n the equation

$$a_1x_1+\cdots+a_mx_m=n \tag{1}$$

has a solution with nonnegative integers $x_1,\dots,x_m$.

The *Frobenius number* $F(a_1,\dots,a_m)$ is the LARGEST integer n such that (1) has no solution in nonnegative integers. For $m=2$, we have

$$F(a,b)=(a-1)(b-1)-1$$

(Sylvester (1884) [17]). For $m\ge 3$, exact determination of the Frobenius number is difficult. The Frobenius number cannot be given by closed formulas of a certain type (Curtis (1990) [6]), the problem of determining $F(a_1,\dots,a_m)$ is NP-hard under Turing reduction (see, e.g., Ramírez Alfonsín [13]). Nevertheless, the Frobenius numbers for some special cases are calculated (e.g., [12, 14, 16]). One convenient formula is by Johnson [9]. An analytic approach to the Frobenius number can be seen in [4, 10]. Some formulae for the Frobenius number in three variables can be seen in [19].

For given a and b with $\gcd(a,b)=1$, let $\mathrm{NR}(a,b)$ denote the set of nonnegative integers nonrepresentable in term of a and b, namely the set of all those nonnegative integers n which cannot be expressed in the form $n=ax+by$, where x and y are nonnegative integers.

There are many kinds of problem related to the Frobenius problem. The problems of the number of solutions (e.g., [18]), and the sum of integer powers of the gaps values in numerical semigroups (e.g., [5, 8, 7]) are popular. Another famous problems is about the so-called *Sylvester sums* $\sum_{n\in\mathrm{NR}(a,b)}n^m$, where m is a nonnegative integer (see, e.g., [20] and references therein). Recently in [3], a more general case is considered, involving the largest integer, the number of integers and the sum of integers whose number of representation is exactly equal to a given number k, and is tackled using similar power sums.

2020 *Mathematics Subject Classification.* 05A15, 11D07, 11B68, 11P81, 05A17.

Key words and phrases. Frobenius problem, weighted sums, Sylvester sums, Apostol-Bernoulli numbers.

Received on 28-2-2021; revised 13-5-2021.

In this paper, we consider the weighted sum

$$S_m^{(\lambda)}(a,b) := \sum_{n\in\mathrm{NR}(a,b)} \lambda^{n-1}n^m \quad (\lambda\neq 0)\,.$$

Sylvester [17] showed that $S_0^{(1)}(a,b)=(a-1)(b-1)/2$, and Brown and Shuie showed [5] that

$$S_1^{(1)}(a,b)=\frac{1}{12}(a-1)(b-1)(2ab-a-b-1)\,.$$

Rødseth [15] obtained a general formula for $S_m^{(1)}$ in terms of Bernoulli numbers and deduced

$$S_2^{(1)}(a,b)=\frac{1}{12}(a-1)(b-1)ab(ab-a-b)\,.$$

Tuenter [20] also investigated $S_m^{(1)}$ by taking a different approach. He established relations between Sylvester sums and the power sums over the natural numbers. Wang and Wang [21] considered the alternating Sylvester sums

$$T_m(a,b)=\sum_{n\in\mathrm{NR}(a,b)}(-1)^n n^m$$

by using Bernoulli and Euler numbers.

The purpose of this paper is to give an explicit expression for $S_m^{(\lambda)}(a,b)$. For $m=1$, we can give the following formula.

Theorem 1.1. *For $\lambda\neq 0$ with $\lambda^a\neq 1$ and $\lambda^b\neq 1$,*

$$S_1^{(\lambda)}(a,b)=\frac{1}{(\lambda-1)^2}+\frac{ab\lambda^{ab-1}}{(\lambda^a-1)(\lambda^b-1)}-\frac{(\lambda^{ab}-1)\big((a+b)\lambda^{a+b}-a\lambda^a-b\lambda^b\big)}{\lambda(\lambda^a-1)^2(\lambda^b-1)^2}\,.$$

We also give a general expression for $S_m^{(\lambda)}(a,b)$ in terms of the Apostol-Bernoulli numbers. The alternating Sylvester sums in [21] can be also expressed as $T_m(a,b)=-S_m^{(-1)}(a,b)$.

