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A duplication theorem for the Hermite polynomials

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Abstract. The generalized Hermite polynomials $H_{n,p}(z)$ are generated by

$$\exp(pzt - t^p) = \sum_{n=0}^{\infty} \frac{H_{n,p}(z)}{n!} t^n \quad (p \in \mathbb{N}).$$

We prove that the formula

$$H_{n,p}(az) = n! \sum_{k=0}^{[n/p]} \frac{(a^p - 1)^k a^{n-pk}}{k!(n-pk)!} H_{n-pk,p}(z)$$

holds for all integers $n \geq 0$, $p \geq 1$ and $a, z \in \mathbb{C}$. The special case $p = a = 2$ leads to the following duplication theorem for the classical Hermite polynomials:

$$H_n(2z) = n!\, 2^n \sum_{k=0}^{[n/2]} \left(\frac{3}{4}\right)^k \frac{H_{n-2k}(z)}{k!\,(n-2k)!}.$$

The classical Hermite polynomials $H_n(z)$ $(n = 0, 1, 2, ...; z \in \mathbb{C})$, named after the French mathematician Charles Hermite (1822 - 1901), are generated by

$$\exp(2zt - t^2) = \sum_{n=0}^{\infty} \frac{H_n(z)}{n!} t^n.$$

They can be written explicitly as

$$H_n(z) = n! \sum_{k=0}^{[n/2]} (-1)^k \frac{2^{n-2k}}{k!(n-2k)!} z^{n-2k}$$

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or by using the Rodrigues formula as

$$H_n(z) = (-1)^n e^{z^2} \frac{d^n}{dz^n} e^{-z^2}.$$

In particular, H_n is a polynomial of degree n with leading coefficient 2^n. The first few polynomials are

$$H_0(z) = 1, \quad H_1(z) = 2z, \quad H_2(z) = 4z^2 - 2,$$
$$H_3(z) = 8z^3 - 12z, \quad H_4(z) = 16z^4 - 48z^2 + 12.$$

We have the symmetry relation

$$H_n(-z) = (-1)^n H_n(z),$$

the recurrence relations

$$H_{n+1}(z) = 2zH_n(z) - 2nH_{n-1}(z), \quad H_n'(z) = 2nH_{n-1}(z)$$

and the pseudo-addition formula

$$H_n(x+y) = 2^{-n/2} \sum_{k=0}^{n} \binom{n}{k} H_k(\sqrt{2}x) H_{n-k}(\sqrt{2}y).$$

In the theory of differential equations H_n appears as a solution of the second-order linear differential equation

$$y'' - 2xy' + 2ny = 0.$$

There is a large body of literature on these functions. Indeed, since the Hermite polynomials have remarkable applications in the theory of special functions, probability theory, physics and other fields, they have attracted the attention of numerous researchers. Their main properties are collected, for example, in [1, chapter 22], [2, chapter 13.1], [5, chapter 1].

In this note, we present a duplication theorem for the Hermite polynomials which we could not locate in the literature.

Theorem 1. *For all nonnegative integers n and complex numbers z we have*

$$H_n(2z) = n!\, 2^n \sum_{k=0}^{[n/2]} \left(\frac{3}{4}\right)^k \frac{H_{n-2k}(z)}{k!\,(n-2k)!}. \tag{1}$$

Actually, we prove a bit more. We offer an identity satisfied by the generalized Hermite polynomials $H_{n,p}(z)$ which are given by

$$\exp(pzt - t^p) = \sum_{n=0}^{\infty} \frac{H_{n,p}(z)}{n!} t^n \quad (p \in \mathbb{N}). \tag{2}$$

Obviously, $H_{n,2} = H_n$. We have the explicit representation

$$H_{n,p}(z) = n! \sum_{k=0}^{[n/p]} (-1)^k \frac{p^{n-pk}}{k!\,(n-pk)!} z^{n-pk}.$$

See [3] and [4] for more information on these functions.

We show that the following extension of (1) is valid.

Theorem 2. *For all integers $n \geq 0$, $p \geq 1$ and complex numbers a, z we have*

$$H_{n,p}(az) = n! \sum_{k=0}^{[n/p]} \frac{(a^p - 1)^k a^{n-pk}}{k!(n-pk)!} H_{n-pk,p}(z). \tag{3}$$

Proof We have

$$\exp(p \cdot az \cdot t - t^p) = \exp\bigl(p \cdot z \cdot at - (at)^p\bigr) \cdot \exp\bigl((a^p - 1)\, t^p\bigr). \tag{4}$$

From (2) and (4) we obtain

$$\sum_{n=0}^{\infty} \frac{H_{n,p}(az)}{n!} t^n = \sum_{n=0}^{\infty} \frac{a^n H_{n,p}(z)}{n!} t^n \cdot \sum_{n=0}^{\infty} \frac{(a^p - 1)^n}{n!} t^{pn}. \tag{5}$$

Let

$$u_n = u_{n,p}(a, z) = \frac{a^n H_{n,p}(z)}{n!}, \quad v_n = v_{n,p}(a) = \frac{(a^p - 1)^n}{n!}$$

and

$$\delta_n = \delta_{n,p} = \begin{cases} v_{n/p}, & \text{if } p|n, \\ 0, & \text{otherwise.} \end{cases}$$

We have

$$\sum_{n=0}^{\infty} v_n t^{pn} = \sum_{n=0}^{\infty} \delta_n t^n. \tag{6}$$

Applying (6) yields

$$\begin{aligned}\sum_{n=0}^{\infty} u_n t^n \cdot \sum_{n=0}^{\infty} v_n t^{pn} &= \sum_{n=0}^{\infty} \sum_{\nu=0}^{n} u_{n-\nu} \delta_\nu t^n \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{[n/p]} u_{n-pk} \delta_{pk} t^n \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{[n/p]} u_{n-pk} v_k t^n. \qquad (7)\end{aligned}$$

A comparison of the coefficients of the power series given in (5) and (7) reveals that

$$\frac{H_{n,p}(az)}{n!} = \sum_{k=0}^{[n/p]} u_{n-pk} v_k = \sum_{k=0}^{[n/p]} \frac{a^{n-pk} H_{n-pk,p}(z)}{(n-pk)!} \frac{(a^p-1)^k}{k!}$$

which is (3).

Remark 1. If we set $p = a = 2$ in (3), then we obtain the duplication formula (1).

Remark 2. The referee wrote: "The choice $a = i$ is also interesting when $p = 2$." In this case we get from (3)

$$H_n(iz) = i^n n! \sum_{k=0}^{[n/2]} \frac{2^k}{k!(n-2k)!} H_{n-2k}(z).$$

Remark 3. From (3) we conclude that for $n \geq 0$, $p \geq 1$ and $a, z \in \mathbb{C} \setminus \{0\}$ we have

$$a^n \sum_{k=0}^{[n/p]} \left(1 - \frac{1}{a^p}\right)^k \frac{H_{n-pk,p}(z)}{k!(n-pk)!} = z^n \sum_{k=0}^{[n/p]} \left(1 - \frac{1}{z^p}\right)^k \frac{H_{n-pk,p}(a)}{k!(n-pk)!}.$$

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