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The aim of the *Bulletin* is to inform Society members, and the mathematical community at large, about the activities of the Society and about items of general mathematical interest. It appears twice each year. The *Bulletin* is published online free of charge.

The *Bulletin* seeks articles written in an expository style and likely to be of interest to the members of the Society and the wider mathematical community. We encourage informative surveys, biographical and historical articles, short research articles, classroom notes, book reviews and letters. All areas of mathematics will be considered, pure and applied, old and new.

Correspondence concerning the *Bulletin* should be sent, in the first instance, by e-mail to the Editor at

`mailto:ims.bulletin@gmail.com`

and only if not possible in electronic form to the address

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Submission instructions for authors, back issues of the *Bulletin,* and further information about the Irish Mathematical Society are available on the IMS website

`http://www.irishmathsoc.org/`

EDITORIAL

Recently, the editors of zbMATH informed us that the article by Elke Wolf: *Composition operators between weighted Bergman spaces and weighted Banach spaces of holomorphic functions* which we published in Bulletin 79 (2017) 75–85, has almost identical content to her article with the same title published earlier in Mathematica 57(80), No. 1–2, 126–134 (2015). See Zbl 1389.47077 `https://zbmath.org/?q=an%3A1389.47077`. They wrote: "The papers are identical to the letter apart from a remark in Section 2.2 in the IMS Bulletin version that was apparently added at the referee's suggestion. Actually, the IMS Bulletin version was submitted after the other version had been accepted." We learned that this author published another article multiple times. We are grateful to zbMATH for detecting the issue and pointing it out. Our Editorial Board was appalled. We contacted the author, and she apologized unreservedly. We contacted the editors of Mathematica, whose copyright we had inadvertently violated, and they were understanding of our position, stating that they considered the author bore the main responsibility for the breach.

There are established guidelines for handling such issues, set out in the article *Retraction Guidelines*, by COPE (Committee on Publication Ethics), available at
`www.publicationethics.org/files/retractionguidelines.pdf`.
Adhering to these, we decided:

- to retract the article on the grounds of self-plagiarism, i.e. redundant publication,
- to retain the online electronic copy of the article on our website, but to mark each page with an indication of the withdrawal,
- to record here in this editorial the basis and details of our decision,
- to publish a statement of the retraction in the format of an ordinary article in this issue of our Bulletin, with a view to having the retraction indexed in zbMath and MathSciNet and any other indexing resources.

Members will wish to note that there are some changes and new classes in the Mathematics Subject Classification 2020. There are searchable versions at the zbMATH website:
`https://zbmath.org/classification/`
and also at the MathSciNet site:
`https://mathscinet.ams.org/mathscinet/searchMSC.html`
Additionally, there is a PDF version of MSC2020 at
`https://mathscinet.ams.org/msnhtml/msc2020.pdf`
I encourage authors to employ MSC2020 from now on. We have provided revised versions of our class file bimsart.cls and article and review templates to accommodate MSC2020 classes.

I remind Irish schools to contribute news, ideally through the local representatives. Please send reports for 2020 by mid-December to

`mailto://ims.bulletin+news@gmail.com`

so they can be included in the Winter Bulletin.

As before, to facilitate members who might wish to print the whole issue, the website will carry a pdf file of the whole Bulletin 85, in addition to the usual pdf files of the individual articles. As a further convenience (which may suit some Departments and

Libraries), for a limited time a printed and bound copy of this Bulletin may be ordered online on a print-on-demand basis at a minimal price[1].

This year's IMS Annual Scientific Meeting (also known as the "September meeting") will be held in DCU. **Due to the disruption caused by the COVID-19 pandemic, the meeting has been rescheduled, and will held in late December or early January.**

Recently the Royal Irish Academy confirmed its decision to cease underwriting the cost of Ireland's participation in the International Mathematical Union (IMU). The main costs are the subscription, currently €2840 per annum, and the expenses of a delegate to the ICM, every four years. The IMS Committee has decided to keep our participation going, and to take over from the RIA the rôle of liaison with the IMU. Our President, Pauline Mellon, will be writing to IMS members in a little while about the challenges this poses. It may be useful to recall some facts about Ireland's connection to the International Union to date:

The main purposes of the IMU are (cf. `https://www.mathunion.org/`)

- To promote international cooperation in mathematics.
- To support and assist the International Congress of Mathematicians (ICM) and other international scientific meetings or conferences.
- To encourage and support other international mathematical activities considered likely to contribute to the development of mathematical science in any of its aspects, pure, applied, or educational.

The permanent secretariat of the Union is in Berlin, since 2011. The Union has 88 member countries.

A country's subscription depends on its level of mathematical activity and development. Ireland pays the same subscription as countries of comparable size (in terms of mathematics) such as Austria, Denmark, Portugal and Slovakia. Big players such as France, Germany, Italy, Israel, UK, Japan, Russia, and USA pay Euro 17,160 per annum.

The first ICM took place at the Chicago World's Fair of 1893, and apart from interruptions due to world wars they continue on a quadrenniel basis. Hilbert's famous 'Problems' address at the 1900 Congress in Paris set out agenda that guided Mathematical research for much of the twentieth century. The IMU was set up in 1920 to look after the administration of the ICM. As part of this work, the IMU could be said to canonise the current priorities in mainstream mathematical research. The programme for the ICM is organised on the basis of a division of the discipline into sections, and this division is dynamic, evolving over time according to the appearance of new developments and applications.

From 1950 onward, the Fields Medals are awarded at the ICM, and more recently other prizes.

Colm Mulcahy has set out a comprehensive record of Irish participation in the International Congress (`http://www.mathsireland.ie/blog/2018_09_cm` and `http://www.mathsireland.ie/blog/2018_11_cm`). Looking at these blogs, one sees that Irish mathematicians were prominent at some key stages in the evolution of the ICM. For instance, Larmor, who was present at Hilbert's address and who gave a plenary address in 1912, was Vice-President of the Congress by 1920, when the IMU was set up. Synge was influential in the administration of several congresses, and played a key rǫle in the creation of the Fields Medal. The first official representative of the Irish state was Conway, at the 1924 ICM in Toronto. At that meeting Ireland had two delegates and members present, out of 444, of whom 298 were from the US or Canada. By contrast,

[1]Go to `www.lulu.com` and search for *Irish Mathematical Society Bulletin*.

the 2018 ICM in Rio de Janeiro had 3018 full members. Other speakers over the years included F.Edgeworth, E.T.Whittaker, William McCrea, John Todd, F.D.Murnaghan, P.de Brun, A.J. McConnell, J.R. McConnell, Lanczos, Paddy Kennedy, Paddy Barry, Cathleen Morawetz, and Don McQuillan.

The visibility of Irish-based people in the key IMU committees (the executive, the programme, the medal committees) and in the lists of invited speakers is substantially less than in the past. To a large extent this follows the same pattern as the sports olympics movement, where rising participation from around the world and differential population growth has reduced our relative proportion of top-level candidates. Recent ICM speakers included Kevin Costello b. Cork (Hyderabad 2010), Samson Shatashvili (TCD), and David Conlon b. Sligo (Seoul 2014).

In the past thirty years or so, the ICM has been accompanied by a large number of 'satellite conferences', held in the same country before or after the Congress, and focussing on specific fields. Although it does not appear on the record of the ICM, Irish mathematicians have been included as invited speakers at some of these meetings, and they are an important benefit to the country, in terms of new ideas learned and international collaborations initiated and maintained.

The most important benefit of Irish participation in the IMU and ICM lies in the maintenance and development of useful contacts, and the integration of the Irish mathematical community into the worldwide community.

The IMU has three commissions: ICMI (founded 1908) for Mathematics Education, CDC for Developing Countries, and ICHM for History of Mathematics, and two main Committees[2]: CEIC for Electronic Information and CWM for Women in Mathematics. Merrilyn Goos (UL) is Vice-President of ICMI, and with Maurice O'Reilly (DCU) Thérèse Dooley (UL) and others helped bring the major CERME10[3] conference, held in Croke Park in February 2017. The ICHM co-sponsored the Fifth Joint Conference of the British Society for the History of Mathematics and the Canadian Society for History and Philosophy of Mathematics held in TCD in July 2011. Romina Gaburro (UL) is local ambassador for the CWM, and related activity includes the annual Women in Mathematics Day.

In my opinion, continued Irish membership in the International Mathematical Union is vital for the health of Irish mathematics and for the reputation of the country. To cease participation would be to declare that we are just giving up. Granted, support within Ireland for research in basic science, particularly in mathematical science, and even more particularly in pure mathematics, is just lamentable, but we are not at rock bottom.

[2] `https://www.mathunion.org/activities/imu-commissions-and-committees`
[3] `https://www.mathunion.org/news-and-events/2017-02-01/cerme-10`

Links for Postgraduate Study

The following are the links provided by Irish Schools for prospective research students in Mathematics:

DCU: mailto://maths@dcu.ie
DIT: mailto://chris.hills@dit.ie
NUIG: mailto://james.cruickshank@nuigalway.ie
MU: mailto://mathsstatspg@mu.ie
QUB: http://www.qub.ac.uk/puremaths/Funded_PG_2016.html
TCD: http://www.maths.tcd.ie/postgraduate/
UCC: http://www.ucc.ie/en/matsci/postgraduate/
UCD: mailto://nuria.garcia@ucd.ie
UL: mailto://sarah.mitchell@ul.ie
UU: http://www.compeng.ulster.ac.uk/rgs/

The remaining schools with Ph.D. programmes in Mathematics are invited to send their preferred link to the editor. All links are live, and hence may be accessed by a click, when read in a suitable pdf reader.

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Irish Mathematical Society Bulletin 85, Summer 2020

CONTENTS

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(1) The Irish Mathematical Society has reciprocity agreements with the American Mathematical Society, the Deutsche Mathematiker Vereinigung, the Irish Mathematics Teachers Association, the London Mathematical Society, the Moscow Mathematical Society, the New Zealand Mathematical Society and the Real Sociedad Matemática Española.

(2) The current subscription fees are given below:

Institutional member	€200
Ordinary member	€30
Student member	€15
DMV, I.M.T.A., NZMS or RSME reciprocity member	€15
AMS reciprocity member	$20

The subscription fees listed above should be paid in euro by means of electronic transfer, a cheque drawn on a bank in the Irish Republic, or an international money-order.

(3) The subscription fee for ordinary membership can also be paid in a currency other than euro using a cheque drawn on a foreign bank according to the following schedule:

If paid in United States currency then the subscription fee is US$ 40.
If paid in sterling then the subscription is £30.
If paid in any other currency then the subscription fee is the amount in that currency equivalent to US$ 40.00.

The amounts given in the table above have been set for the current year to allow for bank charges and possible changes in exchange rates.

(4) Any member with a bank account in the Irish Republic may pay his or her subscription by a bank standing order using the form supplied by the Society.

(5) Any ordinary member who has reached the age of 65 years and has been a fully paid up member for the previous five years may pay at the student membership rate of subscription.

(6) Subscriptions normally fall due on 1 February each year.

(7) Cheques should be made payable to the Irish Mathematical Society.

(8) Any application for membership must be presented to the Committee of the I.M.S. before it can be accepted. This Committee meets twice each year.

(9) Please send the completed application form with one year's subscription to:

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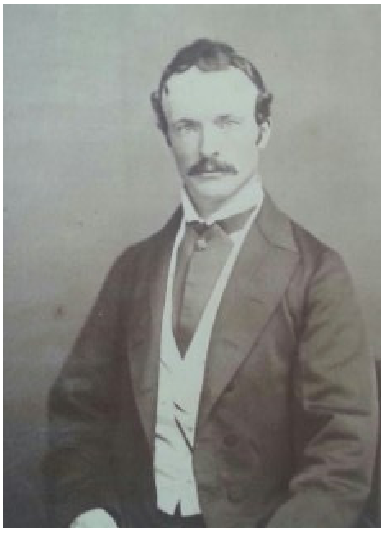
Martin Gardiner: the first Irish–Australian mathematician

GRAEME L. COHEN

Abstract. Martin Gardiner was one of the first students enrolled in Queen's College, Galway, in 1850. He performed exceptionally well but ended his studies in civil engineering after just two years. While there, he developed his own set of correspondents, such as Richard Townsend in Trinity College, Dublin, through which he maintained a comprehensive knowledge of current developments in geometry. By 1857, he was in Melbourne, Australia, and was soon promoting himself as more capable than any local mathematician. He took surveying positions around the country, never for more than five years at a time, and everywhere he lived advertised himself as a private tutor, or as conducting a school for mathematicians and engineers. None was successful. Gardiner had over twenty publications in the proceedings of the Royal Societies of Victoria and New South Wales, as well as a few in leading English journals.



Martin Gardiner, from his paper *The three sections* ....

Introduction

Until the appearance of my book, *Counting Australia In* (Cohen [2]), and more recently my bibliography of Australian mathematics (Cohen [3]), very little had been written about the surveyor, mathematician and perennial combatant, Martin Gardiner. The online *Encyclopedia of Australian Science* says little more than that he "was a mathematician who specialised in geometry," and refers only to his time in Victoria (see http://www.eoas.info/biogs/P001569b.htm). In fact, Gardiner, born in Ireland around 1833, was a surveyor who worked in all states of Australia except Tasmania, rarely holding a job for more than five years. He was aggressive and insulting towards

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Received on 5-3-2020.

his colleagues, highly critical of his superiors, a bankrupt, and violent towards his wife. There is some evidence that he falsified his academic qualifications.

Yet he was indeed a capable and knowledgable mathematician, apparently largely self-taught, and well aware of the current research in his field, with twenty or so publications to his credit including some in the best English journals. The full list of his publications, as far as they are known to me, is given in the Appendix. In *Counting Australia In* (p. 75), I wrote: "Gardiner's work is generally very detailed and difficult to fathom, and none of it has lasted." But in his day he received lavish praise from some of the most highly regarded English mathematicians.

This article is based in large measure on newspaper reports of Royal Society and other meetings and reports of New South Wales parliamentary proceedings, on government *Gazette* notices, on newspaper advertisements, and on letters to the editors of dozens of newspapers. Gardiner was a prolific letter writer, always ready to criticise the colonists among whom he lived and worked. Other sources include journal notices and a number of birth, death and marriage certificates.

Over the past five years or so, there has been a lively email discussion ascertaining various facets of Gardiner's life and work. I very much appreciate the efforts and advice in this period of professors Alan Atkinson, Colm Mulcahy and James Tattersall.

1. First, Melbourne

Martin Gardiner arrived in Melbourne on 9 December 1856 aboard the *Royal Charter* with his wife, maiden name Bridget Mary Maguire, two-year old son Charles Napoleon and infant daughter Margaret. Just a few months later, on 17 March 1857, the baby Margaret died from dysentery, aged two years and two months; and a month after that, on 20 April, another child, James, was born. The *Royal Charter*'s passenger list gives Martin's age on board as 24 and Bridget's as 18, but the latter could hardly be correct. Indeed, James' birth certificate states that Bridget was then aged 22, and was born in Limerick, Ireland. Gardiner's age is given as 25 and his birthplace as Galway, Ireland (but evidence below will suggest the more likely possibility that he was born in Dublin). The birth certificate states also that Martin and Bridget had married on 7 December 1852 at Sherbrook, near Montréal, Canada.

Details of Gardiner's education are sketchy. Alan Atkinson found that *The Dublin Evening Post* reported on 31 December 1850 that a science scholarship for a Martin Gardiner in the Faculty of Arts at Queens College, Galway, had been "granted" on the 23rd; and the Dublin *Advocate* reported the same on 1 January 1851. Colm Mulcahy passed on photographs of the handwritten log from the Registrar's Office at the college, which show a Martin Gardiner from "Dublin School" admitted as "A Scholar in Engineering" on 7 January 1850 for the session 1849–50, and being "A Scholar 1st Year Arts" for 1850–1851.[1] The details for those two years are confirmed by entries in the *Calendar* for Queen's College, Galway, for the year 1899 (available online). There are no further mentions and, in particular, the *Calendar* does not include Gardiner among its list of all graduates of the college up to 1898. So Gardiner, who would very often sign himself as "Martin Gardiner, C.E., Scholar of Queen's College, Galway", does not appear from this to have completed a full course of study there. (See Mitchell [10] for a description of the College's early years.)

However, he brought with him to Australia a certificate from the college, dated September 1852 and stating: "We ... do hereby certify that Mr. Martin Gardiner has diligently pursued the Course of Study prescribed in the Queen's University in Ireland, and is duly qualified to act as Assistant-Civil-Engineer." Alongside that certificate is another, "to Certify that at an Examination held on the 12th day of June, 1851, Mr.

[1]The photographs were obtained by Professor Ted Hurley, there. I am most grateful.

Martin Gardiner has been awarded a Prize of First Rank for Proficiency in Natural Philosophy during the second year of his academic course."[2] There is no evidence that he undertook further tertiary studies anywhere else.

Gardiner, possibly with Bridget, left for Canada soon after his time at Queen's College. The *County Galway Surname List*[3] confirms this in very brief terms. The list says of Gardiner no more than: "born about 1832 emigrated to Canada by about 1850". In Canada, Gardiner worked for the Grand Trunk Railway, and that is about all that is known of his time there.

In contrast to her husband, Bridget may have been illiterate: on James' birth certificate, she signed with her mark, "X".

Within a year of his arrival in Australia, Gardiner issued a challenge: "To the Mathematicians of Australia. — £10 for any Person who shall (before the 1st of January, 1858,) present to the Public Library of Melbourne ORIGINAL GEOMETRICAL PAPERS to equal or excel those which I shall have contributed. — Martin Gardiner, C.E."[4] Just about all of his contributions to newspapers, over 40 years and from five Australian states, exhibited the same self-promoting arrogance and deprecating of the local colonists.

On 28 May 1859, he gave a public lecture entitled "The necessity of having the Mathematics and Applied Mechanics occupy a more prominent place in public instruction"[5] and soon after, he was advertised as giving regular lectures for the Mechanics Institution: classes in "Mathematics and Mechanical Philosophy . . . every Tuesday and Saturday Evening, 7 to 9 o'clock. Two guineas for the session, and no extra charges for instructions in calculus, descriptive geometry, framing, skew bridges, &c."[6] These classes did not eventuate, as "those who were so eager to join the classes have not been forthcoming."[7] This was the first of many instances of failed attempts to establish himself as a teacher.

Gardiner was elected a member of the Philosophical Institute of Victoria in June 1859. He had a paper, "Improvements in Fundamental Ideas and Elementary Theorems of Geometry",[8] in Volume 4 of the *Transactions* of the Institute published that year. Although this, his first paper, is dated 1859, he had earlier presented a bound volume of three papers in 40 pages entitled *Geometrical Papers* to the Melbourne Public Library. These no doubt formed the basis for his challenge, two years before. He wrote in his introduction to these *Papers* that he had endeavoured to bring the work to the notice of William Parkinson Wilson (1826?–1874), professor of mathematics in the recently established University of Melbourne, but, as I wrote in *Counting Australia In* (p. 75), "in the manner of countless academics when similarly approached, 'the learned proffessor [*sic*] did not wish to enter into the details of my methods'."[9]

At the meeting of the Philosophical Institute on 30 November 1859, Gardiner read a paper "Hints for Field Practice, in the Laying-out of Compound-circular and Parabolic Railway Curves". The Institute was transformed into the Royal Society of Victoria in the following year, and Gardiner stood unsuccessfully for election to its council in April.

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[2] The certificates are among the collected correspondence of Sir Henry Parkes, Volume 27, call no. A897 (State Library of NSW), pp. 236–240.