The main new results (Theorems 4.1 and 4.3 below) cover all values of m and λ, and express $S_m^{(\lambda}(a,b)$ in terms of the Apostol-Bernoulli numbers. In case $m=1$ and $\lambda^a\neq 1$ the expressions reduce to those given explicitly in Theorem 1.1.

2. An explicit expression for $m=1$

As in [5], define

$$f(x)=\sum_{n=0}^{ab-a-b}\big(1-r(n)\big)x^n\,,$$

where $r(n)$ denotes the number of representations of n in the form $n=sa+tb$, where s and t are nonnegative integers. Since $r(n)=0$ or 1 for $0\le n\le ab-1$, we have

$$\begin{aligned} f'(\lambda) &= \sum_{n=1}^{ab-a-b} n\big(1-r(n)\big)\lambda^{n-1} = \sum_{\substack{1\le n\le ab-a-b\\ r(n)=0}} n\lambda^{n-1}\\ &= \sum_{n\in\mathrm{NR}(a,b)}\lambda^{n-1}n = S_1^{(\lambda)}(a,b)\,. \end{aligned}$$

We use the following fact from [5].

Lemma 2.1.

$$f(x) = \frac{g(x)}{h(x)}\,,$$

where

$$g(x) = \sum_{k=1}^{b-1} \frac{x^{ak} - x^k}{x - 1} \quad \textit{and} \quad h(x) = \sum_{k=0}^{b-1} x^k\,.$$

Suppose that $\lambda \neq 1 \neq \lambda^a$. Then

$$h(\lambda) = \frac{\lambda^b - 1}{\lambda - 1}$$

and

$$h'(\lambda) = \sum_{k=0}^{b-1} k\lambda^{k-1} = \frac{b\lambda^{b-1}}{\lambda - 1} - \frac{\lambda^b - 1}{(\lambda - 1)^2}\,.$$

Also, we have

$$g(\lambda) = \frac{(\lambda^{ab} - 1)(\lambda - 1) - (\lambda^a - 1)(\lambda^b - 1)}{(\lambda^a - 1)(\lambda - 1)^2}$$

and

$$\begin{aligned} g'(\lambda) = {} & \frac{(ab+1)\lambda^{ab} - ab\lambda^{ab-1} - (a+b)\lambda^{a+b+1} + a\lambda^{a-1} + b\lambda^{b-1} - 1}{(\lambda^a - 1)(\lambda - 1)^2} \\ & - \frac{a\lambda^{a-1}}{\lambda^a - 1} g(\lambda) - \frac{2}{\lambda - 1} g(\lambda)\,. \end{aligned}$$

Hence, we finally get

$$\begin{aligned} S_1^{(\lambda)}(a,b) = f'(\lambda) &= \frac{g'(\lambda)h(\lambda) - g(\lambda)h'(\lambda)}{\bigl(h(\lambda)\bigr)^2} \\ &= \frac{1}{(\lambda - 1)^2} + \frac{ab\lambda^{ab-1}}{(\lambda^a - 1)(\lambda^b - 1)} - \frac{(\lambda^{ab} - 1)\bigl((a+b)\lambda^{a+b} - a\lambda^a - b\lambda^b\bigr)}{\lambda(\lambda^a - 1)^2(\lambda^b - 1)^2}\,. \end{aligned}$$

In particular, for $\lambda = 2$, we have the following.

Corollary 2.2.

$$\begin{aligned} \sum_{n \in \mathrm{NR}(a,b)} 2^{n-1} n = 1 &+ \frac{ab2^{ab-1}}{(2^a - 1)(2^b - 1)} \\ &- \frac{(2^{ab} - 1)\bigl((a+b)2^{a+b} - 2^a a - 2^b b\bigr)}{2(2^a - 1)^2(2^b - 1)^2}\,. \end{aligned}$$

