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PROBLEMS

IAN SHORT

Problems

The first problem uses arithmetic involving ∞ (such as $1/\infty = 0$).

Problem 83.1. Find positive integers a, b, c, d, e such that

$$\cfrac{1}{a-\cfrac{1}{b-\cfrac{1}{c-\cfrac{1}{d-\cfrac{1}{e}}}}} \;=\; 0,$$

and such that this equation remains true if a, b, c, d, e is replaced by any cyclic permutation of those five letters in that order.

The second problem was sent to the Open University by a student, and solved by Phil Rippon, once editor of these problem pages.

Problem 83.2. Prove that for each positive integer m,

$$\tan^{-1} m = \sum_{n=0}^{m-1} \tan^{-1}\left(\frac{1}{n^2+n+1}\right).$$

The third problem is a classic.

Problem 83.3. Find all positive integers x and y such that $x^y = y^x$.

A more difficult problem is to find all positive rational solutions of the same equation.

Solutions

Here are solutions to the problems from *Bulletin* Number 81.

Problem 81.1. Find a homogenous linear ordinary differential equation of order two that is satisfied by the function

$$y(x) = \int_0^\pi \sin(x\cos t)\,dt. \qquad \square$$

We are grateful to Omran Kouba of the Higher Institute for Applied Sciences and Technology, Damascus, Syria for pointing out that y is the zero function, making this question somewhat absurd. The problem pages editor accepts the blame for this one. Apologies to all!

We move swiftly on to a problem suggested by Finbarr Holland of University College Cork. This uses the standard notation

$$f(x) \sim g(x) \quad \text{as} \quad x \to \infty,$$

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where f and g are positive functions, to mean that

$$\frac{f(x)}{g(x)} \to 1 \quad \text{as} \quad x \to \infty.$$

It was solved by Omran Kouba, the North Kildare Mathematics Problem Club and the proposer, Finbarr Holland. We present Omran's solution.

Problem 81.2. Let

$$a_n = \sum_{k=0}^{n} \binom{n}{k}^2, \quad n = 0, 1, 2, \dots.$$

Prove that

$$\sum_{n=0}^{\infty} \frac{a_n x^n}{(n!)^2} \sim \frac{e^{4\sqrt{x}}}{4\pi\sqrt{x}} \quad \text{as} \quad x \to \infty.$$

Solution 81.2. Observe that a_n is the coefficient of x^n in the polynomial expansion of

$$(1+x)^n(1+x)^n = (1+x)^{2n}.$$

Hence $a_n = \binom{2n}{n}$. We define

$$b_n = \frac{a_n}{(n!)^2} \quad \text{and} \quad c_n = \frac{1}{2\pi} \cdot \frac{4^{2n+1}}{(2n+1)!}.$$

The c_n's are chosen so that for all t we have

$$\sum_{n=0}^{\infty} c_n t^{2n+1} = \frac{1}{2\pi} \sinh(4t).$$

In particular, we may define $g(x)$ for $x > 0$ by

$$g(x) = \frac{1}{2\pi} \frac{\sinh(4\sqrt{x})}{\sqrt{x}} = \sum_{n=0}^{\infty} c_n x^n.$$

Using Stirling's formula we see that

$$b_n \cdot \frac{(2n+1)!}{4^{2n}} = (2n+1)\frac{((2n)!)^2}{2^{4n}(n!)^4} \sim 2n \frac{(\sqrt{4\pi n}(2n/e)^{2n})^2}{(\sqrt{2\pi n}(n/e)^n)^4 2^{4n}} = \frac{2}{\pi}.$$

Thus

$$\lim_{n\to\infty} \frac{b_n}{c_n} = 1.$$

This proves that the series

$$\sum_{n=0}^{\infty} b_n x^n$$

defines an entire analytic function f.