[3] www.rootsweb.ancestry.com/∼irlgalway/galway.htm.

[4] *The Argus*, 19 October 1857.

[5] *The Age*, 25 May 1859. Many newspaper advertisements were repeated in the following days. Generally, only the first occurrence is recorded here.

[6] *The Age*, 2 June 1859.

[7] *The Argus*, 30 June 1859.

[8] "Read before the Institute 13th July and 3rd August, 1859."

[9] Before his appointment to the chair in Melbourne, Wilson had been the foundation professor of mathematics in Queen's College, Belfast.

He read a paper to its meeting on 4 June 1860 ("Analogous Solutions to the Section of Ratio, Section of Space, and Determinate Section of Apollonius, with their Generating Problems"), and on 30 September 1861 his paper "A few observations on the tangencies of Apollonius" was "laid upon the table".

2. Sydney

In August 1860 Gardiner, with Bridget and their two surviving children, moved to Sydney with the expectation that he would obtain employment in the department of railways. Just five months later, on 23 January, Bridget died of a lung disease, aged 27. According to the death certificate, the burial was witnessed by a young friend, Emma Guile, who was to become Martin's second wife soon after, on 13 August. Their marriage certificate gives Emma's age as 18 at the time and her birthplace as London. Martin's birthplace is given as Dublin, not Galway as stated above, and his parents are stated as William ("a Gentleman") and Mary. On 12 June 1862, Emma gave birth to a son, Martin. There would be two more children: daughters, Emma on 18 September 1864 and Mary Louisa in Newcastle, New South Wales, on 25 October 1867. This marriage ended in violence by Gardiner against Emma, then desertion and finally divorce.

Gardiner was exceptionally proud of the paper he presented to the Philosophical Society of New South Wales on 9 July 1862. It was titled "Improvements in geometrical science, with their applications in solutions to celebrated problems, and in the investigations of new Porisms."[10] Two months earlier, he had written to the editor of the Sydney critical newspaper, the *Empire*, "As I have determined on publishing a model solution to a celebrated geometrical problem, I hope you will be so kind as to insert its history in your philanthropic journal, and be a benefactor to pure science in this colony, where the highest intellectual ambition of the rising generation is almost entirely directed to what I consider the useless, inglorious, art of 'cricket-batting.'"

The problem in its simplest form is: To inscribe a triangle in a given circle, such that its sides (produced if necessary) shall pass through three given points. The *Empire* did indeed publish (in almost 1300 words) Gardiner's history of the Cramer–Castillon problem, as it is now known. Much of this history was taken, with very similar wording in some parts, from a paper by Thomas Davies [5] that had appeared a dozen years before. Gardiner acknowledged Davies' paper in a more detailed history of the "celebrated problem" at the end of the third part of his "Geometrical Researches" (Appendix (e)), where his complete analysis was given. Generalisations of the Cramer–Castillon problem were the subject of a number of Gardiner's papers, and undoubtedly constituted his best work in mathematics.

Regarding the presentation to the Philosophical Society, the *Herald* (10 July 1862) wrote, "Last night, the monthly meeting of the Philosophical Society of New South Wales was held in the hall of the Australian Library ... Two papers were read—the first by Mr. Martin Gardiner, C.E., ... Mr. G. introduced supplementary ideas concerning the modes of formation of lines in respect of points, and of surfaces in respect to lines and points; and, although the subject was of a very abstruse character, he succeeded in rendering it, by the aid of diagrams, somewhat interesting ..."

In a letter to the *Empire*, on 27 January 1863, in which he signs himself as working for the "Railway Department", Gardiner paid tribute to "the newly appointed Chief Justice of Queensland ... [who] is well known as a very eminent mathematician." Not named by Gardiner in the letter, this was Sir James Cockle (1819–1895). (See Bennett [1], Deakin [6].) In 1867 Gardiner read a paper of Cockle's to a meeting of the Royal

[10]From *Wikipedia*: "A porism is a mathematical proposition or corollary. In particular, the term porism has been used to refer to a direct result of a proof, analogous to how a corollary refers to a direct result of a theorem."

Society of New South Wales. In 1870, Cockle supported Gardiner in his unsuccessful application for the chair of mathematics in the newly established University of Otago, New Zealand (to be described in detail below).

Gardiner presented a paper to a meeting of the Philosophical Society of New South Wales on 17 June 1863 ("Complete Solution to a Celebrated Problem"). Another paper was scheduled for presentation at the next monthly meeting of the Society on 8 July ("The correct scientific method of forming railway curves and railways, with an exposition of the injurious effects of the system adopted in this colony"), but, perhaps because of the controversial nature of the paper, it was deferred to the subsequent meeting on 12 August. Gardiner wrote to the editor of the *Empire* on 14 August, asking that "in order to ... advance the interests of the colony," the paper be printed in the newspaper in full; and this was done.

Gardiner read a paper, "On improved analytical geometry," to a meeting of the Royal Society of New South Wales on 17 August 1864.[11]

In an announcement in *The Sydney Morning Herald* on 6 March 1865, "Mr. Sheridan Moore, Licensed Tutor of the University", informed "undergraduates and others" that Gardiner was to "take charge of all his advanced Mathematical Classes." Joseph Sheridan Moore was a prolific local poet and essayist. He was involved in a number of educational ventures (as was Gardiner), none of which succeeded. An Irishman who migrated to Sydney in 1847, Moore "was esteemed by some but condemned as a charlatan by the native-born members [of his literary group] who distrusted him" (Glass [7]).

3. Newcastle

By the end of July 1865, Gardiner and family had moved to Newcastle, north of Sydney, where Gardiner had obtained the position of City Surveyor. As usual, wherever he went, he was keen to promote himself as a teacher. In the Newcastle *Chronicle* on 23 September 1865, he advertised his services as follows. "Mathematics, Surveying, Engineering &c, taught to Gentlemen who wish to undergo Civil Service Examinations, or to obtain Certificates of Qualification as Civil Engineers. Martin Gardiner, C.E., City Engineer and Surveyor, Newcastle."

A few months later, on 2 January 1866, there was more self-promotion with the following, from a letter to the *Empire*:

> From the report of the Deputy Surveyor-General on the state of the Survey department, I learn that he recommends the Government to send to England for a scientific trigonometrical and geodetic surveyor, and for an equally qualified draughtsman for the office-work pertaining to trigonometrical surveys.
>
> Now, I can produce abundant testimony (to those who are sufficiently advanced in the science of the profession to be able to understand the worth of such testimony) as to my thorough qualifications to perform either the field or office duties of trigonometrical and geodetic (astronomical) surveying, and to avail myself of the formulae of reduction, &c, contained in the most approved French and English authors, as Puissant Airy, and the published accounts of the ordinance survey of Britain and Ireland ... I maintain that at such work I have no superior in this colony, or in England ... I hope Mr. Adams will not overlook capable men who are now in the colony; and I would call his attention to the two following letters the first of which is from his friend Mr. Hodgkinson, the most scientific and accomplished professional surveyor in the service of the Victorian Government.

The reference in the last lines is to Clement Hodgkinson (1818–1893), a public servant and surveyor, heavily involved with the Philosophical Institute and then the Royal Society of Victoria. Gardiner's description of him is apt.[12] His letter to Gardiner is dated

[11]See royalsoc.org.au/council-members-section/91-philsoc1856-65#1866.

[12]Hodgkinson is described in *Counting Australia In* (p. 76) as the author of the "first paper of mathematical interest published in Victoria" (Hodgkinson [8]).

9 June 1860 and reads as follows: "As you state that you are about to proceed to Sydney, with a view to employment as a railway surveyor, I beg to assure you that, in my humble opinion, your qualifications are of the very highest order. As a member of the Royal Society of Victoria, you have not only contributed to the transactions of that society some valuable information on professional subjects, but also, some papers displaying profound knowledge of the higher branches of mathematics and great originality and genius."

The other letter that Gardiner mentions was from Richard Townsend (1821–1884), an eminent geometer and professor of natural philosophy at Trinity College, Dublin. The letter, dated 17 October 1865, is highly praising of Gardiner's work. For example, referring to one of Gardiner's papers, he wrote: "I can assure you I have not for a long time enjoyed a greater treat or experienced more pleasure than its study has afforded me. You need not ask me, my dear sir, to help to rescue your papers from obscurity—that paper will immortalise your name and hand it down to history, as that of a pure geometer of the first order."

Townsend's letter is addressed to Gardiner at "St. John's College, Sydney". In Volume IV (1866) and from Volume VII (1867) to Volume XXXVI (1881), Gardiner was listed as a contributor to *Mathematical Questions, with their Solutions, from the "Educational Times"* and he described himself there (from Volume VIII onwards) as "late Professor of Mathematics in St. John's College, Sydney." There is no evidence, according to the archivist at St John's College in the University of Sydney (personal communication) of any such association. The title of professor would only have been allowed by the college if it had been bestowed by some other institution and there is no evidence of that, either. I have noticed only the following two contributions from Gardiner to *Mathematical Questions*: Unsolved Question No. 1882 in Volume VI (1866) and No. 2255 in Volume VII (1867). In contrast, for example, Sir James Cockle had numerous contributions in most volumes in this period.

As their titles suggest, these volumes consist largely of mathematical problems and their solutions from *The Educational Times*.[13] In the occasional lists of contributors that the journal published (which were distinct from those in the *Mathematical Questions*) from 1870 to 1875, Gardiner was listed as "Martin Gardiner, F.R.A.S., St. John's College, Sydney, Australia." It is known (indirect personal communication) that Gardiner was not at any time a Fellow of the Royal Astronomical Society, and nothing else seems plausible. He did not use those initials in any other context.

Gardiner contributed four questions to *The Educational Times*: No. 1855 in Volume XVIII (December 1865); No. 1882 in Volume XVIII (January 1866) (repeated as No. 4056 in Volume XXVI (April 1873)); No. 1897 in Volume XVIII (February 1866) (repeated as No. 4106 in Volume XXVI (June 1873)); and No. 2255 in Volume XIX (October 1866). Nos 1882 and 2255 subsequently appeared with the same numbers in *Mathematical Questions*, as recorded above. No. 1882, slightly reworded and ascribed to "the Editor" rather than Gardiner, was solved by Âsûtosh Mukhopâdhyây in Volume XLIII (1885) of *Mathematical Questions*. I am grateful to Jim Tattersall for this information; he asserts further that Gardiner did not produce solutions for any of his questions.

Under the heading "FENIAN EXCITEMENT AT NEWCASTLE, N.S.W.", the *Newcastle Chronicle* on 28 March 1868 reported that Gardiner was before the Court, "on summons, to answer a charge of using insulting words to one Charles Edward Thurlow, on the 21st instant, whereby a breach of the peace might have been occasioned." He

[13]Full title: *The Educational Times and Journal of the College of Preceptors*, "A Monthly Journal of Education, Literature and Science", published in London in varying forms from 1847 to 1923.

pleaded not guilty. Not long before, on 12 March, there had been the attempted assassination in Sydney of Prince Alfred, Duke of Edinburgh, during the first ever royal visit to Australia. It was seen as an instance of the pervading tension between Irish Catholics and non-Catholics during what was termed "Fenian terrorism" in England. (The Fenians sought the establishment of an independent Irish republic.) Thurlow complained that, while walking with Gardiner and another man the previous Saturday, and having expressed the view that "the colony had been disgraced by the attempt", Gardiner had immediately replied, "The whole of you in the colonies are a set of cowards and toadies." Thurlow's response was "Gardiner, I'd rather be one of the cowards and toadies that you call the colonists than such an unhappy mortal as yourself; you are always grumbling, or abusing somebody or something." And so it continued, with Thurlow calling Gardiner a Fenian, among other things. The Bench considered the case to be "not a serious one", but "found the defendant guilty, and sentenced him to pay a fine of 20s. and 4s. 6d. court costs, in default of payment, to be imprisoned in the lock-up for forty-eight hours." The fine was paid.

Gardiner continued working in Newcastle until the end of 1868.

4. Sydney again; City College

Gardiner returned to Sydney. On 2 June 1869 he presented three papers to the monthly meeting of the Royal Society of New South Wales — papers (h), (i) and (j) in the Appendix. The meeting and details of all three papers were reported in a number of newspapers in the following week.

He then embarked on his most ambitious educational venture. With Sheridan Moore, he made extensive plans for the City College, to be situated near Hyde Park, Sydney, and to open on 19 July 1869. The College was to have two departments, a School of Engineering, Surveying, and Architecture; and a Classical, English, and Commercial School. The former was advertised as "exclusively under the charge and direction of Mr. Martin Gardiner, C.E., Queen's University, Ireland, Mem. of Math. Society of London". Here we have the first mention of Gardiner as a member of the London Mathematical Society. Elizabeth Fisher, the Membership & Activities Officer there, has confirmed to me (personal communication) that Gardiner was elected as a member on 27 June 1867 and, at that meeting, "Prof. Hirst communicated a Paper 'On the determination of Double Entities in Uniquadric Homographics,' by Mr. Gardiner." I am grateful to Ms Fisher for bringing the paper to my attention; see Appendix (g).

Many of the advertisements included the college's prospectus, of which the following is a very small part.

> Mr. Gardiner obtained the highest distinction in Mathematics, Natural Philosophy, and Civil Engineering, during his University career, and has since then been professionally engaged as draughtsman, land-surveyor, engineer and surveyor on railways and other works. His elementary mathematical and engineering papers have been published by the Royal Societies of Sydney and Melbourne, and his recent researches in the higher branches have had the honour of publication by "The Mathematical Society" and "The Royal Society," of London.

Accompanying references included Morris Birkbeck Pell (1827–1879), first professor of mathematics in the University of Sydney, Sir James Cockle, and "Professor Townsend, (Trinity College Dublin.) Professor Hirst, (London University). Professor Cayley, (University of Cambridge) President of the Mathematical Society of London."

The City College was first advertised on 12 June, five weeks before it was due to open, but within a month or so, before it had opened, there was an unexplained falling out between Gardiner and Moore. Many of the later advertisements did not mention Gardiner, and for the opening on 19 July there was again no mention. The opening was in fact postponed for a week because of "the inclement weather", according to a newspaper advertisement.

On 7 August there was a further advertisement in *Freeman's Journal* for the City College, along the lines of some of the earlier ones but with Gardiner's name absent. In *The Sydney Morning Herald* on the same day, and often over the next few weeks, Gardiner advertised his own services separately: "Gentlemen instructed in all branches of Mathematics, Surveying, and Engineering. Proficiency guaranteed to articled pupils." The split was fully apparent by 19 August with an advertisement in *The Sydney Morning Herald* announcing that the "entire course" in the School of Civil Engineering, Surveying, and Architecture, of the City College, had been "re-constructed" under the direction of "EDWARD HUGHES, Esq., C.E., late Resident Engineer on the Punjaub Railways, formerly (for four years) Assistant Engineer on Brassey and Co.'s French Railway Contracts, Messrs. Grissell's Contracts, Great Yarmouth Bridge Works, Roads, &c."

Then the scheme collapsed entirely, and it cannot be claimed that it would have succeeded with Gardiner remaining as one of the headmasters.

Throughout this time, Gardiner's researches in geometry continued. The monthly meeting of the Royal Society of New South Wales for October 1869 was reported on as follows by *The Sydney Morning Herald*: "Mr. Martin Gardiner, C.E., then read to the members of the Royal Society a paper on 'Improved Solutions to Important Problems in Trigonometrical Surveying.' Mr. Gardiner demonstrated the principles he contended for in the paper by means of diagrams, and was listened to with great attention by all present. The subject treated of was evidently much appreciated, but was of rather too abstruse a character to be made intelligible in a popular form."

At the end of that year, 1869, the Borough of Balmain in Sydney announced Gardiner's appointment as Council Clerk and Surveyor, but the following August saw him declared insolvent and he was obliged to resign from his position.

5. Otago chair; divorce

The University of Otago in Dunedin, New Zealand, the country's first university, was established in 1869, and Gardiner was an applicant for the chair of Mathematics and Natural Philosophy. I have copies of four letters that he wrote to the University Council over the period 27 May to 31 August 1870. Garry Tee from the University of Auckland received these from David Murray, a university archivist in Otago, and forwarded them on, and I am grateful to both for this. The four letters are written from "Darling Street, Balmain", and in one, dated 18 June, there is the following remarkable passage.

> I think it proper, under present circumstances, to observe that the Professors of Mathematics & Natural Philosophy in the Sydney & Melbourne Universities have contributed nothing theoretical or practical to science in these colonies, or elsewhere, since their arrivals, nor, as a natural consequence, have the graduates or scholars done anything to give the universities a name or to prove them to be successful institutions.

To set the record straight, if that is required, Pell from Sydney University was "regarded as the most important commentator on mortality in Australia before 1900" (Lancaster [9]), and Wilson, a fervent astronomer, was instrumental in bringing what became known as the Great Melbourne Telescope to Australia in 1869 ([2], p. 56).

Two references accompanied Gardiner's application and they are worth reproducing in full. The first was from William Hearn at the University of Melbourne, dated 2 May 1870.

> Mr. Martin Gardiner, C.E., was a student in Queen's College, Galway, when I was connected with that institution. Mr. Gardiner, obtained the highest distinctions in the College, in its

> engineering department (including the higher branches of Mathematics and Physics) and subsequently obtained the diploma of Civil Engineer.
>
> I subsequently knew Mr. Gardiner in Victoria, when he was a member of the Royal Society. He pursued, often in circumstances of great difficulty and discouragement, his mathematical studies with great success.
>
> I do not consider myself competent to express an opinion on the value of Mr. Gardiner's papers, but from my general knowledge of his powers if he were able to devote his whole time without interruption to his favourite pursuits, he would be speedily known as one of the most successful Mathematicians of the day.

The other was from Sir James Cockle in Brisbane, dated 4 May 1870.

> I have to beg you to accept my apologies for not having written to you long since. Your letter dated April 21st, 1870 informs me that you are a candidate for the Professorship of Mathematics and Natural Philosophy in the University of Otago, New Zealand.
>
> It would give me pleasure to hear that you had succeeded in obtaining the appointment.
>
> From papers which you have sent me long since, and from a more recent one by you which I have seen in (I think) "The Quarterly Journal of Mathematics," I believe you have been and to be an able and persevering investigator in an abstruse and difficult portion of Mathematical Science. And in the fact of your having pursued these advanced studies, I find an ample warrant for sending you this testimonial accompanied by my best wishes for your success in your application, and that in such an appointment you may find a fitting and agreeable field for the exercise of your Mathematical talents.

The letters appear in the collected correspondence of Sir Henry Parkes.[14] With these is the letter from Clement Hodgkinson that we have seen before, but bearing the date "January 1860" rather than 9 June 1860 as before, and with a few other minor differences.[15] There is also a letter from T. Archer Hirst of University College, London, dated 1 January 1867, commending Gardiner's memoir, "Researches in the Geometry of Three Dimensions".

There were 62 applicants for the position in Otago, including nine specified as "From the Colonies". Success went to a Scot, John Shand, who is remembered in New Zealand mainly for his support for teaching and secondary education.