For example, for $a = 3$ and $b = 17$,

$$\begin{aligned} S_1^{(2)}(3,17) &= 2^0 \cdot 1 + 2^1 \cdot 2 + 2^3 \cdot 4 + 2^4 \cdot 5 + 2^6 \cdot 7 + 2^7 \cdot 8 + 2^9 \cdot 10 \\ &\quad + 2^{10} \cdot 11 + 2^{12} \cdot 13 + 2^{13} \cdot 14 + 2^{15} \cdot 16 + 2^{18} \cdot 19 + 2^{21} \cdot 22 \\ &\quad + 2^{24} \cdot 25 + 2^{27} \cdot 28 + 2^{30} \cdot 31 \\ &= 37515351605\,. \end{aligned}$$

From Theorem 1.1 (or the above Corollary),

$$\begin{aligned} S_1^{(2)}(3,17) = {} & \frac{1}{(2-1)^2} + \frac{3 \cdot 17 \cdot 2^{3 \cdot 17 - 1}}{(2^3 - 1)(2^{17} - 1)} \\ & - \frac{(2^{3 \cdot 17} - 1)\bigl((3+17)2^{3+17} - 3 \cdot 2^3 - 17 \cdot 2^{17}\bigr)}{2(2^3 - 1)^2(2^{17} - 1)^2} \end{aligned}$$

$$= 37515351605\,.$$

Similarly, by replacing 2 by another value, we can obtain that

$$\begin{aligned} S_1^{(5)}(3,17) &= 900879734470832437423896\,,\\ S_1^{(1/2)}(3,17) &= \frac{8822132865}{1073741824}\,,\\ S_1^{(-1)}(3,17) &= 408\,,\\ S_1^{(-5/3)}(3,17) &= \frac{7605085294789029411119864}{205891132094649}\,,\\ S_1^{(\pm\sqrt{2})}(3,17) &= 34250061 \pm 6965604\sqrt{2}\,. \end{aligned}$$

3. Weighted sums of higher power

Since

$$\begin{aligned} f''(x) &= \frac{g''(x)}{h(x)} - \frac{2g'(x)h'(x) + h(x)''(x)}{\bigl(h(x)\bigr)^2} + \frac{2g(x)\bigl(h'(x)\bigr)^2}{\bigl(h(x)\bigr)^3}\\ &= \sum_{n=2}^{ab-a-b} n(n-1)\bigl(1-r(n)\bigr)x^{n-2}\,, \end{aligned}$$

we get

$$xf''(x) + f'(x) = \sum_{n=0}^{ab-a-b} n^2\bigl(1-r(n)\bigr)x^{n-1}\,.$$

Hence,

$$S_2^{(\lambda)}(a,b) = \lambda f''(\lambda) + f'(\lambda)\,.$$

For simplicity, put $X_1 = (a+b)\lambda^{a+b} - a\lambda^a - b\lambda^b$ and $X_2 = (a+b)^2\lambda^{a+b} - a^2\lambda^a - b^2\lambda^b$. Since

$$f'(\lambda) = \frac{1}{(\lambda-1)^2} + \frac{ab\lambda^{ab-1}}{(\lambda^a-1)(\lambda^b-1)} - \frac{(\lambda^{ab}-1)X_1}{\lambda(\lambda^a-1)^2(\lambda^b-1)^2}\,,$$

we get

$$\begin{aligned} f''(\lambda) = &-\frac{2}{(\lambda-1)^3} + \frac{ab(ab-1)\lambda^{ab-2}}{(\lambda^a-1)(\lambda^b-1)} - \frac{2ab\lambda^{ab-2}X_1}{(\lambda^a-1)^2(\lambda^b-1)^2}\\ &- \frac{(\lambda^{ab}-1)(X_2-X_1)}{\lambda^2(\lambda^a-1)^2(\lambda^b-1)^2} + \frac{2(\lambda^{ab}-1)X_1}{\lambda^3(\lambda^a-1)^3(\lambda^b-1)^3}\,. \end{aligned}$$

Therefore, we obtain

$$\begin{aligned} S_2^{(\lambda)}(a,b) = &-\frac{\lambda+1}{(\lambda-1)^2} + \frac{a^2b^2\lambda^{ab-1}}{(\lambda^a-1)(\lambda^b-1)} - \frac{2ab\lambda^{ab}X_1 + (\lambda^{ab}-1)X_2}{\lambda(\lambda^a-1)^2(\lambda^b-1)^2}\\ &+ \frac{2(\lambda^{ab}-1)X_1}{\lambda^2(\lambda^a-1)^3(\lambda^b-1)^3}\,. \end{aligned}$$

