Irish Math. Soc. Bulletin
Number 82, Winter 2018, 105–108
ISSN 0791-5578

PROBLEMS

IAN SHORT

Problems

We begin with two problems by Finbarr Holland of University College Cork. The first of these uses the usual notation $x_1, x_2, \dots, x_n$ for the components of a vector x in $\mathbb{R}^n$.

Problem 82.1. Suppose that u and v are linearly independent vectors in $\mathbb{R}^n$ with

$$0 < u_1 \leqslant u_2 \leqslant \cdots \leqslant u_n \quad \text{and} \quad v_1 > v_2 > \cdots > v_n.$$

Given $x \in \mathbb{R}^n$, let y be the orthogonal projection of x onto the subspace spanned by u and v; thus $y = \lambda u + \mu v$, for uniquely determined real numbers λ and μ. Prove that if

$$x_1 > x_2 > \cdots > x_n,$$

then μ is positive.

Problem 82.2. Prove that

$$\int_0^\infty \frac{\sinh x - x}{x^2 \sinh x}\, dx = \log 2.$$

Readers may also like to attempt to prove the integral formula

$$\int_0^\infty \frac{\sinh x - x}{x^2 \sinh x}\, e^{-x}\, dx = \log \pi - 1.$$

The third problem appeared in the magazine of the M500 Society a few years ago. The M500 society is a mathematical society for those associated to the Open University.

Problem 82.3. Prove that

$$\sum_{n=1}^\infty \frac{1}{(5n-3)(5n-2)} = \frac{\pi}{5}\sqrt{1 - \frac{2}{\sqrt{5}}}.$$

Received 9-1-2019.

Solutions

Here are solutions to the problems from *Bulletin* Number 80. The first problem was solved by the North Kildare Mathematics Problem Club and the proposer, Finbarr Holland. We present the solution of the problem club.

Problem 80.1. Let $x_0, x_1, x_2, \dots$ be a null sequence generated by the recurrence relation

$$(n+1)(x_{n+1}+x_n)=1, \quad n=0,1,2,\dots .$$

Prove that the series

$$\sum_{n=0}^{\infty}(-1)^n x_n$$

converges, and determine its sum.

Solution 80.1. For $n=0,1,2,\dots$, we have

$$x_{n+1}=-x_n+\frac{1}{n+1},$$

so

$$x_{n+2}=x_n-\frac{1}{n+1}+\frac{1}{n+2}.$$

Therefore, by induction,

$$x_{2n}=x_0-S_{2n}, \quad \text{for } n=0,1,2,\dots,$$

where S_n is the sum to n terms of the alternating harmonic series:

$$S_n=1-\frac{1}{2}+\frac{1}{3}-\cdots+(-1)^{n+1}\frac{1}{n}.$$

Since $x_0, x_1, x_2, \dots$ is a null sequence, we have $x_0=\lim_{n\to\infty} S_n=\ln 2$, so now we just have to show that

$$\sum_{n=0}^{\infty}(x_{2n}-x_{2n+1})$$

converges, and to determine its sum.

We have

$$x_{2n}-x_{2n+1}=2(\ln 2-S_{2n})-\frac{1}{2n+1}.$$

For the next part we use an argument of N. Kazarinoff (*Analytic Inequalities*, Holt, Reinhart and Winston, New York, 1961, 45–46) that evaluates $\ln 2 - S_n$ in terms of the integral

$$I(n) = \int_0^{\frac{\pi}{4}} \tan^n \theta \, d\theta.$$

In fact, a simple recursive argument gives

$$I(2n+1) = \frac{1}{2}\ln 2 - \frac{1}{2}S_n,$$

so

$$x_{2n} - x_{2n+1} = 4I(4n+1) - \frac{1}{2n+1}.$$

Observing that

$$\frac{1}{2n+1} = 2\int_0^{\pi/4} \tan^{4n+1}\theta \sec^2\theta = 2I(4n+1) + 2I(4n+3),$$

we obtain

$$x_{2n} - x_{2n+1} = 2(I(4n+1) - I(4n+3)).$$

Hence

$$\sum_{n=0}^{\infty}(x_{2n} - x_{2n+1}) = 2\int_0^{\pi/4} \sum_{m=0}^{\infty}(-1)^m \tan^{2m+1}\theta \, d\theta,$$

which equals

$$2\int_0^{\pi/4} \frac{\tan\theta}{1+\tan^2\theta}\, d\theta = \int_0^{\pi/4} \sin 2\theta \, d\theta = \frac{1}{2}. \qquad \square$$

No correct solutions have been received for the second problem, besides the solution of the proposer, J.P. McCarthy of the Cork Institute of Technology. Here is the problem again in case you want a second crack at it.

Problem 80.2. Let

$$S(\sigma) = \sum_{i=1}^{n} \frac{1}{\sqrt{n^{i+\sigma(i)}}},$$

where σ is a nonidentity permutation of $\{1, 2, \ldots, n\}$. Find the maximum of S over all such permutations.

The third problem was quoted verbatim from *Lectures and Problems: A Gift to Young Mathematicians*, by V.I. Arnold. It was solved by Henry Ricardo of the Westchester Area Math Circle, New York, USA and the North Kildare Mathematics Problem Club. Both solutions were essentially the same.

Problem 80.3. Two volumes of Pushkin, the first and the second, are side-by-side on a bookshelf. The pages of each volume are 2cm thick, and the front and back covers are each 2mm thick. A bookworm has gnawed through (perpendicular to the pages) from the first page of volume 1 to the last page of volume 2. How long is the bookworm's track?

Solution 80.3. It was pointed out that the problem is ambiguous in various ways; however, with the reasonable assumption that volume 1 is to the left of volume 2 on the shelf the answer is 4mm, since the bookworm gnaws through two covers. □

Arnold comments in his book that he posed this problem in a published paper, along with the solution 4mm. However, apparently the journal editors did not trust this answer, so they edited the second-last sentence of the question to say 'from the *last* page of volume 1 to the *first* page of volume 2', thereby creating an error which appeared in the published manuscript!

We invite readers to submit problems and solutions. Please email submissions to `imsproblems@gmail.com` in any format (we prefer Latex). Submissions for the summer Bulletin should arrive before the end of April, and submissions for the winter Bulletin should arrive by October. The solution to a problem is published two issues after the issue in which the problem first appeared. Please include solutions to any problems you submit, if you have them.

School of Mathematics and Statistics, The Open University, Milton Keynes MK7 6AA, United Kingdom