Two and a half years later, apparently living the whole while in Sydney, Gardiner was again granted an "insolvent's certificate",[16] and later that year he issued a notice: "I will not be responsible for any Debts contracted by any person whomsoever without my own authority. Martin Gardiner, C. E.".[17]

For Gardiner's wife Emma, it was much worse than financial hardship. On 22 October 1872, the *Evening News* reported that, in the Central Police Court, "Martin Gardiner was fined 40s, with 6s 6d costs of court and 21s professional costs, or seven days, for assaulting Emma Gardiner." Three days later, in the *The Sydney Morning Herald*, we find: "In Emma Gardiner v. Martin Gardiner, an order was made by consent of defendant for a weekly payment of 30s for the separate maintenance of his deserted wife." Two years later, Gardiner was on another charge before the Central Police Court for "assaulting and beating his wife Emma Gardiner." It was another nineteen years, December 1893, before Emma "commenced a suit" against Martin Gardiner for divorce on the ground of desertion.[18] In an appearance before the Supreme Court, she asserted that Gardiner "had deserted her since 1874, and that she had never heard from or of him from that date." The presiding Justice granted the decree nisi, to be made absolute after three months.

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[14] Parkes, *op. cit.*

[15] Such letters would, of course, have been reprinted by the recipient for subsequent distribution, and might well be altered in content.

[16] *Empire*, 2 April 1873.

[17] *The Sydney Morning Herald*, 23 September 1873.

[18] *NSW Government Gazette*, 8 December 1893, p. 9285.

6. Final wanderings

In fact, in 1874 Gardiner was in Brisbane, as evidenced by the following letter from Sir James Cockle to his equivalent Chief Justice in New South Wales, Sir Alfred Stephen, seeking Stephen's help to find employment for Gardiner.

> There is living in New South Wales, I believe in Sydney, Mr. Martin Gardiner, C.E. If it should happen to come in your way to be able to render him a service I think that such favour would be well bestowed. He is a mathematician of a very high order. It was only a day or two ago that I met with his name in an English mathematical publication and in connection not with an isolated problem, but with a recondite class of researches. I believe that he has contributed at least two papers to the *Transactions* of New South Wales. He is not however a theorist alone, but he is a practical man and visited Queensland lately in quest of professional employment. There was however no opening for him there and I failed in an effort I made for him. I then thought of some endeavor to interest the New South Wales officials in his behalf, but I gathered from what he said that he thought that your good word or influence would be more powerful ... Accordingly after some delay, caused in part by pressures of business, I write. And I do so because the picture of such a man as Mr. Martin Gardiner working hard for inadequate remuneration is one of the most striking instances that I know of how occasionally cruel Fortune is in her caprices. With but a fair chance I doubt not that he would make his mark in the world.[19]

Soon after, Gardiner was back in Melbourne. In May 1876, he presented to the Royal Society a paper on geodetic surveying, but "as, owing to its length, it would have taken several hours to read, Mr. Gardiner gave a *précis* of the contents of the paper, which he explained by diagrams."[20]

In Queensland again, a year or two later, there are some minor newspaper references suggesting that Gardiner worked as a surveyor until the middle of 1883. During 1883 and 1884, preliminary surveys were being undertaken for the famous Cairns Range Railway to Kuranda and Gardiner is listed as one "of the various officers who were directly responsible for the survey of and the building of the Range Railway." So is his son Charles Napoleon, who was also a licensed surveyor (Collinson [4]). The first section of the railway was completed in October 1887, but Gardiner was long gone from Queensland by then. He had travelled to Adelaide where he was noted as presenting to the South Australian Institute of Surveyors his works on "Practical Geodesy" and "Dynamics".[21] Soon after, he opened a "School of Civil Engineering and Surveying, and of Pure and Applied Mathematics". Advertisements for his new school appeared regularly until 30 August, and then no more was heard of the school, or of Gardiner in South Australia.

In the early 1890s, Gardiner was in Melbourne yet again. There was an advertisement in *The Argus* for "ELEMENTARY ORGANIC GEOMETRY (Preparatory to Quaternions), By Martin Gardiner, C.E., Member of the London Mathematical Society."[22] I cannot find any other mention of this book. On 10 March, according to *The Age*, he attended the Annual General Meeting of the Royal Society of Victoria and presented a paper there. The paper's title is not given in *The Age*, but it was presumably Appendix (v), his last mathematical publication.

The final newspaper references to Gardiner are from Perth, Western Australia. From *The West Australian* (18 April 1898), and similarly in other papers there in the following

[19]Correspondence from Sir James Cockle to Sir Alfred Stephen (23 December 1874) in "Public Men of Australia", call no. MS C4872 (State Library of NSW), pp. 202–205. I am grateful to Jim Tattersall for bringing this item to my attention.

[20]*The Argus*, 12 May 1876.

[21]*South Australian Register* and *The South Australian Advertiser*, 17 July 1884.

[22]*The Argus*, 17 January 1891.

days, we find that Gardiner was employed teaching elementary engineering in "the James-street Central School" and was again advertising classes of his own.[23]

That is all. As to Gardiner's death, the only possibility that has been found is in the West Australian Registry. A Martin Gardiner, engineer, died on 3 April 1899 at Helena Weir, Mundaring, about 50km east of Perth. His age at death is given as 60, which does not accord with being born in 1832 or 1833, but may be a common error of the times.

7. Other Irish–Australian mathematicians

The second such, after Martin Gardiner, was Henry Charles Kingsmill (1843–1909). He was born in Donegal, studied in Cambridge, and came to Australia in 1873. Kingsmill was instrumental in the formation of the University of Tasmania and held lecturing positions there in mathematics and surveying.

John Henry MacFarland (1851–1935) was also prominent in university administration, becoming chancellor of the University of Melbourne in 1918. He was born in Omagh, County Tyrone, and studied mathematics at Queen's College, Belfast, and then St John's College, Cambridge. MacFarland first went to the University of Melbourne, as Master of Ormond College, in 1881. He was knighted in 1919.

Then there was John Frederick Adair (1852–1913), from Dublin, who studied at Trinity College, Dublin, and then Pembroke College, Cambridge. He vied with Thomas Lyle (below) and William Henry Bragg to succeed Horace Lamb as Elder Professor of Mathematics in the University of Adelaide. (Bragg won, and was the winner in 1915 of the Nobel Prize in Physics, jointly with his son.) Adair won some fame as a cricketer playing for Ireland in 1883, before taking a demonstratorship in physics at the University of Sydney for three years.

Much better-known than these was Thomas Ranken Lyle (1860–1944), born in Coleraine. After distinguished study at Trinity College, Dublin, he came to Australia in 1889 as professor of natural philosophy in the University of Melbourne. He retired from that position in 1915 because of latent injuries received playing rugby for Ireland before going to Australia. Lyle was made a fellow of the Royal Society in 1912 and he gained a knighthood in 1922.

Hugh Davison Erwin (1879–1957), with a BA, BSc from the Royal University of Ireland, was, like Kingsmill, also involved with mathematics at the University of Tasmania.

There are no others, closely associated with Ireland and then actively associated with Australian mathematics, who are known to me up to the appointment of Vincent Hart to the University of Queensland in 1964. Hart was born in Hull, Yorkshire, in 1930 and moved to Cork in 1940.

All of these, except MacFarland, are discussed in some detail in *Counting Australia In*. See also Mulcahy [11] for an interview with Vincent Hart. I apologise for not having a comprehensive knowledge of others who would be more recent arrivals in Australia.

Appendix

The following are all the papers and pamphlets (and a book that never saw the light of day) by Martin Gardiner that I am aware of. These were listed in *Counting Australia In* ([2], pp. 387–388), except for (d), (f), (o) and (q), which were included in [3]. The titles of the "four papers" in (e) were not given in the former list.

(a) *Geometrical Papers by Martin Gardiner C.E.*, State Library of Victoria, Australian Manuscripts Collection (MS 9947, MSB125) (1857).

[23] *The West Australian*, 6 June 1898.

(b) *Improvements in fundamental ideas and elementary theorems of geometry*, Trans. Phil. Inst. Vic. (4) (1859), 76–96. Also published as a reprint by Mason and Firth, Melbourne (1859), 24 pages.
(c) *The "three sections," the "tangencies," and a "loci problem" of Apollonius, and porismatic developments*, Trans. Roy. Soc. Vic., (5) (1860), 19–89. Also published as a reprint by Mason and Firth, Melbourne (1860), 73 pages.
(d) *A paper concerning polygons inscribed in curves and surfaces of the second degree*, Quart. J. Pure Appl. Math., (7) (1866), 146–154, 284–301. With regard to this paper, Richard Townsend wrote, "A very elegant construction, at once simple, direct, and general, has recently been given for the case of n odd, by Mr. M. Gardiner, C.E., late Scholar of Queen's College, Galway, Ireland, and since Professor of Mathematics in St. John's College, Sydney, New South Wales." (See the footnote in Townsend [13].) Elizabeth Fisher of the London Mathematical Society informed me of this.
(e) *"Geometrical Researches" in four papers, comprising numerous new theorems and porisms, and complete solutions to celebrated problems. 1. Researches concerning figures particularly derived from other figures, 2. Researches concerning n'gons inscribed in other n'gons, 3. Researches concerning n'gons inscribed in curves of the second degree, 4. Researches concerning n'gons inscribed in surfaces of the second degree*, Trans. Phil. Soc. NSW, (1862–1865) (1866), 61–126.
(f) *On the inscription, by a simplification of Sir W. R. Hamilton's process of reduction, of closed n-gons in any quadric, so that the sides of each shall pass in order through n given points*, Proc. Lond. Math. Soc., (s1-2) (1866), 63–69.
(g) *Memoir on "Undevelopable uniquadric homographics"*, Proc. Roy. Soc. Lond., (16) (1867–68), 389–398.
(h) *Analytical solution to Sir William Hamilton's problem on the inscription of closed N'gons in any quadric*, Trans. Roy. Soc. NSW, (3) (1869), 38–41.
(i) *Important new theorem in the geometry of three dimensions*, Trans. Roy. Soc. NSW, (3) (1869), 41–42.
(j) *An exposition of the American method of levelling for sections—its superiority to the English and French methods as regards actual field practice and subsequent plotting of the section*, Trans. Roy. Soc. NSW, (3) (1869), 43–45.
(k) *Improved solutions to important problems in trigonometrical surveying*, Trans. Roy. Soc. NSW, (3) (1869), 129–133.
(l) *Properties of quadrics having common intersection, and of quadrics inscribed in the same developable, (being an extension of Chapter XVI. of Chasles' Conics)*, Quart. J. Pure Appl. Math., (10) (1870), 132–147.
(m) *On the solution of certain geodesic problems*, Trans. Roy. Soc. NSW, (7) (1873), 53–72.
(n) *On geodesic investigations*, Trans. Roy. Soc. NSW (7) (1873), 149–182.
(o) *Leichhardt* (cartographic material) (1874). A ms. map, 54 x 130cm, held by the State Library of NSW.
(p) *On practical geodesy*, Trans. Proc. Roy. Soc. Vic., (13) (1878), 1–66.
(q) *Disproof of the "Parallelogram of Rotative Velocities"*, The Week (Brisbane), (2 June 1883), 21.[24]
(r) *Solution to the celebrated fundamental question (hitherto unsolved) of dynamics*, Woodcock and Co., Brisbane (1883), 10 pages. This booklet was noted as received by the Royal Society of Queensland as its first "donation", Proc. Roy. Soc. Queensland, (1) (1884), 1.

[24]The argument is invalid, according to physicists I have consulted.

(s) *Solution to one of the most celebrated fundamental questions (hitherto unsolved) in dynamics*, Woodcock and Co., Brisbane (July 1883), 8 pages.
(t) *Determination of the motion of the solar system in fixed unalterable space*, Woodcock and Co., Brisbane (1883), 14 pages.
(u) Elementary organic geometry (preparatory to quaternions). Described as "Ready for Publication", *The Argus* (17 January 1891). Unseen; no other references.
(v) *On "confocal quadrics of moments in inertia" pertaining to all planes in space, and loci and envelopes of straight lines whose "moments of inertia" are constant*, Proc. Roy. Soc. Vic., (5) (1893), 200–208.

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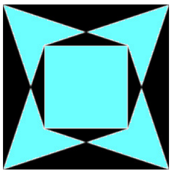

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Values of $f(G)$ for groups G of odd order with $\mathbf{Pr}(G) \geq 11/75$

ROBERT HEFFERNAN AND DESMOND MACHALE

Abstract. We augment the 2011 table of Das and Nath by finding all possible values of the commutativity ratio $f(G)$ for a finite group G of odd order, where another commutativity ratio $\Pr(G)$ satisfies $\Pr(G) \geq 11/75$.

1. Introduction

Throughout, let G be a finite group and let $\Pr(G)$ be the probability that two elements of G, chosen at random with replacement, commute with each other. Since $\Pr(G) = 1$ if and only if G is abelian, $\Pr(G)$ may be regarded as a commutativity ratio for groups. It is well known that $\Pr(G) = \frac{k(G)}{|G|}$, where G has $k(G)$ conjugacy classes. In 2011, Das and Nath [3] found all possible values of $\Pr(G)$ where $|G|$ is odd and $\Pr(G) \geq \frac{11}{75}$. They also found the structures for G', $G' \cap Z(G)$ and $G/Z(G)$ corresponding to each of these values of $\Pr(G)$.

We define $f(G)$ to be

$$\frac{1}{|G|} \sum_{i=1}^{k(G)} d_i$$

where d_i, $1 \leq i \leq k(G)$, are the degrees of the irreducible complex representations of G. Since $f(G) = 1$ if and only if G is abelian, $f(G)$ may also be regarded as a commutativity ratio for finite groups.

The commuting probability $\Pr(G)$ has been extensively studied [5, 9, 12, 10, 13, 11, 4] and the ratio $f(G)$ has also been considered by several authors [8, 7, 1, 13].

One's intuitive feeling is that if the values of one commutativity ratio $\Pr(G)$ for a given set of groups are 'large', then the values of another commutativity ratio $f(G)$ should be 'large' also. For the groups G of odd order with $\Pr(G) \geq \frac{11}{75}$, we find the corresponding values of $f(G)$ and show that if $\Pr(G) \geq \frac{11}{75}$, then $f(G) > \frac{15}{75}$.

In general

$$(f(G))^2 \leq \Pr(G) \leq f(G)$$

with equality if and only G is abelian [2].

We note that, for non-abelian G, saying $\Pr(G)$ and $f(G)$ are 'large' is another way of saying that G is close to being abelian.

Finally, it is clear that $\Pr(G) = 1 = f(G) = |G'| = |G/Z(G)|$ if and only if G is abelian and this corresponds to row 1 of the table in [3]. So, from now on we may assume that G is non-abelian of odd order.

We employ Philip Hall's very useful concept of isoclinism [6], which is not specifically mentioned in [3]. Two groups H and K are said to be *isoclinic* if there exist isomorphisms $\theta : H/Z(H) \to K/Z(K)$ and $\phi : H' \to K'$ such that the isomorphism ϕ is induced by the isomorphism θ. Isoclinism is an equivalence relation on finite groups and

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the isoclinism classes are called families. Each family contains a stem group G, with the property that $G' \supseteq Z(G)$. Thus, for a stem group G we have $G' \cap Z(G) = Z(G)$ and $|G| = |Z(G)||G/Z(G)| = |G' \cap Z(G)||G/Z(G)|$ and these values of the orders of stem groups can be read off from the following table taken from [3]:

Row	$\Pr(G)$	G'	$G' \cap Z(G)$	$G/Z(G)$	$f(G)$
1	1	$\{1\}$	$\{1\}$	$\{1\}$	1
2	$\frac{1}{3}\left(1+\frac{2}{3^{2s}}\right)$	C_3	C_3	$(C_3 \times C_3)^s$	$\frac{3^{2s}+2}{3^{2s+1}}$
3	$\frac{1}{5}\left(1+\frac{4}{5^{2s}}\right)$	C_5	C_5	$(C_5 \times C_5)^s$	$\frac{5^{2s}+4}{5^{2s+1}}$
4	$\frac{5}{21}$	C_7	$\{1\}$	$C_7 \rtimes C_3$	$\frac{3}{7}$
5	$\frac{55}{343}$	C_7	C_7	$C_7 \times C_7$	$\frac{13}{49}$
6	$\frac{17}{81}$	C_9 or $C_3 \times C_3$	C_3	$(C_3 \times C_3) \rtimes C_3$	$\frac{11}{27}$
6A	$\frac{17}{81}$	$C_3 \times C_3$	$C_3 \times C_3$	$C_3 \times C_3 \times C_3$	$\frac{11}{27}$
7	$\frac{121}{729}$	$C_3 \times C_3$	$C_3 \times C_3$	$C_3 \times C_3 \times C_3 \times C_3$	$\frac{25}{81}$
8	$\frac{7}{39}$	C_{13}	$\{1\}$	$C_{13} \rtimes C_3$	$\frac{5}{13}$
9	$\frac{3}{19}$	C_{19}	$\{1\}$	$C_{19} \rtimes C_3$	$\frac{7}{19}$
10	$\frac{29}{189}$	C_{21}	C_3	$C_3 \times (C_7 \rtimes C_3)$	$\frac{23}{63}$
11	$\frac{11}{75}$	$C_5 \times C_5$	$\{1\}$	$(C_5 \times C_5) \rtimes C_3$	$\frac{9}{25}$

We aim to justify the values of $f(G)$ appearing in the final column of this augmented table. Both $\Pr(G)$ and $f(G)$ are isoclinic invariants [10, 2], so we may confine our attention in general to the case where G is a stem group.

2. Values of $f(G)$

Consider the unique non-abelian group G_{pq} of order pq, where $p < q$ are odd primes and p divides $q - 1$.

It is easy to see that $Z(G_{pq})$ is trivial and that $|G_{pq} : G'_{pq}| = p$, since the Sylow q-subgroup is normal with abelian factor group. Furthermore, each representation of G_{pq} has degree 1 or p, since the Sylow q-subgroup is normal and abelian.

Routine calculations show that G_{pq} has $p + (q-1)/p$ conjugacy classes so that

$$\Pr(G_{pq}) = \frac{p^2 + q - 1}{p^2 q}.$$

The degree equation

$$|G| = \sum_{i=1}^{k(G)} d_i^2$$

of G_{pq} is now given by

$$|G_{pq}| = p + \left[\frac{q-1}{p}\right] p^2$$

so

$$f(G_{pq}) = \frac{p + [(q-1)/p]\, p}{pq} = \frac{p + q - 1}{pq}.$$

We are now in a position to fill in the values of $f(G)$ for several rows of the table.

Row 4. $\Pr(G) = \frac{5}{21}$; a stem group G has order $21 = 3 \cdot 7$, so $f(G) = \frac{7+3-1}{7 \cdot 3} = \frac{9}{21} = \frac{3}{7}$.

Row 8. $\Pr(G) = \frac{7}{39}$; a stem group G has order $39 = 3 \cdot 13$, so $f(G) = \frac{3+13-1}{3 \cdot 13} = \frac{15}{39} = \frac{5}{13}$.

Row 9. $\Pr(G) = \frac{3}{19}$; a stem group G has order $57 = 3 \cdot 19$, so $f(G) = \frac{3+19-1}{3 \cdot 19} = \frac{7}{19}$.

Row 11. $\Pr(G) = \frac{11}{75}$; a stem group has order $|Z(G)||G/Z(G)| = 75$ and is the unique non-abelian group of this order. Since the Sylow 5-subgroup is abelian, normal and of index 3, each $d_i = 1$ or 3 for all i. Thus G has eleven conjugacy classes, so the degree equation can only be

$$75 = 1 + 1 + 1 + 8 \cdot 3^2.$$

Thus, $f(G) = \frac{3+8\cdot 3}{75} = \frac{27}{75} = \frac{9}{25}$.