Similarly, we see that

$$\begin{aligned} S_3^{(\lambda)}(a,b) &= \lambda^2 f'''(\lambda) + 3\lambda f''(\lambda) + f'(\lambda)\,,\\ S_4^{(\lambda)}(a,b) &= \lambda^3 f^{(4)}(\lambda) + 6\lambda^2 f'''(\lambda) + 7\lambda f''(\lambda) + f'(\lambda)\,,\\ S_5^{(\lambda)}(a,b) &= \lambda^4 f^{(5)}(\lambda) + 10\lambda^3 f^{(4)}(\lambda) + 25\lambda^2 f'''(\lambda) + 15\lambda f''(\lambda) + f'(\lambda)\,. \end{aligned}$$

4. Apostol-Bernoulli numbers

Though one may obtain explicit expressions of $S_m^{(\lambda)}(a,b)$ for small positive integers m, it is harder to obtain the formulas for large m. In this section, using the so-called Apostol-Bernoulli numbers, we give an expression of $S_m^{(\lambda)}(a,b)$ for general positive integral m.

The Apostol-Bernoulli polynomials $\mathcal{B}_n(x,\lambda)$ are defined by the generating function [1, p.165, (3.1)]:

$$\frac{ze^{xz}}{\lambda e^z-1}=\sum_{n=0}^\infty\mathcal{B}_n(x,\lambda)\frac{z^n}{n!}\quad(|z+\log\lambda|<2\pi)\,. \tag{2}$$

When $\lambda=1$ in (2), $B_n(x)=\mathcal{B}_n(x,1)$ are the classical Bernoulli numbers. When $x=0$ in (2), $\mathcal{B}_n(\lambda)=\mathcal{B}_n(0,\lambda)$ are *Apostol-Bernoulli numbers* [11, Definition 1.2], defined by

$$\frac{z}{\lambda e^z-1}=\sum_{n=0}^\infty\mathcal{B}_n(\lambda)\frac{z^n}{n!}\quad(|z+\log\lambda|<2\pi)\,. \tag{3}$$

They seem to be also called λ-Bernoulli numbers. When $\lambda=1$, the generating function of the left-hand side in (3) is exactly the same as that of the classical Bernoulli numbers B_n. But it does not imply that $\mathcal{B}_n(1)=B_n$ on the right-hand side though quite a few authors misunderstand. In fact, as seen in [1, p.165], the first several values are given by

$$\mathcal{B}_0(\lambda)=0,\quad \mathcal{B}_1(\lambda)=\frac{1}{\lambda-1},\quad \mathcal{B}_2(\lambda)=-\frac{2\lambda}{(\lambda-1)^2},\quad \mathcal{B}_3(\lambda)=\frac{3\lambda(\lambda+1)}{(\lambda-1)^3},$$
$$\mathcal{B}_4(\lambda)=-\frac{4\lambda(\lambda^2+4\lambda+1)}{(\lambda-1)^4},\quad \mathcal{B}_5(\lambda)=\frac{5\lambda(\lambda^3+11\lambda^2+11\lambda+1)}{(\lambda-1)^5}\,.$$

But,

$$B_0=1,\ B_1=-\frac{1}{2},\ B_2=\frac{1}{6},\ B_3=0,\ B_4=-\frac{1}{30},\ B_5=0,\ B_6=\frac{1}{42},\ \dots\,.$$

For $\lambda\ne1$, Apostol-Bernoulli polynomials $\mathcal{B}_n(x,\lambda)$ can be expressed explicitly by

$$\mathcal{B}_n(x,\lambda)=\sum_{k=1}k\binom{n}{k}\sum_{j=0}^{k-1}(-1)^j\lambda^j(\lambda-1)^{-j-1}j!\left\{{k-1\atop j}\right\}x^{n-k}\quad(n\ge0) \tag{4}$$

[11, Remark 2.6], where the Stirling numbers of the second kind $\left\{{n\atop k}\right\}$ are given by

$$\left\{{n\atop k}\right\}=\frac{1}{k!}\sum_{j=0}^k(-1)^{k-j}\binom{k}{j}j^n\,.$$

