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PROBLEMS

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Problems

The first two problems this month come courtesy of Des MacHale of University College Cork.

Problem 85.1. Dissect an equilateral triangle into four pieces that can be reassembled, without flips, to form three equilateral triangles of different sizes. Can this be accomplished with just three pieces?

Problem 85.2. An absent-minded professor of mathematics cannot remember her debit card PIN. However, she remembers that the PIN lies between 4129 and 9985 and it cannot be expressed as the sum of two or more consecutive integers. Can you help her determine the PIN?

The third problem is a classic, which I encountered recently in the magazine of the M500 Society, a mathematical society of the Open University.

Problem 85.3. Arrange the integers 1 to 27 in a $3 \times 3 \times 3$ cube in such a way that any row of three integers (excluding diagonals) has sum 42.

Solutions

Here are solutions to the problems from *Bulletin* Number 83.

The first problem was solved by Ibae Aedo of the Open University, Omran Kouba of the Higher Institute for Applied Sciences and Technology, Damascus, Syria, and the North Kildare Mathematics Problem Club. We present the solution of the Problem Club.

*Problem 83.1***.** Find positive integers a, b, c, d, e such that

$$\cfrac{1}{a - \cfrac{1}{b - \cfrac{1}{c - \cfrac{1}{d - \cfrac{1}{e}}}}} = 0,$$

and such that this equation remains true if a, b, c, d, e is replaced by any cyclic permutation of those five letters in that order.

(Note that this problem uses arithmetic involving ∞, such as $1/\infty = 0$).

Solution 83.1. One solution is

$$(a, b, c, d, e) = (2, 2, 1, 3, 1).$$

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To obtain this, observe that the continued fraction equation is equivalent to

$$bcde + 1 = bc + be + de,$$

so we are asked for a solution in positive integers of the system of five equations obtained by combining this with the other four equations obtained by cyclic permutation of (a, b, c, d, e). Parity considerations show that exactly two of the variables must be even, and they must be adjacent in cyclic order, so without loss of generality we may consider a and b to be even and c, d, and e to be odd. Testing $e = 1$ in the equation, we see that $d \neq 1$, so $d \geq 3$, and we are led to the inequality $b(2c-1) \leq 2$, which forces $b = 2$, $c = 1$, $d = 3$. Looking at the other permuted equations, we see that all are satisfied if $a = 2$. □

The next problem was solved by Ibae Aedo, Henry Ricardo of the Westchester Area Math Circle, New York, USA, Daniel Văcaru of Piteşti, Romania, Brendan and Ronan Wallace, and the North Kildare Mathematics Problem Club. Solutions were similar; we use the wording of Henry Ricardo.

Problem 83.2. Prove that for each positive integer m,

$$\tan^{-1} m = \sum_{n=0}^{m-1} \tan^{-1}\left(\frac{1}{n^2+n+1}\right).$$

Solution 83.2. The familiar identity $\tan(\alpha - \beta) = (\tan\alpha - \tan\beta)/(1 + \tan\alpha\tan\beta)$ leads to the following identity for the principal value of the inverse tangent function:

$$\tan^{-1} u - \tan^{-1} v \;=\; \tan^{-1}\left(\frac{u-v}{1+uv}\right).$$

Let $u = n+1$ and $v = n$ to give

$$\tan^{-1}\left(\frac{1}{n^2+n+1}\right) = \tan^{-1}(n+1) - \tan^{-1} n.$$

Thus we have the telescoping series

$$\begin{aligned}\sum_{n=0}^{m-1} \tan^{-1}\left(\frac{1}{n^2+n+1}\right) &= \sum_{n=0}^{m-1}(\tan^{-1}(n+1) - \tan^{-1} n)\\ &= \tan^{-1} m - \tan^{-1} 0 = \tan^{-1} m.\end{aligned}$$ □

Henry points out that by taking limits we obtain

$$\sum_{n=1}^{\infty} \tan^{-1}\left(\frac{1}{n^2+n+1}\right) \;=\; \frac{\pi}{2}$$

and other, similar results can be obtained by choosing special values of u and v in the inverse tangent identity stated in the solution.

The third problem was solved by Ibai Aedo, Henry Ricardo, Brendan and Ronan Wallace, and the North Kildare Mathematics Problem Club.

Problem 83.3. Find all positive integers x and y such that $x^y = y^x$.

We offer a known solution, presented by some of the contributors, which gives the full set of solutions for positive *rationals* x and y.

Solution 83.3. Naturally, $x = y$ is a solution for any positive number x, so let us assume that $x < y$.

We write the equation in the form $y = x^{y/x}$. Dividing by x gives

$$x^{y/x-1} = \frac{y}{x}. \qquad (*)$$

We let $m/n = y/x - 1$, where m and n are coprime positive integers. Equation $(*)$ becomes $x^{m/n} = (m+n)/n$, or, equivalently,

$$x = \frac{(m+n)^{n/m}}{n^{n/m}}. \qquad (**)$$

Since m and n are relatively prime, so are $(m+n)^n$ and n^n. It follows from $(**)$ that x is rational if and only if both $(m+n)^n$ and n^n are mth powers. So if x is rational, then each of $m+n$ and n must be an mth power, because the exponents m and n are relatively prime. Hence $n = a^m$ and $m + n = b^m$, where a and b are positive integers and $b > a$. This is possible if and only if $m = 1$, because the difference between two consecutive mth powers is greater than m if $m > 1$.

It follows from $(*)$ and $(**)$ that

$$x = (1 + 1/n)^n \quad \text{and} \quad y = (1 + 1/n)^{n+1},$$

where n is any positive integer. The only pair of positive integer solutions with $x < y$ is obtained when $n = 1$, giving $x = 2$ and $y = 4$. □

Thanks to all those who provided references for papers written on this problem. Henry Ricardo notes that the problem was first stated in a letter from Daniel Bernoulli to Christian Goldbach in 1728, in which Bernoulli asserts (without proof) that the equation has only one solution in positive integers and infinitely many rational solutions. Euler later solved the equation over the positive reals and positive integers, and provided rational solutions, without claiming that they were the only ones.

To finish this issue, it was incorrectly stated in Issue 84 that no solutions had been received for the extended version of Problem 82.2, which asks for a proof of the integral formula

$$\int_0^\infty \frac{\sinh x - x}{x^2 \sinh x} e^{-x}\, dx = \log \pi - 1.$$

In fact, Omran Kouba had already submitted a correct solution. I apologise for the error.

We invite readers to submit problems and solutions. Please email submissions to `imsproblems@gmail.com` in any format (we prefer Latex). Submissions for the summer Bulletin should arrive before the end of April, and submissions for the winter Bulletin should arrive by October. The solution to a problem is published two issues after the issue in which the problem first appeared. Please include solutions to any problems you submit, if you have them.

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