Row 10. $\Pr(G) = \frac{29}{189}$; a stem group G has order 189, has 29 conjugacy classes and $|G : G'| = \frac{189}{21} = 9$. The degree equation can only be

$$189 = 9 \cdot 1 + 20 \cdot 3^2.$$

So, $f(G) = \frac{9+20\cdot 3}{189} = \frac{23}{63}$.

Row 6. $\Pr(G) = \frac{17}{81}$; a stem group G has order 81 and 17 conjugacy classes. We have $|G : G'| = 9$, so the only possible degree equation is

$$81 = 9 \cdot 1^2 + 8 \cdot 3^2.$$

Thus $f(G) = \frac{9+8\cdot 3}{81} = \frac{11}{27}$.

Row 6A. $\Pr(G) = \frac{17}{81} = \frac{51}{243}$; a stem group has order $27 \cdot 9 = 243$ and 51 conjugacy classes. $|G'| = 9$, so $|G : G'| = 27$ and there are 24 other conjugacy classes. The only possible degree equation is

$$243 = 27 \cdot 1^2 + 24 \cdot 3^2,$$

so $f(G) = \frac{27+24\cdot 3}{243} = \frac{11}{27}$.

Note that rows 6 and 6A are an example of different families which have the same $\Pr(G)$ and $f(G)$ values.

Row 5. $\Pr(G) = \frac{55}{343}$; a stem group G has order $7^3 = 343$ and 55 conjugacy classes. $|G'| = 7$, so $|G : G'| = 49$ and there are 6 other classes. Thus the only possible degree equation is

$$343 = 49 \cdot 1^2 + 6 \cdot 7^2$$

and $f(G) = \frac{49+6\cdot 7}{343} = \frac{13}{49}$.

Row 7. $\Pr(G) = \frac{121}{729}$; a stem group G has order $3^2 \cdot 3^4 = 729$. $|G'| = 9$ and $|G : G'| = 81$. G has 40 other classes.

Now, $81 + 40 \cdot 9 < 729$, so we must consider the possibility that G has representations of degrees 3 and 9. Thus the degree equation is

$$729 = 81 + a \cdot 3^2 + b \cdot 3^4$$

for some non-negative integers a and b. We get $9a + 81b = 648$ and $a + b = 40$. This gives $a = 36$ and $b = 4$. So, the degree equation is

$$729 = 81 + 36 \cdot 3^2 + 4 \cdot 3^4.$$

Thus

$$f(G) = \frac{81 + 36 \cdot 3 + 4 \cdot 9}{729} = \frac{25}{81} = \left(\frac{5}{9}\right)^2.$$

Now all that remains is to examine the extra-special 3-group and 5-group cases.

Row 2. $\Pr(G) = \left(\frac{1}{3}\right)\left(1 + \frac{2}{3^s}\right)$, $s \geq 1$. Here $|G'| = 3$, $|G' \cap Z(G)| = |Z(G)| = 3$ and $|G/Z(G)| = 3^{2s}$. So, a stem group has order 3^{2s+1}. Now, $|G : G'| = \frac{3^{2s+1}}{3} = 3^{2s}$ and

$$\Pr(G) = 3^{2s}\left(\frac{1 + 2/3^{2s}}{3^{2s+1}}\right) = \frac{3^{2s}+2}{3^{2s+1}}.$$

So G has $3^{2s}+2$ classes, so we have two extra classes to consider. The degree equation can only be

$$3^{2s+1} = 3^{2s} + (3^s)^2 + (3^s)^2 .$$

So

$$f(G) = \frac{3^s+2}{3^{s+1}}$$

after simplification.

Row 3. $\Pr(G) = \frac{1}{5} + \frac{4}{5^{2s+1}}$. In like manner to the above, we find

$$f(G) = \frac{5^s+4}{5^{s+1}}.$$

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A reflection property in Minkowski planes

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Abstract. With the reflection property investigated here we mean a generalization of the following task, also called Heron's problem: Minimize the length of a path joining two points on one side of a line in the Euclidean plane such that this path meets also this line. We present an extension of this reflection property to (normed or) Minkowski planes and use this generalization to discuss also corresponding reflection properties of Minkowskian conics.

1. Introduction

The concept of Minkowski distance defined by means of a convex body centered at the origin was developed by H. Minkowski [21], yielding the notion of *Minkowski spaces*; these are simply finite dimensional real Banach spaces with the planar subcase of *Minkowski planes*. The geometry of such spaces and planes is usually called *Minkowski geometry*, see the monograph [31]. The articles [3] by Busemann, [26] by Petty, the surveys [19] and [18], Chapter 6 from [2] and Chapter 4 from [32] as well as the whole monograph [31] contain useful background material reflecting main directions of Minkowski geometry and also those parts of classical convexity which are needed for it.

In this paper we will deal with the extension of Heron's problem to Minkowski planes and conic sections there. In the Euclidean plane, Heron's problem asks for minimizing the length of a path that joins two points on one side of a line and should meet also this line. Using Fermat's principle of least time and the fact that in a homogeneous medium the time travelled is proportional to the distance travelled one obtains the reflection principle as follows: Consider two points u, v lying in one of the open halfplanes determined by a line $\mathcal{L}$ in the Euclidean plane. A point w on $\mathcal{L}$ such that the Euclidean sum of distances $\|u-w\|_e + \|v-w\|_e$ is minimum has the property that the reflection of a light ray, say from u to w, will pass through v. The angle of incidence is equal to the angle of reflection. Our main objective is it to extend this reflection property to normed planes and to apply it to corresponding conics.

A *convex body* in the Euclidean plane is a compact, convex set having non-empty interior. Any convex body E centered at the origin can be taken to define a *Minkowskian distance* from x to y by

$$\|x-y\| = \frac{\|x-y\|_e}{r}.$$

Here $\|x-y\|_e$ is the *Euclidean distance* from x to y, and r is the value of the Euclidean radial function of E in the direction of the vector $y-x$. (The Euclidean *radial function* is the function on $\mathbb{R}^2$ whose value at each oint z depends only on the distance of z and the origin o.) We will call the standard plane equipped with this new metric a (*normed* or) *Minkowski plane*, having E as *unit circle*.

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2. Preliminaries

Assuming that the unit circle E of a Minkowski plane is smooth and strictly convex, we give in the next section an extension of the reflection principle, and in Section 4 we will discuss the reflection property of conics in Minkowski planes. Note that E is called *smooth* if through any boundary point of E a unique supporting line passes; and it is called *strictly convex* if the boundary of E contains no line segments. Consider a Minkowski plane with unit circle E. An *E-ellipse* is the locus of all points whose sum of Minkowskian distances from two fixed points is constant. An *E-hyperbola* is the locus of all points whose difference of Minkowskian distances from two fixed points is constant, and an *E-parabola* is the locus of all points which are equidistant from a given point and a given line. Both E-ellipses and E-parabolas bound convex regions. Minkowskian analogues of the reflection properties for an ellipse and a hyperbola and the focal property for a parabola are given in Section 4. We refer also to [12], [16], and [14] for various results and properties of Minkowskian analogues of conics.

The following is needed in Section 4. Given a line $\mathcal{L}$ and a point $u \notin \mathcal{L}$, for $v \in \mathcal{L}$ the direction $v - u$ is called *transversal to $\mathcal{L}$* provided $\|u - v\| = \min\|u - w\|$ for all $w \in \mathcal{L}$. It is very easy to see that if a homothetic copy of the unit circle E centered at u *touches* $\mathcal{L}$ at a point v, then $u - v$ is transversal to $\mathcal{L}$. (Note that *homotheties* are transformations with a fixed point x sending each $m \neq x$ to a point n such that $n - s$ is on the same line as $m - s$, but scaled by a real factor λ.) This natural type of transversality is usually called *Birkhoff orthogonality*, see the related expository paper [1].

We finish this preliminary part by proving a lemma needed in Section 4 and saying that the sum of distances from m fixed points in a normed space is a convex function.

Lemma 2.1. *Consider m points $v_1, v_2, ..., v_m$ in a normed space X. The function f defined by $f(x) = \sum_{i=1}^{m} \|x - v_i\|$ is convex.*

Proof. The statement holds since the norm function is convex, and the sum of convex functions is convex, too. ☐

We remark that the level sets of the function discussed here occur as so-called polyellipses or multifocal ellipses and their higher-dimensional analogues, i.e., as respective generalizations of ellipsoids having m foci (see [14]).

3. A reflection property

Our main objective in this section is to prove the following theorem giving the Minkowskian analogue of Heron's problem (see Figure 1, where the Euclidean subcase is shown).

Theorem 3.1. *Consider two points u, v lying in one of the open halfplanes determined by a line $\mathcal{L}$ in a Minkowski plane with continuously differentiable boundary of the unit circle E. A point p on $\mathcal{L}$ such that*

$$\|u - p\| + \|v - p\| = \min_{q \in \mathcal{L}}\{\|u - q\| + \|v - q\|\}$$

has the following reflection property: Let a homothetic copy E' of the unit circle E, which is centered at the point p, intersect the line segments joining p to u and v in the points u' and v', respectively. Let u'' and v'' be intersections of the tangent lines to E' at u' and v' with $\mathcal{L}$. Then

$$\|p - u''\|_e = \|p - v''\|_e \quad (\textit{and so } \|p - u''\| = \|p - v''\|)\,.$$

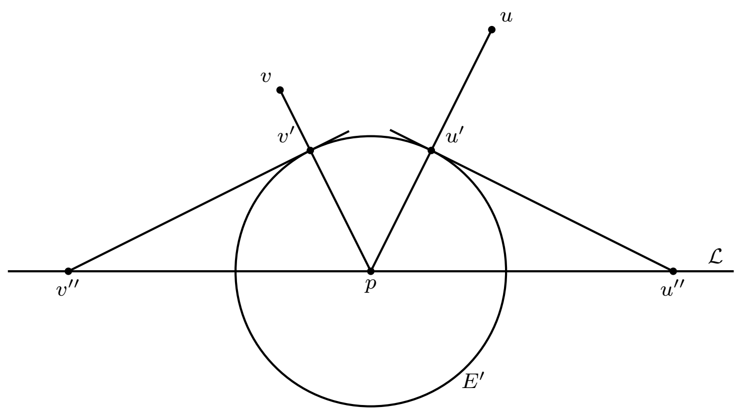

FIGURE 1. E-reflection property

We will use the following Lemma 3.2 (the proof of which is a nice exercise in calculus) to prove Theorem 3.1.

Lemma 3.2. *Let $r = g(\vartheta)$ describe a differentiable curve C in polar coordinates. Then the x-intercept of the tangent line at a point (r, ϑ) is given by*

$$x = \frac{r^2}{r\cos\vartheta + r'\sin\vartheta}\,,$$

where

$$r' = \frac{dg}{d\vartheta}\,.$$

Proof of Theorem 3.1. The function $f(q) = \|q-u\| + \|q-v\|, q \in \mathcal{L}$, is convex and unbounded. Hence f has a minimum on at least one point $p \in \mathcal{L}$. That is, there exists $p \in \mathcal{L}$ with $f(p) \leq f(q)$ for all $q \in \mathcal{L}$. Since the boundary of E is of class C^1, $f(q) = \|q-u\| + \|q-v\| = F(E, q-u) + F(E, q-v)$ is of class C^1. Therefore the function $f = f|_{\mathcal{L}}$ is of class C^1, and the method of Lagrange multipliers can be applied.

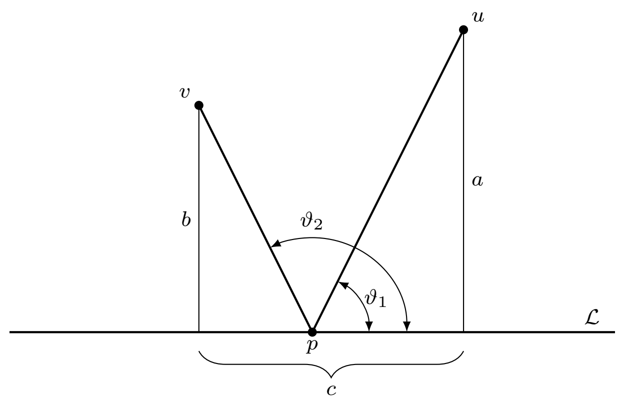

FIGURE 2. For the proof of Theorem 3.1

Using some trigonometry and the definitions of the Minkowski metric, Theorem 3.1 is equivalent to the following constraint optimization problem (see Figure 2):

$$\text{Minimize} \quad F(\vartheta_1, \vartheta_2) = \frac{a\csc\vartheta_1}{r(\vartheta_1)} + \frac{b\csc\vartheta_2}{r(\vartheta_2)}\,, \tag{1}$$

$$\text{subject to} \quad G(\vartheta_1, \vartheta_2) = a \cot \vartheta_1 - b \cot \vartheta_2 - c = 0\,, \tag{2}$$

with a, b, c as shown in Figure 2.

The Lagrangian is denoted by J and given by

$$J(\vartheta_1, \vartheta_2, \lambda) = \frac{a \csc\vartheta_1}{r(\vartheta_1)} + \frac{b \csc\vartheta_2}{r(\vartheta_2)} - \lambda\,(a \cot \vartheta_1 - b \cot \vartheta_2 - c)\,. \tag{3}$$

Setting $\frac{\partial J}{\partial \vartheta_1}$, $\frac{\partial J}{\partial \vartheta_2}$, and $\frac{\partial J}{\partial \lambda}$ equal to zero, we obtain

$$\frac{\partial J}{\partial \vartheta_1} = \frac{-a \csc\vartheta_1 \cot \vartheta_1}{r(\vartheta_1)} - \frac{a \csc\vartheta_1 r'(\vartheta_1)}{r^2(\vartheta_1)} + \lambda a \csc^2\vartheta_1 = 0\,, \tag{4}$$

$$\frac{\partial J}{\partial \vartheta_2} = \frac{-b \csc\vartheta_2 \cot \vartheta_2}{r(\vartheta_2)} - \frac{b \csc\vartheta_2 r'(\vartheta_2)}{r^2(\vartheta_2)} - \lambda b \csc^2\vartheta_2 = 0\,, \tag{5}$$

$$\frac{\partial J}{\partial \lambda} = -a \cot \vartheta_1 + b \cot \vartheta_2 + c = 0\,. \tag{6}$$

Dividing equations (4) and (5) by $a \csc^2\vartheta_1$ and $b \csc^2\vartheta_2$, respectively, and using Lemma 3.2, we obtain

$$\lambda = \frac{r(\vartheta_1) \cos \vartheta_1 + r'(\vartheta_1) \sin \vartheta_1}{r^2(\vartheta_1)} = \frac{1}{\|p - u''\|_e}\,, \tag{7}$$

$$\lambda = -\frac{r(\vartheta_2) \cos \vartheta_2 + r'(\vartheta_2) \sin \vartheta_2}{r^2(\vartheta_2)} = \frac{1}{\|p - v''\|_e}\,, \tag{8}$$

where u'' and v'' are the x-intercepts of tangent lines to a copy of the unit circle centered at p. Points of tangency are intersections of line segments joining p to u and v in the points u' and v', with the unit circle. Hence $\|p - u''\|_e = \|p - v''\|_e$, and consequently $\|p - u''\| = \|p - v''\|$. □

The reflection property for the Euclidean case is a special case of Theorem 3.1. If E has a vertical axis of symmetry, then the angle of incidence will be equal to the angle of reflection. In the next section we use the above reflection property to discuss the reflection property of conics.

Hawkins [9] used variational techniques to find a generalization of Snell's law of refraction for media, where the speed of light depends only on the direction at each point. He also used the method of Lagrange multipliers to generalize Snell's law, taking a constraint optimization problem similar to that in the proof of Theorem 3.1 into consideration. Ghandehari and Golomb [8] have done similar work.

By methods from convex analysis, Heron's problem was generalized in [22] where the sum of distances to m given closed convex sets is studied. From the numerical point of view, generalizations of Heron's problem are investigated in [5]. An extension to Banach spaces is presented in [23], and we also mention the paper [11] containing related results for Hilbert spaces.

4. Conics

In this section we define conics in the Minkowski plane and analyze their reflection properties. The excellent book by Hilbert and Cohn-Vossen has a good introduction to conics in the Euclidean plane, and for Minkowskian conics we refer to [12] and [16].

Suppose E is a smooth, convex and compact body in the plane inducing a Minkowski norm $\|\cdot\|$. If u and v are distinct points and $\mathcal{L}$ is a line, we say that a point $p \in \mathcal{L}$ has the *E-reflection property* with respect to u, v, and $\mathcal{L}$ if $\|u - p\| + \|v - p\| = \min_{z \in \mathcal{L}}\{\|u - z\| + \|v - z\|\}$.

In the Euclidean plane, the line segments joining the two foci to a point on an ellipse will make equal angles with the tangent line to the ellipse at the point of tangency. This is called the *reflection property of an ellipse.* A short and elegant proof of the reflection property of an ellipse using derivatives is given in Schulz and Moore [28]. Recall that we defined an E-ellipse in a Minkowski plane with unit circle E as the locus of all points whose Minkowskian sum of distances from two fixed points is constant. We shall not use the definition of an ellipse by using eccentricity here, but mention that Tamássy and Bélteky [30] showed that if in a Minkowski plane the characterization of an ellipse in terms of sums of distances coincides with that via eccentricity, then the plane is Euclidean.

The following Theorem 4.2 concerns the Minkowskian analogue of the reflection property of an E-ellipse. The proof is the same as in the Euclidean case. Before stating and proving Theorem 4.2, we need the following lemma.

Lemma 4.1. *The region bounded by an E-ellipse is convex.*

Proof. Let $D = \{x \in \mathbb{R}^2 | \, \|x-u\| + \|x-v\| \leq d\}$, where u and v are the foci and d is the constant sum of distances of points of the E-ellipse from u and v. Let $f(x) = \|x-u\| + \|x-v\|$. By Lemma 2.1 we have that if $x \in D, y \in D$, then, for $0 \leq \lambda \leq 1$,

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y) \leq \lambda d + (1-\lambda) d = d\,.$$

Hence D is convex. □

We note that the unit circle E is strictly convex if and only if E-ellipses are strictly convex (see [12]).

Theorem 4.2. *Let u and v be foci of an E-ellipse in a Minkowski plane with smooth and strictly convex unit circle E. Let $\mathcal{L}$ be the tangent line to the E-ellipse at a point p. Then p has the E-reflection property with respect to u, v, and $\mathcal{L}$ (see Figure 3 for the Euclidean subcase).*

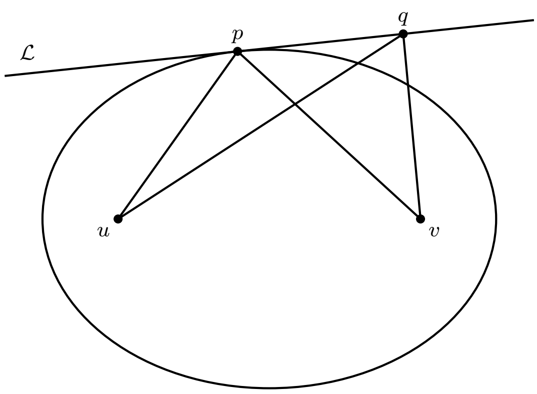

Figure 3. Reflection property of an E-ellipse

Proof. Since the E-ellipse is a convex curve, any point $q \in \mathcal{L}, q \neq p$, lies outside of the E-ellipse. Then $\|q-u\| + \|q-v\| \geq \|p-u\| + \|p-v\|$. Hence p has the E-reflection property. □

In what follows, Theorem 4.3 deals with a property of a point p on the unit circle such that $\|p-u\| + \|p-v\|$ is a minimum for given points u, v in a Minkowski plane. Theorem 4.4 is based on Theorem 4.3 and gives a property of the *Fermat-Torricelli point* for three given points u, v, and w (i.e., of the unique point having minimal sum of distances to u, v, and w). Theorem 4.5 is a generalization of Theorem 3.1 where, given a point u and two lines $\mathcal{L}_1$ and $\mathcal{L}_2$, we find points $p \in \mathcal{L}_1$ and $v \in \mathcal{L}_2$ such that

$\|p-u\|+\|p-v\|$ is minimum. We then use Theorem 4.5 to prove Theorem 4.7, which is the focal property for an E-parabola. Theorem 4.8 gives the reflection property for an E-hyperbola, and we finish the article with a conjecture referring to confocal ellipses.