When $x=0$ in (4), Apostol-Bernoulli numbers $\mathcal{B}_n(\lambda)$ have an explicit expression in terms of the Stirling numbers of the second kind [1, p.166, (3.7)],[11, p.510, (3)][1].

$$\mathcal{B}_n(\lambda)=n\sum_{j=0}^{n-1}(-1)^j\lambda^j(\lambda-1)^{-j-1}j!\left\{{n-1\atop j}\right\}\quad(n\ge0) \tag{5}$$

We use a similar approach to Rødseth in [15]. Let n, r and s be integers with

$$r\equiv n\pmod a\quad(0\le r<a),\qquad bs\equiv r\pmod a\quad(0\le s<a)\,.$$

Notice that

$$n\in\mathrm{NR}(a,b)\iff\exists t\in\mathbb{Z}\ (1\le t\le\lfloor bs/a\rfloor),\ n=-at+bs$$

[1] In both references, the sum begins from $j=1$. However, the value for $n=1$ does not match the correct one $\mathcal{B}_1(\lambda)=1/(\lambda-1)$.

$$\Longleftrightarrow \exists k \in \mathbb{Z}\ (0 \le k \le (bs-r)/a - 1),\ n = ak + r\,.$$

Note that the case $\lambda = 1$ is discussed in [15]. Since

$$S_m^{(\lambda)}(a,b) = \sum_{r=0}^{a-1} \sum_{k=0}^{\frac{bs-r}{a}-1} \lambda^{ak+r-1}(ak+r)^m\,,$$

for $\lambda \ne 1$, we have

$$\begin{aligned}
\sum_{m=0}^{\infty} S_m^{(\lambda)}(a,b)\frac{z^m}{m!} &= \frac{1}{\lambda}\sum_{r=0}^{a-1}\sum_{k=0}^{\frac{bs-r}{a}-1}(\lambda e^z)^{ak+r}\\
&= \frac{1}{\lambda}\frac{1}{(\lambda e^z)^a - 1}\left(\sum_{r=0}^{a-1}(\lambda e^z)^{bs} - \sum_{r=0}^{a-1}(\lambda e^z)^r\right)\\
&= \frac{1}{\lambda}\frac{1}{(\lambda e^z)^a - 1}\left(\sum_{s=0}^{a-1}(\lambda e^z)^{bs} - \sum_{r=0}^{a-1}(\lambda e^z)^r\right)\\
&= \frac{1}{\lambda}\frac{az}{(\lambda e^z)^a - 1}\frac{bz}{(\lambda e^z)^b - 1}\frac{(\lambda e^z)^{ab} - 1}{abz^2} - \frac{1}{\lambda}\frac{1}{\lambda e^z - 1}\,. \qquad (6)
\end{aligned}$$

Assume that $\lambda^a \ne 1$ and $\lambda^b \ne 1$. The second term (without sign) of the right-hand side is equal to

$$\begin{aligned}
\frac{1}{\lambda}\frac{1}{\lambda e^z - 1} &= \frac{1}{\lambda z}\sum_{m=0}^{\infty}\mathcal{B}_m(\lambda)\frac{z^m}{m!}\\
&= \frac{1}{\lambda}\sum_{m=0}^{\infty}\frac{\mathcal{B}_m(\lambda)}{m}\frac{z^{m-1}}{(m-1)!}\\
&= \frac{1}{\lambda}\sum_{m=0}^{\infty}\frac{\mathcal{B}_{m+1}(\lambda)}{m+1}\frac{z^m}{m!} \quad (\mathcal{B}_0(\lambda) = 0)\,.
\end{aligned}$$