Now, given $\varepsilon \in (0,1)$ there exists $n_\varepsilon > 0$ such that if $n \geqslant n_\varepsilon$, then

$$\left(1 - \frac{\varepsilon}{2}\right) c_n \leqslant b_n \leqslant \left(1 + \frac{\varepsilon}{2}\right) c_n.$$

It follows that, for $x > 0$, we have

$$\left(1 - \frac{\varepsilon}{2}\right) \sum_{n=n_\varepsilon}^{\infty} c_n x^n \leqslant \sum_{n=n_\varepsilon}^{\infty} b_n x^n \leqslant \left(1 + \frac{\varepsilon}{2}\right) \sum_{n=n_\varepsilon}^{\infty} c_n x^n.$$

Thus

$$\left(1 - \frac{\varepsilon}{2}\right) g(x) - \left(1 - \frac{\varepsilon}{2}\right) g_{n_\varepsilon}(x) \leqslant f(x) - f_{n_\varepsilon}(x) \leqslant \left(1 + \frac{\varepsilon}{2}\right) g(x),$$

where

$$f_m(x) = \sum_{k=0}^{m-1} b_k x^k \quad \text{and} \quad g_m(x) = \sum_{k=0}^{m-1} c_k x^k.$$

Hence

$$1 - \frac{\varepsilon}{2} - \frac{g_{n_\varepsilon}(x)}{g(x)} \leqslant \frac{f(x)}{g(x)} \leqslant 1 + \frac{\varepsilon}{2} + \frac{f_{n_\varepsilon}(x)}{g(x)}.$$

Now clearly

$$\lim_{x\to\infty} \frac{f_{n_\varepsilon}(x)}{g(x)} = 0 \quad \text{and} \quad \lim_{x\to\infty} \frac{g_{n_\varepsilon}(x)}{g(x)} = 0.$$

Consequently, there exists $x_\varepsilon > 0$ such that for all $x > x_\varepsilon$ we have

$$\frac{f_{n_\varepsilon}(x)}{g(x)} < \frac{\varepsilon}{2} \quad \text{and} \quad \frac{g_{n_\varepsilon}(x)}{g(x)} < \frac{\varepsilon}{2}.$$

Thus, for $x > x_\varepsilon$, we have

$$1 - \varepsilon \leqslant \frac{f(x)}{g(x)} \leqslant 1 + \varepsilon.$$

This proves that $f(x) \sim g(x)$ as $x \to +\infty$. But obviously $g(x) \sim e^{4\sqrt{x}}/(4\pi\sqrt{x})$ as $x \to +\infty$, so

$$f(x) \sim \frac{e^{4\sqrt{x}}}{4\pi\sqrt{x}} \quad \text{as} \quad x \to +\infty,$$

as desired. □

The third problem was solved by Omran Kouba, the North Kildare Mathematics Problem Club, and Neil Dobbs of University College Dublin, and the three solutions were all different. Neil points out that there are detailed discussions of this question on-line with numerous solutions, including his one, which we present here.

Problem 81.3. Find a function $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ such that the restriction of f to any open interval I is a surjective function from I to $\mathbb{R}$.

Solution 81.3. We express each real number x in binary form as $a.x_1x_2x_3\ldots$, where a is an integer and $x_i \in \{0, 1\}$. Let

$$f_N(x) = \sum_{n=1}^{N} \frac{(-1)^{x_n}}{n}.$$

If $f_N(x)$ converges to a finite value as $N \to \infty$, then we denote the limit by $f(x)$; otherwise, we set $f(x) = \pi$. □

The verification that f has the required properties is left to the reader! A non-constructive solution (from MathOverflow, due to Jim Belk) goes as follows. Let $\pi\colon \mathbb{R} \longrightarrow \mathbb{R}/\mathbb{Q}$ be the projection homomorphism, and let $\rho\colon \mathbb{R}/\mathbb{Q} \longrightarrow \mathbb{R}$ be a bijection. Then $f = \rho \circ \pi$ has the required property.

We invite readers to submit problems and solutions. Please email submissions to `imsproblems@gmail.com` in any format (we prefer LaTeX). Submissions for the summer Bulletin should arrive before the end of April, and submissions for the winter Bulletin should arrive by October. The solution to a problem is published two issues after the issue in which the problem first appeared. Please include solutions to any problems you submit, if you have them.

School of Mathematics and Statistics, The Open University, Milton Keynes MK7 6AA, United Kingdom