Theorem 4.3. *Consider two points u and v such that the line segment connecting u and v lies outside of a homothetic copy of the strictly convex unit circle E. A point p on the copy of E minimizes $\|u-p\|+\|v-p\|$ if and only if p has the E-reflection property with respect to u, v, and the line $\mathcal{L}$ tangent to the copy of E at p (see Figure 4).*

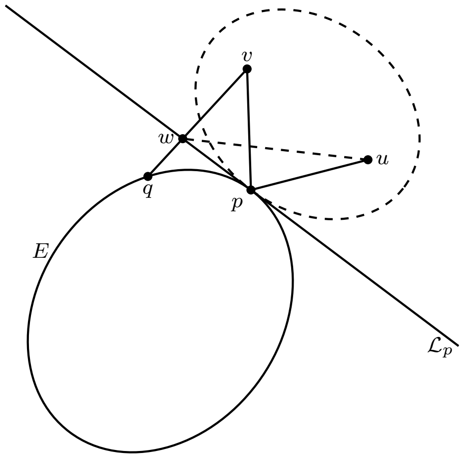

FIGURE 4. p minimizes $\|p-u\|+\|p-v\|$

Proof. We first prove that if $p \in E$ such that $\|u-p\|+\|v-p\| = \min_{z\in\mathcal{L}_p}\{\|u-z\|+\|v-z\|\}$ with $\mathcal{L}_p$ tangent to E at p, then $\|u-p\|+\|v-p\| = \min_{q\in E}\{\|u-q\|+\|v-q\|\}$. For any $q \in E$, let w be the intersection of the line segment, connecting q and v outside of E, with the tangent line $\mathcal{L}_p$ at p (see again Figure 4). Then $\|u-q\|+\|q-v\| = \|u-q\|+\|q-w\|+\|w-v\|$, since $\mathcal{L}_p$ meets the line segment $[q, v]$. This follows from the assumption on p formulated at the beginning of the proof. Further on, we have $\|u-q\|+\|q-w\|+\|w-v\| \geq \|u-w\|+\|w-v\| \geq \|u-p\|+\|v-p\|$, where the last inequality follows since $w \in \mathcal{L}_p$ and $\|p-u\|+\|v-p\| = \min_{z\in\mathcal{L}_p}\{\|z-u\|+\|v-z\|\}$. Hence, if p has the E-reflection property with respect to u, v, and the tangent line $\mathcal{L}_p$ at p, then $\|u-p\|+\|v-p\| = \min_{q\in E}\{\|u-q\|+\|v-q\|\}$.

If $\|u-p\|+\|v-p\| = \min_{q\in E}\{\|u-q\|+\|v-q\|\}$, then $\|u-p\|+\|v-p\| = \min_{z\in\mathcal{L}_p}\{\|u-z\|+\|v-z\|\}$ as follows: The E-ellipse, with foci u and v and the constant sum of distances equal to the minimum, intersects E at only one point. If there are two points of intersection, then for any x inside the line segment joining two points of intersection, x is inside E since E is strictly convex. Then $\|u-x\|+\|v-x\| \leq \|u-p\|+\|v-p\|$ since E is strictly convex. Let y be some chosen point from the intersection of the line segment joining x to v with E. Then $\|u-x\|+\|v-x\| = \|u-x\|+\|x-y\|+\|y-v\| > \|u-y\|+\|u-v\| \geq \|u-p\|+\|v-p\|$, a contradiction. Hence there is only one point p of intersection. The tangent line to E at p is also tangent to the E-ellipse. By Theorem 4.2, p has the E-reflection property with respect to u, v, and the tangent line $\mathcal{L}_p$. $\square$

A problem related to the reflection property is the so-called *Fermat-Torricelli problem.* In its simplest form, it asks for a point in the Euclidean plane minimizing the sum of the distances to three given points. In the article [4] the Fermat-Torricelli problem in Minkowski spaces is investigated, see also [20].

Theorem 4.4. *Consider three points $u, v,$ and w in a given Minkowski plane with smooth and strictly convex unit circle E. Let p be the Fermat-Torricelli point, assuming it exists and is different from $u, v,$ or w. Let u', v' and w' be the intersections of a copy of E centered at p with the line segments joining p to $u, v,,$ and w. Then the triangle formed by drawing tangents to E at u', v', and w' has p as its centroid.*

Proof. Consider a copy of the unit circle E centered at w and passing through p. Since $\|u - p\| + \|v - p\|$ is minimized as p varies on this copy of the unit circle, p has the E-reflection property with respect to u, v, and the tangent line $\mathcal{L}_1$ to this copy of the unit circle. We now consider another copy of the unit circle, centered at p and passing through w. Let u' and v' be the intersection with line segments joining p to u and v, respectively. Let $\mathcal{L}_2$ be a line tangent to a copy of the unit circle at w with $\mathcal{L}_2 \| \mathcal{L}_1$. By the E-reflection property (Theorem 3.1), the line segment passing through p and parallel to $\mathcal{L}_2$ is bisected by the sides of the triangle formed by the tangents. Using the same argument for u and v we see that the line segment passing through p and being parallel to $\mathcal{L}_2$ is bisected by the sides of the triangle formed by tangents. Using the same argument for u and v we see that the line segments passing through p and parallel to the sides of the triangle formed by the tangents are bisected. It is an elementary but interesting exercise to show that the point p has to be the centroid. □

The excellent book by Courant and Robbins [6] has a treatment of extremal distances in the Euclidean plane. Similar results for Minkowski planes can be obtained. We will use the following theorem to discuss the focal property of parabolas.

Theorem 4.5. *Consider a point u and two lines $\mathcal{L}_1$ and $\mathcal{L}_2$ in a Minkowski plane with smooth and strictly convex unit circle E. If $\|v - p\| + \|u - p\|$ is a minimum for $p \in \mathcal{L}_1$ and $v \in \mathcal{L}_2$, then p has the E-reflection property with respect to $u, v,$ and $\mathcal{L}_1$.*

Proof. If $p \in \mathcal{L}_1$ and $v \in \mathcal{L}_2$ attain this minimum, then the minimum of $\|u - z\| + \|v - z\|$ as z runs along $\mathcal{L}_1$ is attained at $z = p$.

Hence p has the E-reflection property with respect to u, v, and $\mathcal{L}_1$. □

For a parabola in the Euclidean plane, the path of reflection of a light ray starting from the focus and going to the boundary is called its *focal property.* Theorem 4.7 below gives an analogue of this focal property in the Minkowski plane. Before this, we show that the region bounded by an E-parabola is convex.

Lemma 4.6. *An E-parabola bounds a convex region.*

Proof. Let $\mathcal{L}$ and u be the line and the point generating the E-parabola in the classical way. Let $D = \{x | \|x - u\| \leq \rho(x, \mathcal{L})\}$ where $\rho(x, \mathcal{L})$ is the distance of x from $\mathcal{L}$. If $x, y \in D$ and $0 \leq \lambda \leq 1$, then $\|\lambda x + (1 - \lambda)y - u\| \leq \lambda\|x - u\| + (1 - \lambda)\|y - u\| \leq \lambda\rho(x, \mathcal{L}) + (1 - \lambda)\rho(y, \mathcal{L}) = \rho(\lambda x + (1 - \lambda)y, \mathcal{L})$. The last equality follows from the fact that transversal directions to $\mathcal{L}$ are all parallel. □

Laatsch [17] gave an interesting treatment of pyramidal sections in taxicab geometry which is a special case of Minkowski geometry, with unit circle E a square centered at the origin and diagonals on the x- and y-axes. For other references on conics in Taxicab geometry see the articles by Iny [13], Moser and Kramer [24], Reynolds [27], and Sowell [29]. For a computerized approach to conics with Taxicab metric we refer to Natsoulas [25].

The following theorem gives the focal property of a Minkowskian parabola, which is generated as locus of all points equidistant from a given point (focus) and a given line (directrix). For a short geometric proof of the focal property of the parabola in the Euclidean plane, see Williams [33].

Theorem 4.7. *(Focal property). For an E-parabola with focus u and directrix $\mathcal{L}$, let $\mathcal{L}'$ be any line parallel to $\mathcal{L}$ such that u is between $\mathcal{L}$ and $\mathcal{L}'$. For any point p on the E-parabola, choose $v \in \mathcal{L}'$ such that $p - v$ is transversal to $\mathcal{L}'$. Then p has the E-reflection property with respect to u, v, and the tangent line to the E-parabola at p.*

Proof. Let $\mathcal{L}''$ be the tangent line to the E-parabola at p. For $q \in \mathcal{L}''$, choose $w \in \mathcal{L}'$ such that $q - w$ is transversal to $\mathcal{L}'$. Let q' be the intersection of the line segment joining q to u with the E-parabola. Choose w' on $\mathcal{L}'$ such that $q' - w'$ is transversal to $\mathcal{L}'$. Then

$$\|u-q\| + \|q-v\| \geq \|u-q\| + \|q-w\| \geq \|u-q'\| + \|q'-w\| \geq$$
$$\|u-q'\| + \|q'-w'\| = \|u-p\| + \|p-v\|\,,$$

where the first inequality follows from the triangle inequality for the points q', q, w, and the second inequality follows since $q'w'$ is transversal to $\mathcal{L}'$. The last equality holds since the points p and q' both lie on the E-parabola. Thus, p has the E-reflection property with respect to u, v, and $\mathcal{L}''$. □

The following theorem gives the reflection property of an E-hyperbola. Chapter 3 of the book by Kazarinoff [15] contains a good treatment of reflection properties in the Euclidean plane. In particular, a proof of the reflection property for a hyperbola is given there. The book by Courant and Robbins [6] contains a treatment of extremal distances in the Euclidean plane and a nice discussion of Heron's reflection principle and the reflection property of conics.

Theorem 4.8. *Let u and v be two foci of an E-hyperbola in a Minkowski plane with smooth and strictly convex unit circle E. Let p be a point on a branch of the hyperbola containing the focus v. Assume v' is the reflection of v through p. Consider a line $\mathcal{L}$ through p such that p has the E-reflection property with respect to u, v', and $\mathcal{L}$. Then $\mathcal{L}$ is tangent to the hyperbola at p (see Figure 5 for the Euclidean subcase).*

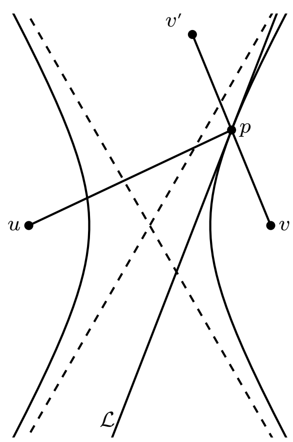

FIGURE 5. The tangent property of the hyperbola

Proof. Suppose the line $\mathcal{L}$ intersects the hyperbola in another point q. Then

$$\begin{aligned} & \|u-q\| + \|q-v'\| = \|u-q\| - \|v-q\| + \|v-q\| + \|q-v'\| \\ = \;& \|p-u\| - \|p-v\| + \|v-q\| + \|q-v'\| \\ > \;& \|p-u\| - \|p-v\| + 2\|p-v\| \\ = \;& \|p-u\| + \|p-v\|\,, \end{aligned}$$

where we have used the definition of an E-hyperbola and the triangle inequality for the triangle vqv'. Thus

$$\|u-q\| + \|q-v'\| > \|p-u\| + \|p-v\|,$$

giving a contradiction to the fact that p has the E-reflection property with respect to u, v' and $\mathcal{L}$. Hence $\mathcal{L}$ is tangent to the hyperbola. □

We conclude the paper by a conjecture which would generalize the following result on *Euclidean reflections for confocal ellipses* (see Figure 6): Consider two confocal ellipses with foci f_1 and f_2. Assume that p is a point on the larger ellipse. Draw the tangent line $\mathcal{L}$ to the larger ellipse at p. From p draw two tangents to the smaller ellipse, with a and b as points of tangency, respectively. Then $\overline{ap}$ and $\overline{bp}$ have the reflection property with respect to $\mathcal{L}$. That is, in Figure 6 the angles α and β are equal.

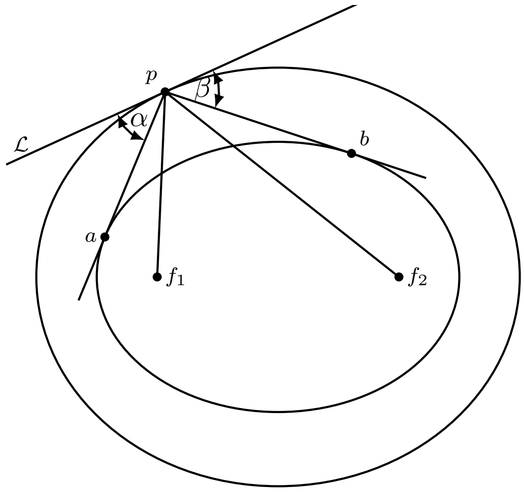

Figure 6. Confocal ellipses

Conjecture. Consider an analogous construction of confocal ellipses in a Minkowski plane with smooth unit circle. Then the Minkowskian reflection property holds between $\overline{ap}, \overline{bp}$, and $\mathcal{L}$.

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Injectivity

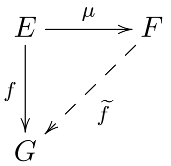
MARTIN MATHIEU AND MICHAEL ROSBOTHAM

To the memory of Edward G. Effros (1935–2019)

Abstract. The concept of dimension is ubiquitous in Mathematics. In this survey we discuss the interrelations between dimension and injectivity in the categorical sense.

1. Introduction

An invariant for an object enables us to distinguish it from a like one up to a suitable notion of isomorphism; 'dimension' is one of the most common invariants. Maybe the best known dimension is the cardinality of a basis of a vector space; which is even a complete invariant in the sense that two vector spaces (over the same field) are isomorphic if and only if they have the same dimension. In the argument that the cardinality of any two bases of a given vector space agree with each other (so that its 'dimension' is well defined) the fact that every basis of a subspace can be extended to a basis of a larger space plays an important role. This is tantamount to the statement that a linear mapping from a subspace of a vector space can always be extended to a linear mapping on the larger space; diagrammatically



(1.1)

where μ is the 'embedding' (an injective linear mapping) of E into F, f is the given linear mapping into a vector space G and $\widetilde{f}$ denotes the extension of f to F. In the context of modules over a (commutative, unital) ring R this is quickly seen to fail in general: given the canonical embedding $\mu\colon 2\mathbb{Z} \to \mathbb{Z}$, the $\mathbb{Z}$-linear map $f\colon 2n \mapsto n$ from $2\mathbb{Z}$ into $\mathbb{Z}$ cannot be extended to the larger module $\mathbb{Z}$ as, otherwise, the extension $\widetilde{f}$ would have to satisfy $1 = f(2) = \widetilde{f}(2) = 2\widetilde{f}(1)$ which is impossible as $\widetilde{f}(1) \in \mathbb{Z}$. The expert already notices at this stage the reason for this is that the $\mathbb{Z}$-module $\mathbb{Z}$ is not 'injective'; equivalently, the ring $\mathbb{Z}$ is not 'semisimple'.

The property of an object I in a category to ensure the 'extension' of a morphism from a 'subobject' E of an object F to F is generally called 'injectivity'; and is used in module categories to define a cohomological dimension which is not tied to the existence of a basis. In this survey article, we will review the interaction between injectivity and dimension in a wider setting of not necessarily abelian categories with a view on categories arising in functional analysis.

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The fundamental ingredients in a category are the morphisms, the 'arrows' between objects. They determine the concept of 'isomorphism' in the category, and similarly important is the choice of 'embeddings' some of which we have observed above. Typical categories we are interested in are the following ones.

Category	*Objects*	*Morphisms/Arrows*
$\mathscr{V}\!ec_{\mathbb{C}}$	complex vector spaces	linear maps
$\mathscr{M}\!od_R$	modules over a ring R	R-module maps
$\mathscr{N}\!or^\infty$	(complex) normed spaces	bounded linear maps
$\mathscr{N}\!or^1$	normed spaces	contractive linear maps
$\mathscr{B}\!an^\infty$	Banach spaces	bounded linear maps
$\mathscr{B}\!an^1$	Banach spaces	contractive linear maps
$\mathscr{T}\!op$	topological spaces	continuous maps

Note, however, that not all objects in a category have to be sets with some additional structure and even if they are, the morphisms need not be mappings. For instance, we could consider homotopy classes of continuous mappings between topological spaces as the morphisms. In functor categories, such as categories of sheaves, e.g., the morphisms are typically given by natural transformations.

In a category $\mathscr{A}$ we will denote by $\mathrm{obj}(\mathscr{A})$ the class of objects of $\mathscr{A}$ and, for any two $E, F \in \mathrm{obj}(\mathscr{A})$, by $\mathrm{Mor}(E, F)$ the set of all morphisms between E and F. In case we need to specify the category explicitly, we write $\mathrm{Mor}_{\mathscr{A}}(E, F)$. Let us recall some basic terminology; for a comprehensive discussion see, e.g., [1].

A morphism $f \in \mathrm{Mor}(E, F)$ is called a *monomorphism* if for any two morphisms $g, h \in \mathrm{Mor}(F, G)$ the identity $fg = fh$ implies that $g = h$, and it is called an *epimorphism* if for any two morphisms $g, h \in \mathrm{Mor}(G, E)$ the identity $gf = hf$ implies that $g = h$. In a concrete category, that is, the objects have an underlying set structure and the morphisms are set mappings (with some additional properties), every injective morphism is a monomorphism and every surjective morphism is an epimorphism however the reverse implications fail in general. The morphism $f \in \mathrm{Mor}(E, F)$ is called an *isomorphism* if there is a morphism $\bar{f} \in \mathrm{Mor}(F, E)$ such that $\bar{f}f = \mathrm{id}_E$ and $f\bar{f} = \mathrm{id}_F$, where id stands for the identity morphism of an object. An isomorphism is always a monomorphism and an epimorphism but the converse often fails; for example, in $\mathscr{N}\!or^\infty$.

The concept of a 'subobject' can be replaced by specifying a class $\mathscr{M}$ of monomorphisms which one usually assumes to be closed under composition and contains all isomorphisms in the category. For $E, F \in \mathrm{obj}(\mathscr{A})$ we will write

$$\mathscr{M}(E, F) = \{\mu \in \mathrm{Mor}_{\mathscr{A}}(E, F) \mid \mu \in \mathscr{M}\}$$

for the set of all morphisms between E and F that belong to the class $\mathscr{M}$. With these preparations we now introduce the main idea.