The first term is divided into two parts. One part (without sign) is given as

$$\begin{aligned}
&\frac{1}{\lambda}\frac{1}{abz^2}\frac{az}{(\lambda e^z)^a - 1}\frac{bz}{(\lambda e^z)^b - 1}\\
&= \frac{1}{\lambda}\frac{1}{abz^2}\left(\sum_{i=0}^{\infty}\mathcal{B}_i(\lambda^a)a^i\frac{z^i}{i!}\right)\left(\sum_{j=0}^{\infty}\mathcal{B}_j(\lambda^b)b^i\frac{z^j}{j!}\right)\\
&= \frac{1}{\lambda}\sum_{m=0}^{\infty}\sum_{i=0}^{m}\binom{m}{i}a^{i-1}b^{m-i-1}\mathcal{B}_i(\lambda^a)\mathcal{B}_{m-i}(\lambda^b)\frac{z^{m-2}}{m!}\\
&= \frac{1}{\lambda}\sum_{m=0}^{\infty}\frac{1}{(m+1)(m+2)}\sum_{i=0}^{m+2}\binom{m+2}{i}a^{i-1}b^{m-i+1}\mathcal{B}_i(\lambda^a)\mathcal{B}_{m-i+2}(\lambda^b)\frac{z^m}{m!}\,.
\end{aligned}$$

Another part is given as

$$\begin{aligned}
&\frac{\lambda^{ab-1}}{abz^2}\frac{az}{(\lambda e^z)^a - 1}\frac{bz}{(\lambda e^z)^b - 1}e^{abz}\\
&= \lambda^{ab-1}\left(\sum_{k=0}^{\infty}a^k b^k\frac{z^k}{k!}\right)
\end{aligned}$$

$$
\times \left(\sum_{\ell=0}^\infty \frac{1}{(\ell+1)(\ell+2)}\sum_{i=0}^{\ell+2}\binom{\ell+2}{i}a^{i-1}b^{\ell-i+1}\mathcal{B}_i(\lambda^a)\mathcal{B}_{\ell-i+2}(\lambda^b)\frac{z^\ell}{\ell!}\right)
$$

$$
\begin{aligned}
&= \lambda^{ab-1}\sum_{m=0}^\infty\sum_{\ell=0}^m\binom{m}{\ell}\frac{1}{(\ell+1)(\ell+2)}\\
&\quad\times\sum_{i=0}^{\ell+2}\binom{\ell+2}{i}a^{m-\ell+i-1}b^{m-i+1}\mathcal{B}_i(\lambda^a)\mathcal{B}_{\ell-i+2}(\lambda^b)\frac{z^m}{m!}\,.
\end{aligned}
$$

Comparing the coefficients on both sides of (6), we get the following expression.

Theorem 4.1. *For $\lambda\ne 0$ with $\lambda^a\ne 1$ and $\lambda^b\ne 1$, and a nonnegative integer m,*

$$
\begin{aligned}
S_m^{(\lambda)}(a,b) &= \lambda^{ab-1}\sum_{\ell=0}^m\sum_{i=0}^{\ell+2}\binom{\ell+2}{i}\binom{m}{\ell}\frac{a^{m-\ell+i-1}b^{m-i+1}}{(\ell+1)(\ell+2)}\mathcal{B}_i(\lambda^a)\mathcal{B}_{\ell-i+2}(\lambda^b)\\
&\quad-\frac{1}{(m+1)(m+2)\lambda}\sum_{i=0}^{m+2}\binom{m+2}{i}a^{i-1}b^{m-i+1}\mathcal{B}_i(\lambda^a)\mathcal{B}_{m-i+2}(\lambda^b)\\
&\quad-\frac{\mathcal{B}_{m+1}(\lambda)}{(m+1)\lambda}\,.
\end{aligned}
$$

Remark 4.2. When $m=1$ in the expression of Theorem 4.1, that of Theorem 1.1 is obtained.

If $\lambda^a=1$ or $\lambda^b=1$ in (6), without loss of generality, we can assume that $\lambda^a=1$ and $\lambda^b\ne 1$. Because $\gcd(a,b)=1$, $\lambda^a=\lambda^b=1$ is impossible for $\lambda\ne 1$. Then, the first term of the right-hand side of (6) is equal to