2. Injective Objects

An object $I \in \mathrm{obj}(\mathscr{A})$ is called *$\mathscr{M}$-injective* (for a specified class $\mathscr{M}$ of monomorphisms in the category $\mathscr{A}$) if, whenever $E, F \in \mathrm{obj}(\mathscr{A})$ and $\mu \in \mathscr{M}(E, F)$ are given, every $f \in \mathrm{Mor}(E, I)$ can be 'extended' to a morphism $\widetilde{f} \in \mathrm{Mor}(F, I)$, that is, $f = \widetilde{f}\mu$ as shown in the diagram (1.1) above (with $I = G$). Equivalently, if the mapping $\mu^*\colon \mathrm{Mor}(F, I) \to \mathrm{Mor}(E, I)$, $g \mapsto g\mu$ is surjective. (We shall take up this point of view in more detail in Section 3 below.)

Example 2.1. In the category $\mathscr{Nor}^1$ we choose as $\mathscr{M}$ all linear isometries. An $\mathscr{M}$-injective object I therefore has the property that, whenever E is (linearly isometric to) a subspace of a normed space F, every linear contraction from E into I can be extended to a contraction from F to I. Let $f\colon E \to I$ be a bounded linear mapping with $\|f\| \neq 0$. The contraction $f_1 = \frac{f}{\|f\|}$ is extended to a contraction $\widetilde{f_1}\colon F \to I$. With $\mu\colon E \to F$ the embedding we have $\widetilde{f_1}\mu = f_1$, equivalently, $\|f\|\widetilde{f_1}\mu = f$. Hence $\big\|\|f\|\widetilde{f_1}\big\| = \|f\|\,\|\widetilde{f_1}\| \leq \|f\|$, in other words, $\widetilde{f} := \|f\|\widetilde{f_1}$ is a 'Hahn–Banach extension' of f: it has the same norm as f. The Hahn–Banach theorem now states that $\mathbb{C}$ is an $\mathscr{M}$-injective object in $\mathscr{Nor}^1$.

Taking the same class $\mathscr{M}$ in the full subcategory $\mathscr{Ban}^1$, the $\mathscr{M}$-injectives in $\mathscr{Ban}^1$ are the completions of the $\mathscr{M}$-injectives in $\mathscr{Nor}^1$.

Sometimes it is possible to characterise all injective objects in a category. For example, in $\mathscr{Ban}^1$, an object E is $\mathscr{M}$-injective (where $\mathscr{M}$ is the class of all linear isometries) if and only if E is isomorphic in $\mathscr{Ban}^1$ to a space $C(X)$ of continuous complex-valued functions on an extremally disconnected compact Hausdorff space X [11, Chapter 3, Section 11, Theorem 6].

We say that the category $\mathscr{A}$ has *enough $\mathscr{M}$-injectives* if, for every $E \in \operatorname{obj}(\mathscr{A})$, there are an $\mathscr{M}$-injective object I and a morphism $\mu \in \mathscr{M}(E, I)$; in other words, every object can be embedded into an $\mathscr{M}$-injective object.

Example 2.2. The category $\mathscr{Ban}^1$ has enough $\mathscr{M}$-injectives. The reason for this is two-fold. Firstly, every Banach space can be isometrically embedded into a space of the form

$$\ell^\infty(\Omega) = \{\varphi\colon \Omega \to \mathbb{C} \mid \varphi \text{ is bounded}\}.$$

This is a consequence of the Hahn–Banach theorem. Let $E \in \operatorname{obj}(\mathscr{Ban}^1)$ and let E_1' denote its dual unit ball, that is, the set of all bounded linear functionals on E with norm at most one. Then $x \mapsto \hat{x}$, $E \to \ell^\infty(E_1')$, where $\hat{x}(f) = f(x)$ for all $f \in E_1'$ is a linear isometry. So we may take $\Omega = E_1'$ and $\mu \in \mathscr{M}(E, \ell^\infty(\Omega))$ this isometry.

Secondly, $\mathscr{Ban}^1$ has arbitrary products, namely, for any family $\{E_\omega \mid \omega \in \Omega\}$ of Banach spaces, the space

$$\prod_{\omega\in\Omega} E_\omega = \big\{\varphi \in \mathop{\times}_{\omega\in\Omega} E_\omega \mid \sup_{\omega\in\Omega} \|\varphi(\omega)\| < \infty\big\},$$

where $\times_{\omega\in\Omega} E_\omega$ denotes the cartesian product of the family $\{E_\omega \mid \omega \in \Omega\}$. Setting $E_\omega = \mathbb{C}$ for each ω, we clearly have $\prod_{\omega\in\Omega} E_\omega = \ell^\infty(\Omega)$. Each E_ω is $\mathscr{M}$-injective (Example 2.1) and it is a general fact that products of injectives are injective in a category with products; thus $\ell^\infty(\Omega)$ is $\mathscr{M}$-injective.

Since every normed space can be isometrically embedded into a Banach space (its completion), $\mathscr{Nor}^1$ has enough $\mathscr{M}$-injectives as well.

The following terminology is useful in understanding the relations between injective and non-injective objects.

Definition 2.3. (i) Let $E, F \in \operatorname{obj}(\mathscr{A})$. We say E is a *retract of* F if there exist morphisms $s \in \operatorname{Mor}(E, F)$ and $r \in \operatorname{Mor}(F, E)$ such that $rs = \mathrm{id}_E$. In this case we call s a *section* and r a *retraction*.

(ii) An object $E \in \operatorname{obj}(\mathscr{A})$ is an *absolute $\mathscr{M}$-retract* if every $\mu \in \mathscr{M}(E, F)$ for any $F \in \operatorname{obj}(\mathscr{A})$ is a section.

Proposition 2.4. *Every $\mathscr{M}$-injective object is an absolute $\mathscr{M}$-retract. Every retract of an $\mathscr{M}$-injective object is $\mathscr{M}$-injective. If $\mathscr{A}$ has enough $\mathscr{M}$-injectives then every absolute $\mathscr{M}$-retract is $\mathscr{M}$-injective.*

Proof. Let $I \in \mathrm{obj}(\mathscr{A})$ be $\mathscr{M}$-injective and let $\mu \in \mathscr{M}(I, F)$ for some $F \in \mathrm{obj}(\mathscr{A})$. Then, for id_I, there is $r \in \mathrm{Mor}(F, I)$ such that $\mathrm{id}_I = r\mu$, so μ is a section. If $E \in \mathrm{obj}(\mathscr{A})$ and $s \in \mathrm{Mor}(E, I)$, $r \in \mathrm{Mor}(I, E)$ satisfy $rs = \mathrm{id}_E$ then, for every $f \in \mathrm{Mor}(G, E)$, G any object in $\mathscr{A}$, and $\mu \in \mathscr{M}(G, H)$, $H \in \mathrm{obj}(\mathscr{A})$, there is $\widetilde{f} \in \mathrm{Mor}(H, I)$ with $sf = \widetilde{f}\mu$ and hence, $f = \mathrm{id}_E f = rsf = r\widetilde{f}\mu$ so that E is $\mathscr{M}$-injective as a retract of I.

Suppose $\mathscr{A}$ has enough $\mathscr{M}$-injectives. Then every absolute $\mathscr{M}$-retract is a retract of an $\mathscr{M}$-injective and hence is $\mathscr{M}$-injective. □

The above result is effective in deciding which objects can be injective.

Example 2.5. Let F be a Banach space. Suppose E is a retract of F and $s\colon E \to F$ and $r\colon F \to E$ are the section and retraction, respectively. Then $(sr)^2 = srsr = sr$ is a projection of norm one from F onto E. In other words, E is a (topological) direct summand of F. As $\mathscr{Ban}^1$ has enough $\mathscr{M}$-injectives (Example 2.2), a Banach space E is injective if and only if, whenever E is (isometrically isomorphic to) a subspace of an injective Banach space F, there is a norm-one projection from F onto E, by Proposition 2.4 above.

Let $c_0(\Omega)$ be the closed subspace of $\ell^\infty(\Omega)$ consisting of those bounded functions φ such that, for every $\varepsilon > 0$, the set $\{\omega \in \Omega \mid |\varphi(\omega)| \geq \varepsilon\}$ is finite. By a well-known result of Phillips, see, e.g., [9, Theorem 5.6], there is no bounded projection from $\ell^\infty(\Omega)$ onto $c_0(\Omega)$; as a result, $c_0(\Omega)$ is not $\mathscr{M}$-injective.

3. Additive Categories

So far the categories we considered had very few additional properties; in order to be able to define a dimension efficiently we need some more structure.

Definition 3.1. A category $\mathscr{A}$ is called *additive* if it has a zero object (a unique object 0 such that, for every $E \in \mathrm{obj}(\mathscr{A})$, both $\mathrm{Mor}_{\mathscr{A}}(E, 0)$ and $\mathrm{Mor}_{\mathscr{A}}(0, E)$ are singleton sets each); for all $E, F \in \mathrm{obj}(\mathscr{A})$ the morphism set $\mathrm{Mor}_{\mathscr{A}}(E, F)$ has the structure of an (additive) abelian group (in which case it is usually denoted by $\mathrm{Hom}_{\mathscr{A}}(E, F)$) such that composition of morphisms is bilinear; and for every pair of objects $E, F \in \mathrm{obj}(\mathscr{A})$ their biproduct exists (that is, there exists $D \in \mathrm{obj}(\mathscr{A})$ together with morphisms $\mu_E \in \mathrm{Mor}_{\mathscr{A}}(E, D)$, $\pi_E \in \mathrm{Mor}_{\mathscr{A}}(D, E)$, $\mu_F \in \mathrm{Mor}_{\mathscr{A}}(F, D)$, $\pi_F \in \mathrm{Mor}_{\mathscr{A}}(D, F)$ such that $\pi_E\mu_E = \mathrm{id}_E$, $\pi_F\mu_F = \mathrm{id}_F$ and $\mu_E\pi_E + \mu_F\pi_F = \mathrm{id}_D$. In this case, the unique biproduct is usually denoted by $D = E \oplus F$ and called *the direct sum of* E and F.

More details on additive categories can be found, for example, in [12].

Example 3.2. Probably the most commonly known additive categories are module categories. Let R be a unital ring. Let $\mathscr{Mod}_R$ denote the category whose objects are the right R-modules and the morphisms are the R-module maps (also called R-linear maps). Usually, $\mathrm{Mor}_{\mathscr{Mod}_R}(E, F)$ is denoted by $\mathrm{Hom}_R(E, F)$, for $E, F \in \mathrm{obj}(\mathscr{Mod}_R)$, and it is evidently an abelian group. The zero object is the zero module. The direct sum of E and F consists of all pairs (x, y) with $x \in E$ and $y \in F$ with coordinatewise operations, the R-module maps μ and π are the inclusions and the projections into and onto the respective coordinate. Hence, the direct sum $E \oplus F$ is isomorphic to the direct product $E \times F$ (as is the case in any additive category, where the terminology *coproduct* is used instead of direct sum).

The canonical choice for the class $\mathscr{M}$ is the one consisting of all monomorphisms in $\mathscr{Mod}_R$; these agree with the one-to-one R-module maps. The category $\mathscr{Mod}_R$ has enough $\mathscr{M}$-injectives [10, Proposition I.8.3].

Example 3.3. Since the sum of two contractions is not a contraction, the category $\mathscr{Ban}^1$ is not additive. However, the larger category $\mathscr{Ban}^\infty$ is: the sum of two bounded linear operators is bounded, the zero object is the zero Banach space, and the direct sum of two Banach spaces exists. In this case, for $E, F \in \mathrm{obj}(\mathscr{Ban}^\infty)$, $\mathrm{Mor}_{\mathscr{Ban}^\infty}(E,F)$ is typically written as $\mathscr{L}(E,F)$ and is another object in $\mathscr{Ban}^\infty$ (a difference to $\mathscr{Mod}_R$). The monomorphisms in $\mathscr{Ban}^\infty$ are the one-to-one bounded operators and the epimorphisms those with dense range. Thus, for $f \in \mathscr{L}(E,F)$ to be an isomorphism in $\mathscr{Ban}^\infty$ it is not sufficient to be both a monomorphism and an epimorphism.

For the class $\mathscr{M}$ one could take the same as in $\mathscr{Ban}^1$; but then not all isomorphisms (bijective bounded operators) would be in $\mathscr{M}$. So the canonical choice is the one-to-one bounded operators with closed range. By Example 2.2, $\mathscr{Ban}^\infty$ has enough $\mathscr{M}$-injectives. Let us point out a subtle difference in the notions of injectivity in $\mathscr{Ban}^1$ and in $\mathscr{Ban}^\infty$. A Banach space I which is injective in $\mathscr{Ban}^1$ is also injective in $\mathscr{Ban}^\infty$: see the normalisation argument in Example 2.1. But if I is injective in $\mathscr{Ban}^\infty$ it need not be injective in $\mathscr{Ban}^1$ as the extension may not preserve the norm.

There is a neat way to describe injectivity in a category by 'comparison' with the category $\mathscr{Ab}$ of abelian groups with group homomorphisms; this is done via the concept of an 'exact functor'. To introduce this notion, we firstly look at module categories. A sequence in $\mathscr{Mod}_R$,

$$0 \longrightarrow E \xrightarrow{\ \mu\ } F \xrightarrow{\ \pi\ } G \longrightarrow 0 \tag{3.1}$$

is called *short exact* if μ is a monomorphism (one-to-one), π is an epimorphism (onto) and the image of μ agrees with the kernel of π. We introduce the *contravariant* Hom-*functor* as follows. Let $I \in \mathrm{obj}(\mathscr{Mod}_R)$ be arbitrary and define

$$\begin{aligned} \mathrm{Hom}_R(-,I)\colon \mathscr{Mod}_R &\longrightarrow \mathscr{Ab} \\ E &\longmapsto \mathrm{Hom}_R(E,I) \\ \mathrm{Hom}_R(E,G) \ni f &\longmapsto f^* = \mathrm{Hom}_R(f,I) \end{aligned} \tag{3.2}$$

given by $f^*(g) = gf$ for $g \in \mathrm{Hom}_R(G,I)$. Then $f^*\colon \mathrm{Hom}_R(G,I) \to \mathrm{Hom}_R(E,I)$ is a group homomorphism, and 'contravariant' means that $(f_1 f_2)^* = f_2^* f_1^*$ for composable morphisms f_1 and f_2.

It is easy to check that this functor turns the sequence (3.1) above into the sequence

$$0 \longrightarrow \mathrm{Hom}_R(G,I) \xrightarrow{\ \pi^*\ } \mathrm{Hom}_R(F,I) \xrightarrow{\ \mu^*\ } \mathrm{Hom}_R(E,I) \tag{3.3}$$

where π^* is one-to-one and the image of π^* equals the kernel of μ^* but μ^* need not be surjective. One says the functor $\mathrm{Hom}_R(-,I)$ is *left exact.* In the case that μ^* *is* surjective—so that (3.3) turns into an exact sequence in $\mathscr{Ab}$—one calls the functor *exact.*

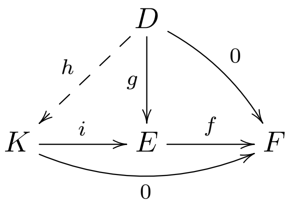
With $\mathscr{M}$ still the class of all monomorphisms in $\mathscr{Mod}_R$ we find that $I \in \mathrm{obj}(\mathscr{Mod}_R)$ is $\mathscr{M}$-injective if and only if the functor $\mathrm{Hom}_R(-,I)$ is exact. The idea behind using a functor is that properties in the image category, such as $\mathscr{Ab}$ for example, may be easier to understand.

Before moving on to more general categories, we wish to make the following point. A morphism $\mu \in \mathrm{Hom}_R(E,F)$ is always 'the first half' of a short exact sequence as in (3.1): we only have to take for G the quotient $F/\mathrm{im}\,\mu$, where $\mathrm{im}\,\mu$ is the image of μ, and π the canonical quotient mapping. This point of view will be stressed very soon below.

Let $\mathscr{A}$ be an additive category and let $f \in \mathrm{Mor}_{\mathscr{A}}(E,F)$ for some $E, F \in \mathrm{obj}(\mathscr{A})$.

Definition 3.4. A morphism $i\colon K \to E$ is a *kernel of* f if $fi = 0$ and for each $D \in \mathrm{obj}(\mathscr{A})$ and $g \in \mathrm{Mor}_{\mathscr{A}}(D,E)$ with $fg = 0$ there is a unique $h \in \mathrm{Mor}_{\mathscr{A}}(D,K)$

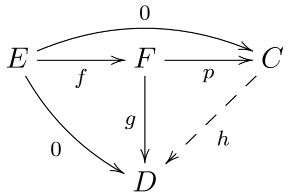
making the diagram below commutative



(3.4)

Any kernel is a monomorphism and is, up to isomorphism, unique. Thus we shall write $i = \ker f$.

Definition 3.5. A morphism $p\colon F \to C$ is a *cokernel of* f if $pf = 0$ and for each $D \in \mathrm{obj}(\mathscr{A})$ and $g \in \mathrm{Mor}_{\mathscr{A}}(F, D)$ with $gf = 0$ there is a unique $h \in \mathrm{Mor}_{\mathscr{A}}(C, D)$ making the diagram below commutative



(3.5)

Any cokernel is an epimorphism and is, up to isomorphism, unique. Thus we shall write $p = \operatorname{coker} f$.

Example 3.6. Let E, F be Banach spaces and let $f \in \mathscr{L}(E, F)$ be a bounded linear operator. A kernel of f is the isometric embedding of $\ker f = \{x \in E \mid f(x) = 0\}$ into E. A cokernel of f is the open quotient mapping $F \mapsto F/\overline{\mathrm{im}\, f}$, where $\overline{\mathrm{im}\, f}$ stands for the closure of the subspace $\mathrm{im}\, f = \{f(x) \mid x \in E\}$.

Since the composition of a kernel with an isomorphism is a kernel, a monomorphism in $\mathscr{Ban}^\infty$ is a kernel if and only if it has closed image (by the Open Mapping Theorem). Likewise, an epimorphism is a cokernel if and only if it is surjective.

Let

$$\ell^1 = \big\{(\xi_n)_{n\in\mathbb{N}} \mid \sum_{n=1}^{\infty} |\xi_n| < \infty\big\}$$

be the space of all absolutely summable complex sequences with its canonical norm and let $c_0 = c_0(\mathbb{N})$. Then the embedding $\ell^1 \hookrightarrow c_0$ is both a monomorphism and an epimorphism but neither a kernel, nor a cokernel, nor an isomorphism.

Good sources of information on categories of Banach spaces are, e.g., [6, Chapter IV] and [7].

It turns out that the correct generalisation of short exact sequences in general additive categories is the concept of 'kernel–cokernel pairs'.

Definition 3.7. In an additive category $\mathscr{A}$, a *kernel–cokernel pair* (μ, π) consists of two composable morphisms in $\mathscr{A}$ such that $\mu = \ker \pi$ and $\pi = \operatorname{coker} \mu$, depicted as

$$E_1 \overset{\mu}{\rightarrowtail} E_2 \overset{\pi}{\twoheadrightarrow} E_3 \tag{3.6}$$

where $E_i \in \mathrm{obj}(\mathscr{A})$. A monomorphism arising in such a pair is called *admissible* and is denoted as

$$E \rightarrowtail F$$

and an epimorphism arising in such a pair is called *admissible* and is denoted as

$$E \twoheadrightarrow F$$

Evidently this is a generalisation of (3.1) in $\mathscr{M}od_R$. The categories that resemble module categories most are the abelian categories which are now discussed in the next section.

4. Abelian vs. Exact Categories

One of the main technical devices in Homological Algebra are the 'diagram lemmas' which allow for (often skillful) manipulations with morphisms. In order for these to be possible one often requires the additive category $\mathscr{A}$ to satisfy two further conditions

(i) every morphism in $\mathscr{A}$ has both a kernel and a cokernel;
(ii) every monomorphism is a kernel and every epimorphism is a cokernel.

In this case, $\mathscr{A}$ is an *abelian* category. These seemingly innocent looking additional requirements have far-reaching consequences. For example, it follows that every morphism which is both a monomorphism and an epimorphism is already an isomorphism. In addition, every morphism f can be uniquely factorised as

$$\begin{array}{ccc} E & \xrightarrow{\quad f \quad} & F \\ & {\scriptstyle\pi}\searrow \quad \nearrow{\scriptstyle\mu} & \\ & G & \end{array} \tag{4.1}$$

where π is an epimorphism and μ is a monomorphism. Clearly, $\mathscr{M}od_R$ is an abelian category and, in fact, every abelian category can, in some sense, be 'embedded' into a module category (the Freyd–Mitchell embedding theorem [14, Section VI.7]). The short exact sequences can then equivalently be expressed by (3.6).

Alas, the categories in functional analysis such as $\mathscr{B}an^\infty$ are typically not abelian, see Example 3.6. Among the many generalisations of abelian categories the one that seems to work best for us is the concept of an exact category in the sense of Quillen; see [5] and [6].

Definition 4.1. An *exact structure* on an additive category $\mathscr{A}$ is a class of kernel–cokernel pairs, closed under isomorphisms, satisfying the following axioms.

[E0] $\forall\, E \in \mathscr{A} \colon \mathrm{id}_E$ is an admissible monomorphism;
[E0$^{\mathrm{op}}$] $\forall\, E \in \mathscr{A} \colon \mathrm{id}_E$ is an admissible epimorphism;
[E1] the class $\mathscr{M}$ of admissible monomorphisms is closed under composition;
[E1$^{\mathrm{op}}$] the class $\mathscr{P}$ of admissible epimorphisms is closed under composition;
[E2] the push-out of an admissible monomorphism along an arbitrary morphism exists and yields an admissible monomorphism;
[E2$^{\mathrm{op}}$] the pull-back of an admissible epimorphism along an arbitrary morphism exists and yields an admissible epimorphism.

Together with an exact structure, $\mathscr{A}$ is called an *exact category.* We will also use the notation $\mathscr{E} = (\mathscr{M}, \mathscr{P})$ to denote an exact structure.

It is not a coincidence that we chose the symbol $\mathscr{M}$ above; this will become clear in the next section. An easy exercise shows that an abelian category equipped with the exact structure given by all monomorphisms and all epimorphisms is an exact category. On the other hand, $\mathscr{B}an^\infty$ is a non-abelian category which is an exact category when endowed with the structure $\mathscr{E}_{\max}$ of *all* kernel–cokernel pairs, see [6, Theorem 2.3.3].

We can now make contact with the notion of retract introduced in Section 2, Definition 2.3.

Definition 4.2. A kernel–cokernel pair in an exact category $\mathscr{A}$,

$$E \overset{\mu}{\rightarrowtail} F \overset{\pi}{\twoheadrightarrow} G$$

is *split* if there exist morphisms $\nu \in \mathscr{M}(F, E)$ and $\iota \in \mathscr{P}(G, F)$ that make F a direct sum of E and G (where $\mathscr{P}(G, F) = \{\rho \in \mathrm{Mor}_{\mathscr{A}}(G, F) \mid \rho \in \mathscr{P}\}$).

The following result is the analogue of the 'Splitting Lemma' in module theory.

Proposition 4.3. *Let $\mathscr{A}$ be an exact category. The following are equivalent for a kernel-cokernel pair* $E \overset{\mu}{\rightarrowtail} F \overset{\pi}{\twoheadrightarrow} G$ *in $(\mathscr{M}, \mathscr{P})$:*
(a) *The kernel–cokernel pair is split;*
(b) *E is a retract of F with section μ;*
(c) *G is a retract of F with retraction π.*

Proof. By definition, (a) implies both (b) and (c). Assume (b) and let $\nu \in \mathrm{Hom}_{\mathscr{A}}(F, E)$ be such that $\nu\mu = \mathrm{id}_E$. Then $(\mathrm{id}_F - \mu\nu)\mu = 0$ so by the property of $\pi = \operatorname{coker}\mu$ there is $\iota \in \mathrm{Hom}_{\mathscr{A}}(G, F)$ such that $\mathrm{id}_F - \mu\nu = \iota\pi$ and hence, $\mathrm{id}_F = \mu\nu + \iota\pi$. Moreover,

$$\pi\iota\pi = \pi(\mathrm{id}_F - \mu\nu) = \pi - \pi\mu\nu = \pi$$

so that $\pi\iota = \mathrm{id}_G$ follows as π is an epimorphism.

The implication (c) $\Rightarrow$ (a) is proved in a similar way. □

In analogy with module theory we introduce the following concept.

Definition 4.4. An object $F \in \mathrm{obj}(\mathscr{A})$ is called *$\mathscr{M}$-semisimple* if all kernel–cokernel pairs of the form $E \overset{\mu}{\rightarrowtail} F \overset{\pi}{\twoheadrightarrow} G$ in $(\mathscr{M}, \mathscr{P})$ split.

Corollary 4.5. *The following are equivalent:*
(a) *Every object in $\mathscr{A}$ is $\mathscr{M}$-injective;*
(b) *Every kernel–cokernel pair in $(\mathscr{M}, \mathscr{P})$ is split;*
(c) *Every object in $\mathscr{A}$ is $\mathscr{M}$-semisimple.*

Proof. This follows immediately from the definitions, Proposition 4.3 and Proposition 2.4. □

Example 4.6. Let R be a unital ring and let $\mathscr{A} = \mathscr{M}od_R$. Let $\mathscr{M}$ be the class of all monomorphisms in $\mathscr{A}$. Then $\mathscr{M}$-injectivity is the usual injectivity considered in module theory, and the statement in Corollary 4.5 above is well known. In addition, see, e.g., [15, Theorem 4.40], every right R-module is projective; every right R-module is a direct sum of simple submodules; and R is a finite direct product of matrix rings over division rings (the Artin–Wedderburn theorem). In this situation, R is termed *semisimple.*

5. Dimension

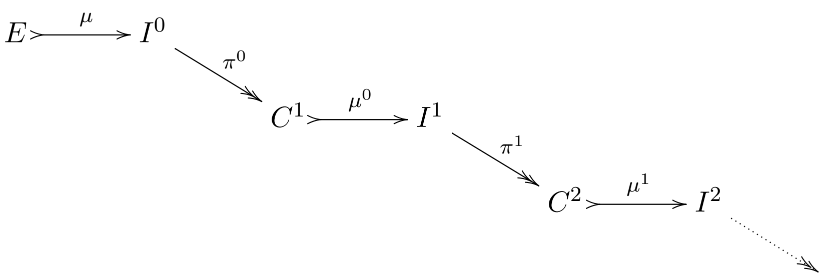
In this section we come back to the topic of dimension. Let us approach it from the point of view of splitting the short exact sequence (3.1):

$$0 \longrightarrow E \overset{\mu}{\longrightarrow} F \overset{\pi}{\longrightarrow} G \longrightarrow 0 \tag{5.1}$$

If G is a free module then π automatically is a retraction; this continues to hold if G is merely projective (a direct summand of a free module). On the other hand, if E is injective, then it is an absolute retract (Proposition 2.4) so μ is a section and the sequence splits too. We have a left–right symmetric situation here and it may thus not come as a surprise that, in module theory, the 'global dimension' of the ring R can be defined equivalently using projective or using injective modules; see, e.g., [15]. In other categories, for example sheaves of modules over ringed spaces or their analogues in C^*-theory, see [3], there are enough injective but not enough projective objects. This is why it may be desirable to focus on injectivity.

The starting point is: if an object is injective, its dimension should be 0. Now, and from now on, suppose we have enough injectives. Then any object can be embedded into an injective one and if it is a retract, then it is itself injective (Proposition 2.4) so the dimension is still 0. But if it is not a retract then its dimension should be at least 1. In this case it makes sense to consider the 'quotient' of the bigger injective object by the smaller non-injective one: if this turns out to be injective, one would say the dimension is equal to 1; otherwise at least 2. And so on ...

Let us formalise this process. Suppose $\mathscr{A}$ is an exact category and $\mathscr{M}$ the class of admissible monomorphisms (kernels of cokernels), cf. Definition 3.7. Take $E \in \mathrm{obj}(\mathscr{A})$. As $\mathscr{A}$ has enough $\mathscr{M}$-injectives, there are an $\mathscr{M}$-injective I^0 and $\mu \in \mathscr{M}(E, I^0)$. If μ is a section we are done. Otherwise let $\pi^0 = \operatorname{coker} \mu$ with codomain C^1. If C^1 is injective we stop. Otherwise there are an $\mathscr{M}$-injective I^1 and $\mu^0 \in \mathscr{M}(C^1, I^1)$. If μ^0 is a section we stop; and so on ...

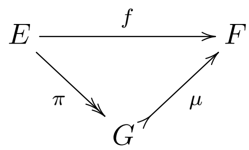


(5.2)

Why have we written this long sequence as a staircase? Note that (μ, π^0), (μ^0, π^1), ..., in general, (μ^{k-1}, π^k) are kernel–cokernel pairs while the morphisms $\mu^k \pi^k$ between I^{k-1} and I^k are compositions of a morphism in $\mathscr{P}$ followed by a morphism in $\mathscr{M}$. This means the sequence is 'exact' at the I^k whereas the morphisms between the $\mathscr{M}$-injective objects are of a special form.

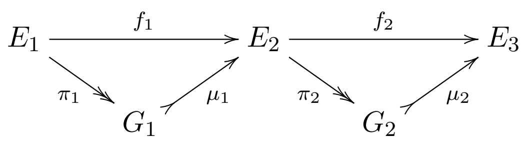
Let's have a look again at the canonical factorisation of a morphism in an abelian category as displayed in (4.1). This is an essential ingredient in the workings of Homological Algebra; however, not every morphism in an exact category can be factorised in such a way. In fact, *if* every morphism can be factorised as in (4.1) in an exact category, then the category is already abelian. So we have to specialise to those morphisms, which is done below. In addition, we have to define 'long exact sequences'.

Definition 5.1. Let $\mathscr{A}$ be an exact category with exact structure $\mathscr{E} = (\mathscr{M}, \mathscr{P})$. The morphism $f \in \mathrm{Hom}_{\mathscr{A}}(E, F)$, $E, F \in \mathrm{obj}(\mathscr{A})$ is called *admissible* if it can be factorised as



for some admissible monomorphism μ and some admissible epimorphism π in $\mathscr{A}$.

A sequence of admissible morphisms in $\mathscr{A}$,



is said to be *exact* if the short sequence $G_1 \overset{\mu_1}{\rightarrowtail} E_2 \overset{\pi_2}{\twoheadrightarrow} \mathfrak{G}_2$ is exact (that is, $(\mu_1, \pi_2) \in (\mathscr{M}, \mathscr{P})$). An arbitrary sequence of admissible morphisms in $\mathscr{A}$ is *exact* if the sequences given by any two consecutive morphisms are exact.

We can now reformulate the above 'staircase' (5.2) by 'straightening it out' as follows.

Definition 5.2. Let $E \in \operatorname{obj}(\mathscr{A})$ for an exact category $\mathscr{A}$. An *$\mathscr{M}$-injective resolution* of E is a sequence of the form

$$E \rightarrowtail I^0 \xrightarrow{d^0} I^1 \xrightarrow{d^1} I^2 \xrightarrow{d^2} \dots \tag{5.3}$$

where all the morphisms d^k are admissible, the sequence is exact (at all I^k) and all I^k are $\mathscr{M}$-injective. (Note that this in particular implies that the sequence is a complex, that is, $d^k d^{k-1} = 0$ for all $k \in \mathbb{N}$.)

We are now in a position to define a dimension using injectivity.

Definition 5.3. Let $E \in \operatorname{obj}(\mathscr{A})$ for an exact category $\mathscr{A}$. We say *E has finite $\mathscr{M}$-injective dimension* if there exists a finite $\mathscr{M}$-injective resolution (5.3) such that d^{k-1} is a section for some $k \in \mathbb{N}$. In this case we define

$$\mathscr{M}\text{-}\dim(E) = \min\{k \in \mathbb{N} \mid d^{k-1} \text{ is a section}\} \tag{5.4}$$

as the *$\mathscr{M}$-injective dimension* of E. In case E does not have a finite $\mathscr{M}$-injective resolution we put $\mathscr{M}\text{-}\dim(E) = \infty$ and say that E has *infinite $\mathscr{M}$-injective dimension.*

Let us return to the staircase (5.2) using the same notation and put $d^{k-1} = \mu^{k-1}\pi^{k-1}$ for all $k \geq 1$ to obtain (5.3). Suppose d^{k-1} is a section with retraction ρ^{k-1} in $\operatorname{Hom}_{\mathscr{A}}(I^k, I^{k-1})$. From $\operatorname{id}_{I^{k-1}} = \rho^{k-1}d^{k-1} = \rho^{k-1}\mu^{k-1}\pi^{k-1}$ we obtain $\operatorname{id}_{G^k}\pi^{k-1} = \pi^{k-1}\rho^{k-1}\mu^{k-1}\pi^{k-1}$ which implies that $\operatorname{id}_{G^k} = \pi^{k-1}\rho^{k-1}\mu^{k-1}$ as π^{k-1} is an epimorphism. Hence μ^{k-1} is a section and G^k is a retract of the $\mathscr{M}$-injective object I^k, thus $\mathscr{M}$-injective by Proposition 2.4. Conversely, if G^k is $\mathscr{M}$-injective, then μ^{k-1} is a section (as G^k is an absolute $\mathscr{M}$-retract) and we can replace I^k by G^k. Therefore finite $\mathscr{M}$-injective dimension really determines the first $k \geq 0$ such that a morphism in an injective resolution is a cokernel just as intended in the explanation of the staircase. (Note also that, by Proposition 4.3, μ^{k-1} is a section if and only if π^k is a retraction.)

It would, however, be tedious to work through all possible injective resolutions in order to find the injective dimension of an object. This is where the Hom-functor comes in.

In the sequel, $\mathscr{A}$ will always denote an exact category with exact structure $(\mathscr{M}, \mathscr{P})$ and with enough injectives. Firstly we observe that *every* object $E \in \operatorname{obj}(\mathscr{A})$ has an $\mathscr{M}$-injective resolution; this is the construction in the staircase (5.2). Secondly, all such resolutions are equivalent in the following sense.

Definition 5.4. A *complex in $\mathscr{A}$*, denoted by $(E^\bullet, d^\bullet)$, is a sequence

$$\dots \longrightarrow E^{n-1} \xrightarrow{d^{n-1}} E^n \xrightarrow{d^n} E^{n+1} \longrightarrow \dots \tag{5.5}$$

such that $(E^n)_{n\in\mathbb{Z}}$ is a sequence of objects in $\mathscr{A}$, $(d^n)_{n\in\mathbb{Z}}$ is a sequence of admissible morphisms $d^n \in \operatorname{Hom}_{\mathscr{A}}(E^n, E^{n+1})$ and $d^{n+1}d^n = 0$ for all $n \in \mathbb{Z}$.

Let $(E^\bullet, d^\bullet)$ and $(F^\bullet, \partial^\bullet)$ be two complexes in $\mathscr{A}$. A *morphism* from $(E^\bullet, d^\bullet)$ to $(F^\bullet, \partial^\bullet)$ is a sequence of morphisms $E^n \to F^n$, $n \in \mathbb{Z}$ making the diagram below commutative

$$\begin{array}{ccccccccc} \dots & \longrightarrow & E^{n-1} & \xrightarrow{d^{n-1}} & E^n & \xrightarrow{d^n} & E^{n+1} & \longrightarrow & \dots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ \dots & \longrightarrow & F^{n-1} & \xrightarrow{\partial^{n-1}} & F^n & \xrightarrow{\partial^n} & F^{n+1} & \longrightarrow & \dots \end{array} \tag{5.6}$$

Definition 5.5. Let $\varphi, \psi \colon (E^\bullet, d^\bullet) \to (F^\bullet, \partial^\bullet)$ be morphisms of complexes in $\mathscr{A}$. Then φ is *homotopic to* ψ, written as $\varphi \simeq \psi$, if there is a sequence $(\sigma^n)_{n\in\mathbb{Z}}$ of morphisms $\sigma^n \in \mathrm{Hom}_{\mathscr{A}}(E^n, F^{n-1})$ such that, for all $n \in \mathbb{Z}$, we have

$$\varphi_n - \psi_n = \partial^{n-1}\sigma^n + \sigma^{n+1}d^n. \tag{5.7}$$

This defines an equivalence relation on the class of morphisms of complexes.

The two complexes $(E^\bullet, d^\bullet)$ and $(F^\bullet, \partial^\bullet)$ are called *homotopic* if there exist morphisms $\varphi \colon (E^\bullet, d^\bullet) \to (F^\bullet, \partial^\bullet)$ and $\bar\varphi \colon (F^\bullet, \partial^\bullet) \to (E^\bullet, d^\bullet)$ such that $\bar\varphi\varphi \simeq \mathrm{id}_{(E^\bullet,d^\bullet)}$ and $\varphi\bar\varphi \simeq \mathrm{id}_{(F^\bullet,\partial^\bullet)}$.

To an $\mathscr{M}$-injective resolution (5.3) one associates a complex $(I^\bullet, d^\bullet)$, where E is deleted from the sequence and all $I^n = 0$ for $n < 0$ (in particular, $d^{-1} = 0$). The following is a standard result in Homological Algebra, see, e.g., [10, Proposition IV.4.5], since the arguments used in abelian categories take over in exact categories, cf. [5] and [16, Chapter 3].

Proposition 5.6. *Any two $\mathscr{M}$-injective resolutions of $E \in \mathrm{obj}(\mathscr{A})$ are homotopic.*

In analogy to the contravariant Hom-functor (3.2) one has the covariant Hom-functor. Let $F \in \mathrm{obj}(\mathscr{A})$ be arbitrary and define

$$\begin{aligned} \mathrm{Hom}_{\mathscr{A}}(F,-) \colon \mathscr{A} &\longrightarrow \mathscr{Ab} \\ E &\longmapsto \mathrm{Hom}_{\mathscr{A}}(F,E) \\ \mathrm{Hom}_{\mathscr{A}}(G,E) \ni f &\longmapsto f_* = \mathrm{Hom}(f,F) \end{aligned} \tag{5.8}$$

given by $f_*(g) = fg$ for $g \in \mathrm{Hom}_{\mathscr{A}}(F,G)$. Then $f_* \colon \mathrm{Hom}_{\mathscr{A}}(F,G) \to \mathrm{Hom}_{\mathscr{A}}(F,E)$ is a group homomorphism, and 'covariant' means that $(f_1f_2)^* = f_1^*f_2^*$ for composable morphisms f_1 and f_2.

Apply this functor to the complex $(I^\bullet, d^\bullet)$ to obtain a complex as below in $\mathscr{Ab}$

$$0 \longrightarrow \mathrm{Hom}_{\mathscr{A}}(F,I^0) \xrightarrow{d_*^0} \mathrm{Hom}_{\mathscr{A}}(F,I^1) \xrightarrow{d_*^1} \mathrm{Hom}_{\mathscr{A}}(F,I^2) \longrightarrow \dots \tag{5.9}$$

In general, this is no longer an exact sequence so one applies homology, that is, takes the quotient group $\ker d_*^{k+1}/\operatorname{im} d_*^k$ which is possible since $d_*^{k+1}d_*^k = 0$.