$$
\begin{aligned}
&\frac{1}{\lambda}\frac{az}{e^{az}-1}\frac{bz}{\lambda^b e^{bz}-1}\frac{e^{abz}-1}{abz^2}\\
&=\frac{1}{\lambda z}\left(\sum_{k=0}^\infty\frac{a^kb^k}{k+1}\frac{z^k}{k!}\right)\left(\sum_{i=0}^\infty B_i a^i\frac{z^i}{i!}\right)\left(\sum_{j=0}^\infty\mathcal{B}_j(\lambda^b)b^j\frac{z^j}{j!}\right)\\
&=\frac{1}{\lambda z}\left(\sum_{k=0}^\infty\frac{a^kb^k}{k+1}\frac{z^k}{k!}\right)\left(\sum_{\ell=0}^\infty\sum_{i=0}^\ell\binom{\ell}{i}a^ib^{\ell-i}B_i\mathcal{B}_{\ell-i}(\lambda^b)\frac{z^\ell}{\ell!}\right)\\
&=\frac{1}{\lambda z}\sum_{m=0}^\infty\sum_{\ell=0}^m\sum_{i=0}^\ell\binom{m}{\ell}\binom{\ell}{i}\frac{a^{m-l+i}b^{m-i}}{m-\ell+1}B_i\mathcal{B}_{\ell-i}(\lambda^b)\frac{z^m}{m!}\\
&=\frac{1}{\lambda}\sum_{m=0}^\infty\sum_{\ell=0}^{m+1}\sum_{i=0}^\ell\binom{m+1}{\ell}\binom{\ell}{i}\frac{a^{m-l+i+1}b^{m-i+1}}{(m-\ell+2)(m+1)}B_i\mathcal{B}_{\ell-i}(\lambda^b)\frac{z^m}{m!}\,.
\end{aligned}
$$

Comparing the coefficients on both sides of (6), we get the following expression.

Theorem 4.3. *For $\lambda\ne 0$ with $\lambda^a=1$ and $\lambda^b\ne 1$, and a nonnegative integer m,*

$$
\begin{aligned}
S_m^{(\lambda)}(a,b) &= \sum_{\ell=0}^{m+1}\sum_{i=0}^\ell\binom{m+1}{\ell}\binom{\ell}{i}\frac{a^{m-l+i+1}b^{m-i+1}}{(m-\ell+2)(m+1)\lambda}B_i\mathcal{B}_{\ell-i}(\lambda^b)\\
&\quad-\frac{\mathcal{B}_{m+1}(\lambda)}{(m+1)\lambda}\,.
\end{aligned}
$$

Remark 4.4. When $\lambda = -1$ in Theorem 4.1 or Theorem 4.3, formulas for Sylvester sums (5.11)–(5.14) in [21] are obtained. For, when a is odd, $\mathcal{B}_n((-1)^a) = -nE_{n-1}(0)/2$ $(n \ge 0)$, where $E_n(x)$ are Euler polynomials defined by

$$\frac{2e^{xz}}{e^z+1} = \sum_{n=0}^\infty E_n(x)\frac{z^n}{n!} \quad (|z| < \pi)\,.$$

In particular, when $\lambda = -1$ and $m = 1, 2$ in Theorem 4.3, we have the following formulas. The first relation is not included in the formula in Theorem 1.1.

Corollary 4.5. *When a is even and b is odd,*

$$S_1^{(-1)}(a,b) = \frac{b(ab-a-b)+1}{4}\,,$$
$$S_2^{(-1)}(a,b) = \frac{ab(b-1)(2ab-a-3b)}{12}\,.$$

For example, for $a = 4$ and $b = 11$, we get

$$\begin{aligned} &S_1^{(-1)}(4,11)\\ &= (-1)^0\cdot 1 + (-1)^1\cdot 2 + (-1)^2\cdot 3 + (-1)^4\cdot 5 + (-1)^5\cdot 6 + (-1)^6\cdot 7\\ &\quad + (-1)^8\cdot 9 + (-1)^9\cdot 10 + (-1)^{12}\cdot 13 + (-1)^{13}\cdot 14 + (-1)^{16}\cdot 17\\ &\quad + (-1)^{17}\cdot 18 + (-1)^{20}\cdot 21 + (-1)^{24}\cdot 25 + (-1)^{28}\cdot 29\\ &= 80\,. \end{aligned}$$

From Corollary 4.5, we also get

$$S_1^{(-1)}(4,11) = \frac{11(4\cdot 11-4-11)+1}{4} = 80\,.$$

Similarly, $S_2^{(-1)}(4,11) = 1870$.

Acknowledgement

The authors thank the anonymous referees for useful comments and suggestions.

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