Definition 5.7. Let $\mathscr{A}$ be an exact category with enough injectives. Let $F \in \mathrm{obj}(\mathscr{A})$ be fixed. Let $E \in \mathrm{obj}(\mathscr{A})$ and $(I^\bullet, d^\bullet)$ be the complex associated to an $\mathscr{M}$-injective resolution (5.3) of E. Each $\ker d_*^{k+1}/\operatorname{im} d_*^k$ is called the *k-th cohomology group* and will be denoted by $\mathrm{Ext}^k(F,E)$.

Remark 5.8. Either by definition or left exactness of $\mathrm{Hom}_{\mathscr{A}}(F,-)$ we have $\mathrm{Ext}^0(F,E) \cong \mathrm{Hom}_{\mathscr{A}}(F,E)$ for all F and E.

Though it appears that the above definition depends on the choice of the injective resolution, in fact, by Proposition 5.6, any two injective resolutions of E are homotopic and this is preserved by the functor $\mathrm{Hom}_{\mathscr{A}}(F,-)$. As a consequence, the homology is the same. For details, see, e.g., [10, Section IV.3]. Moreover, for each $\varphi \in \mathrm{Hom}_{\mathscr{A}}(E,E')$ one can define a homomorphism $\varphi_* \colon \mathrm{Ext}^k(F,E) \to \mathrm{Ext}^k(F,E')$, $k \in \mathbb{N}$ and hence obtains the *k-th right derived functor of* $\mathrm{Hom}_{\mathscr{A}}(F,-)$. See [10, Section IV.5] for more details.

We finally state how these gadgets can help to determine the injective dimension.

5.9 Injective Dimension Theorem. *Let $\mathscr{A}$ be an exact category with enough injectives. Let $n \in \mathbb{N}$. The following are equivalent for an object $E \in \mathrm{obj}(\mathscr{A})$:*

(a) *$\mathscr{M}\text{-}\dim(E) \le n$;*
(b) *$\mathrm{Ext}^m(F,E) = 0$ for all $m > n$ and all $F \in \mathrm{obj}(\mathscr{A})$;*
(c) *$\mathrm{Ext}^{n+1}(F,E) = 0$ for all $F \in \mathrm{obj}(\mathscr{A})$;*

(d) $\operatorname{Ext}^n(-,E)\colon \mathscr{A} \to \mathscr{Ab}$ *is exact;*
(e) *there exists an $\mathscr{M}$-injective resolution of E whose n-th cokernel is $\mathscr{M}$-injective;*
(f) *for every $\mathscr{M}$-injective resolution of E the n-th cokernel is $\mathscr{M}$-injective.*

A proof of this result for module categories and more general abelian categories can be found in [15] and its extension to exact categories in [16].

6. Operator Modules

In [16], the above theory is applied to the category of operator modules over a C^*-algebra and, in more general form, to sheaves of operator modules over C^*-ringed spaces in [3]; see also [13]. Throughout this section A will denote a unital C^*-algebra.

Definition 6.1. A unital right A-module E which at the same time is an operator space is a *right operator A-module* if it satisfies either of the following equivalent conditions:
(a) There exist a complete isometry $\Phi\colon E \longrightarrow B(H,K)$, for some Hilbert spaces H, K, and a *-homomorphism $\pi\colon A \longrightarrow B(H)$ such that $\Phi(x\cdot a) = \Phi(x)\pi(a)$ for all $x \in E$, $a \in A$.
(b) The bilinear mapping $E \times A \longrightarrow E$, $(x,a) \mapsto x\cdot a$ extends to a complete contraction $E \otimes_h A \longrightarrow E$.
(c) For each $n \in \mathbb{N}$, $M_n(E)$ is a right Banach $M_n(A)$-module in the canonical way.

Our general reference for operator modules is [4], where, for instance, the Haagerup tensor product in the above definition, part (b) is treated in great detail. See also [16, Appendix A] for an in-depth discussion of this type of module and comparisons to other kinds of 'operator space modules'. We will denote by $\mathscr{OMod}_A^\infty$ the category with objects the right operator A-modules and morphisms the completely bounded A-module maps. It is similar to the category $\mathscr{Ban}^\infty$, in particular it is not abelian, but the morphisms respect the so-called matricial structure of a C^*-algebra, which has become important in that area since the 1970s.

In $\mathscr{OMod}_A^\infty$, a morphism T is a kernel iff it is a completely bounded isomorphism onto its image, and it is a cokernel iff it is surjective and completely open. (Note that there is no Open Mapping Theorem for operator spaces.)

Theorem 6.2 ([16], Theorem 4.40; see also [3])**.** *The class $(\mathscr{M},\mathscr{P})$ of all kernel–cokernel pairs in $\mathscr{OMod}_A^\infty$ is an exact structure on $\mathscr{OMod}_A^\infty$.*

Consequently, and since $\mathscr{OMod}_A^\infty$ has enough injectives, by Wittstock's Hahn–Banach theorem [8, Theorem 4.1.5], we can apply the ideas developed above. One is particularly interested in an invariant for the C^*-algebra A, and hence defines a 'global dimension' in analogy to the concept from ring theory.

Definition 6.3. The *global C^*-dimension* of a (unital) C^*-algebra A is defined by

$$C^*\text{-}\dim(A) = \sup\{\mathscr{M}\text{-}\dim(E) \mid E \in \mathscr{OMod}_A^\infty\}.$$

Recall, from Example 4.6, that a unital ring R is semisimple (in the classical sense) if and only if every module in $\mathscr{Mod}_R$ is injective; that is, has global dimension equal to zero. These rings are described by the Artin–Wedderburn theorem. One might hope that a similar class of C^*-algebras could also be identified; however, this is not the case!

Example 6.4. The unital C^*-algebra $\mathbb{C}$ has global C^*-dimension greater than 0. This follows immediately from the fact that c_0, viewed as a $\mathbb{C}$-module in a canonical way, is an operator module and is completely isometrically embedded into ℓ^∞. The latter is injective as an operator module (as every bounded linear map into ℓ^∞ is completely bounded [8, Proposition 2.2.6]) and thus, if c_0 was injective, it would have to be a retract of ℓ^∞ (Proposition 2.4) which it is not (Example 2.5). Thus c_0 is not $\mathscr{M}$-injective in $\mathscr{OMod}_{\mathbb{C}}^\infty$.

In fact, the same statement holds for every unital C^*-algebra A; one can use the compact operators on an infinite-dimensional Hilbert space in place of c_0. But let us move on to dimension 1.

Proposition 6.5. *The global C^*-dimension of A is at most one if and only if every complete quotient of an $\mathscr{M}$-injective object in $\mathscr{OMod}_A^\infty$ is $\mathscr{M}$-injective.*

This follows immediately from the Injective Dimension Theorem (5.9) as a complete quotient is nothing but the image F of a cokernel so we can apply the equivalence of (a), (e) and (f) to an injective presentation $E \overset{\mu}{\rightarrowtail} I \overset{\pi}{\twoheadrightarrow} F$ with I $\mathscr{M}$-injective.

But it turns out that the condition in the above proposition always fails.

Theorem 6.6. *The global C^*-dimension of every unital C^*-algebra is at least* 2.

The details of the proof can be found in [16, Chapter 5]; an important ingredient is the injective presentation $K(H) \rightarrowtail B(H) \twoheadrightarrow B(H)/K(H)$ for an infinite-dimensional Hilbert space H and the classical fact that ℓ^∞/c_0 is not injective in $\mathscr{Ban}^\infty$ [2].

At this moment, no C^*-algebra with global C^*-dimension equal to 2 is known; in fact, it is unclear whether there is any C^*-algebra with finite dimension.

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Retraction

THE EDITOR

Abstract. We retract an article redundantly published in issue 79 of this *Bulletin.*

Elke Wolf: *Composition operators between weighted Bergman spaces and weighted Banach spaces of holomorphic functions* Bulletin IMS 79 (2017) 75–85.

Retracted because of self-plagiarism. This paper was first published in Mathematica 57(80), No. 1–2, 126–134 (2015). The author has apologized.

The IMS Bulletin regrets its unintentional violation of Mathematica's copyright in the paper.

See `http://www.irishmathsoc.org/bull85/editorial.pdf` for more details.

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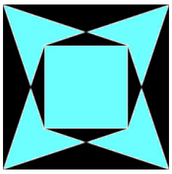

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Annalisa Crannell, Marc Frantz and Fumiko Futamura: Perspective and Projective Geometry, Princeton University Press, 2019. ISBN:9780-6911-9655-8, USD 49.95, x+280 pp.

REVIEWED BY PETER LYNCH

This book, essentially a course text, has lofty aspirations. It opens with the claim that it will "change the way that you look at the world, and we mean that literally". Seeing the world in new ways is mind-expanding and empowering, and this course may open the doors of perception just a little wider.

The challenge addressed in the book is to elucidate techniques used in graphical art by revealing the geometric principles underlying them. These techniques emerged from the Italian Renaissance and enabled artists to create strikingly realistic images. Among the most notable were Piero della Francesca and Leon Battista Alberti, who invented the method of perspective drawing. Artists were ahead of mathematicians, who only later codified the techniques in projective geometry. But the relationship became symbiotic, with each group learning from and teaching the other.

The book comprises twelve chapters and three appendices. For centuries, artists have painted scenes on a sheet of glass. In the opening lesson, students stick masking-tape on a large window, guided by an "artist", whose head is held in a fixed position. They discover that some lines that are parallel in the physical scene converge when marked on the window. There are many questions and exercises for the students to develop the capacity to visualize on a plane scenes in space, and many practical exercises where they must make drawings or take and use photographs, usually working together in groups.

Chapter 2 includes exercises on drawing large block letters in three dimensions. Chapter 3 answers the question "What is the image of a line?" The basics of Euclidean geometry are introduced in Chapter 4, and several theorems relevant to perspective are presented. In most cases, gaps are left in the proofs; this is deliberate and is intended to lead the students to make discoveries themselves. Students are asked to prove Ceva's Theorem and Menelaus's Theorem. Although generous hints are given, this will be daunting for many.

Chapter 5 introduces extended Euclidean space, with points and lines at infinity. This removes the need for special consideration of non-generic cases. In the context of perspective, the extended space includes the vanishing points of parallel lines. Chapter 6 discusses meshes and maps. Chapter 7 introduces Desargues's Theorem. Once again, the proof is gappy. This reviewer suspects that most readers will not have the tenacity (or inclination) to work through all the proofs and fill in all the gaps.

Collineations are considered in Chapter 8. The following two chapters are about drawing boxes and cubes. In Chapter 11, the cross-ratio, of key importance in both perspective drawing and projective geometry, is defined and applied. Eve's Theorem is proved (in outline) and Casey's Angle, a projective invariant, is introduced and illustrated by application to a perspective drawing. The invariance of this angle was first proved by the Irish geometer John Casey. I found the discussion in this chapter

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less lucid than elsewhere in the book, and somewhat confusing. Students will probably have the same reaction.

Chapter 12, on coordinate geometry, is more algebraic in character, enabling us to do perspective "by the numbers". The final chapter is on arcana like the topology of the perspective plane and the shape of space in the large. It is more abstract and less likely to assist artists directly, but perhaps it will give them a "broader perspective on perspective".

There are three appendices. The first gives complete beginners an introduction to the GeoGebra graphics program. The second, for reference, collects all the main definitions and results in one place. The third, on writing mathematical prose, has practical advice that should benefit all students. A Bibliography with more than fifty references and a two-page Index conclude the book.

⋆ ⋆ ⋆

It is difficult to know precisely who will benefit most from this book. There will be little new to mathematicians interested in projective geometry. And most artists will be strongly deterred by symbolic formulations of the type $\exists P \in \mathbb{R}^3 : \mathcal{P} = \{p \subset \mathbb{R}^3 : P \in p\}$ occurring in the definition of a pencil of lines. Students will need to work hard to benefit from the course. The absence of solutions means that they will struggle unless they have the guidance of an instructor, so the book is not really suitable for self-study except by especially talented readers.

The authors have undertaken a formidable task: to teach mathematics (geometry) to artists and (graphic) art to geometers. They have been only partially successful. They are not the first to struggle with such a task. We recall Euler's book *Tentamen novæ theoræ musicæ*, completed when he was 23 years old. This work was described as "too mathematical for musicians and too musical for mathematicians".

Peter Lynch is emeritus professor at UCD. His interests include all areas of mathematics and its history. He writes an occasional mathematics column in The Irish Times and has published a book of articles entitled *Thats Maths*. His blog is at `http://thatsmaths.com`.

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PROBLEMS

IAN SHORT

Problems

The first two problems this month come courtesy of Des MacHale of University College Cork.

Problem 85.1. Dissect an equilateral triangle into four pieces that can be reassembled, without flips, to form three equilateral triangles of different sizes. Can this be accomplished with just three pieces?

Problem 85.2. An absent-minded professor of mathematics cannot remember her debit card PIN. However, she remembers that the PIN lies between 4129 and 9985 and it cannot be expressed as the sum of two or more consecutive integers. Can you help her determine the PIN?

The third problem is a classic, which I encountered recently in the magazine of the M500 Society, a mathematical society of the Open University.

Problem 85.3. Arrange the integers 1 to 27 in a $3 \times 3 \times 3$ cube in such a way that any row of three integers (excluding diagonals) has sum 42.

Solutions

Here are solutions to the problems from *Bulletin* Number 83.

The first problem was solved by Ibae Aedo of the Open University, Omran Kouba of the Higher Institute for Applied Sciences and Technology, Damascus, Syria, and the North Kildare Mathematics Problem Club. We present the solution of the Problem Club.

Problem 83.1. Find positive integers a, b, c, d, e such that

$$\cfrac{1}{a-\cfrac{1}{b-\cfrac{1}{c-\cfrac{1}{d-\cfrac{1}{e}}}}} = 0,$$

and such that this equation remains true if a, b, c, d, e is replaced by any cyclic permutation of those five letters in that order.

(Note that this problem uses arithmetic involving ∞, such as $1/\infty = 0$).

Solution 83.1. One solution is

$$(a, b, c, d, e) = (2, 2, 1, 3, 1).$$

.

To obtain this, observe that the continued fraction equation is equivalent to

$$bcde + 1 = bc + be + de,$$

so we are asked for a solution in positive integers of the system of five equations obtained by combining this with the other four equations obtained by cyclic permutation of (a, b, c, d, e). Parity considerations show that exactly two of the variables must be even, and they must be adjacent in cyclic order, so without loss of generality we may consider a and b to be even and c, d, and e to be odd. Testing $e = 1$ in the equation, we see that $d \neq 1$, so $d \geq 3$, and we are led to the inequality $b(2c - 1) \leq 2$, which forces $b = 2$, $c = 1$, $d = 3$. Looking at the other permuted equations, we see that all are satisfied if $a = 2$. □

The next problem was solved by Ibae Aedo, Henry Ricardo of the Westchester Area Math Circle, New York, USA, Daniel Văcaru of Piteşti, Romania, Brendan and Ronan Wallace, and the North Kildare Mathematics Problem Club. Solutions were similar; we use the wording of Henry Ricardo.

Problem 83.2. Prove that for each positive integer m,

$$\tan^{-1} m = \sum_{n=0}^{m-1} \tan^{-1}\left(\frac{1}{n^2 + n + 1}\right).$$

Solution 83.2. The familiar identity $\tan(\alpha - \beta) = (\tan\alpha - \tan\beta)/(1 + \tan\alpha\tan\beta)$ leads to the following identity for the principal value of the inverse tangent function:

$$\tan^{-1} u - \tan^{-1} v \; = \; \tan^{-1}\left(\frac{u - v}{1 + uv}\right).$$

Let $u = n + 1$ and $v = n$ to give

$$\tan^{-1}\left(\frac{1}{n^2 + n + 1}\right) = \tan^{-1}(n + 1) - \tan^{-1} n.$$

Thus we have the telescoping series

$$\begin{aligned} \sum_{n=0}^{m-1} \tan^{-1}\left(\frac{1}{n^2 + n + 1}\right) \; &= \; \sum_{n=0}^{m-1} (\tan^{-1}(n + 1) - \tan^{-1} n) \\ &= \; \tan^{-1} m - \tan^{-1} 0 = \tan^{-1} m. \end{aligned}$$ □

Henry points out that by taking limits we obtain

$$\sum_{n=1}^{\infty} \tan^{-1}\left(\frac{1}{n^2 + n + 1}\right) \; = \; \frac{\pi}{2}$$

and other, similar results can be obtained by choosing special values of u and v in the inverse tangent identity stated in the solution.

The third problem was solved by Ibai Aedo, Henry Ricardo, Brendan and Ronan Wallace, and the North Kildare Mathematics Problem Club.

Problem 83.3. Find all positive integers x and y such that $x^y = y^x$.

We offer a known solution, presented by some of the contributors, which gives the full set of solutions for positive *rationals* x and y.

Solution 83.3. Naturally, $x = y$ is a solution for any positive number x, so let us assume that $x < y$.

We write the equation in the form $y = x^{y/x}$. Dividing by x gives

$$x^{y/x-1} = \frac{y}{x}. \qquad (*)$$

We let $m/n = y/x - 1$, where m and n are coprime positive integers. Equation $(*)$ becomes $x^{m/n} = (m+n)/n$, or, equivalently,

$$x = \frac{(m+n)^{n/m}}{n^{n/m}}. \qquad (**)$$

Since m and n are relatively prime, so are $(m+n)^n$ and n^n. It follows from $(**)$ that x is rational if and only if both $(m+n)^n$ and n^n are mth powers. So if x is rational, then each of $m+n$ and n must be an mth power, because the exponents m and n are relatively prime. Hence $n = a^m$ and $m+n = b^m$, where a and b are positive integers and $b > a$. This is possible if and only if $m = 1$, because the difference between two consecutive mth powers is greater than m if $m > 1$.

It follows from $(*)$ and $(**)$ that

$$x = (1+1/n)^n \quad \text{and} \quad y = (1+1/n)^{n+1},$$

where n is any positive integer. The only pair of positive integer solutions with $x < y$ is obtained when $n = 1$, giving $x = 2$ and $y = 4$. □

Thanks to all those who provided references for papers written on this problem. Henry Ricardo notes that the problem was first stated in a letter from Daniel Bernoulli to Christian Goldbach in 1728, in which Bernoulli asserts (without proof) that the equation has only one solution in positive integers and infinitely many rational solutions. Euler later solved the equation over the positive reals and positive integers, and provided rational solutions, without claiming that they were the only ones.

To finish this issue, it was incorrectly stated in Issue 84 that no solutions had been received for the extended version of Problem 82.2, which asks for a proof of the integral formula

$$\int_0^\infty \frac{\sinh x - x}{x^2 \sinh x}\, e^{-x}\, dx = \log \pi - 1.$$

In fact, Omran Kouba had already submitted a correct solution. I apologise for the error.

We invite readers to submit problems and solutions. Please email submissions to `imsproblems@gmail.com` in any format (we prefer Latex). Submissions for the summer Bulletin should arrive before the end of April, and submissions for the winter Bulletin should arrive by October. The solution to a problem is published two issues after the issue in which the problem first appeared. Please include solutions to any problems you submit, if you have them.

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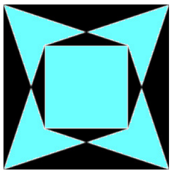

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