

## PROBLEMS

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The first problem this issue was contributed by Peter Danchev of the Institute of Mathematics & Informatics of the Bulgarian Academy of Sciences.

**Problem 81.1.** Find a homogenous linear ordinary differential equation of order two that is satisfied by the function

$$y(x) = \int_0^\pi \sin(x \cos t) dt.$$

The second problem was suggested by Finbarr Holland of University College Cork. To state this problem, we use the standard notation

$$f(x) \sim g(x) \quad \text{as } x \rightarrow \infty,$$

where  $f$  and  $g$  are positive functions, to mean that

$$\frac{f(x)}{g(x)} \rightarrow 1 \quad \text{as } x \rightarrow \infty.$$

**Problem 81.2.** Let

$$a_n = \sum_{k=0}^n \binom{n}{k}^2, \quad n = 0, 1, 2, \dots$$

Prove that

$$\sum_{n=0}^{\infty} \frac{a_n x^n}{(n!)^2} \sim \frac{e^{4\sqrt{x}}}{4\pi\sqrt{x}} \quad \text{as } x \rightarrow \infty.$$

I learned the last problem from a friend recently.

**Problem 81.3.** Find a function  $f: \mathbb{R} \rightarrow \mathbb{R}$  such that the restriction of  $f$  to any open interval  $I$  is a surjective function from  $I$  to  $\mathbb{R}$ .

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## SOLUTIONS

Here are solutions to the problems from *Bulletin* Number 79. The first problem was solved by Henry Ricardo of the Westchester Area Math Circle, New York, USA, as well as the North Kildare Mathematics Problem Club and the proposer, Peter Danchev. The solution we give is an amalgamation of these solutions.

*Problem 79.1.* Suppose that  $k$  and  $n$  are positive integers with  $1 \leq k \leq n$ . Find the largest integer  $m$  such that the binomial coefficient  $\binom{2^n}{k}$  is divisible by  $2^m$ .

*Solution 79.1.* We have

$$\binom{2^n}{k} = \frac{2^n}{k} \binom{2^n - 1}{k - 1}.$$

Observe that, for  $1 \leq a < 2^n$ , the highest power of 2 that is a factor of  $a$  is equal to the highest power of 2 that is a factor of  $2^n - a$ . From this we see that the integer

$$\binom{2^n - 1}{k - 1} = \frac{(2^n - 1)(2^n - 2) \cdots (2^n - (k - 1))}{1 \cdot 2 \cdots (k - 1)}$$

is odd. Let us write  $k = 2^r s$ , where  $r$  is a nonnegative integer and  $s$  is an odd positive integer. It follows that  $\binom{2^n}{k}$  is divisible by  $2^{n-r}$ , and it is divisible by no higher power of 2, so  $m = n - r$ .  $\square$

The second problem was solved by Finbarr Holland, Ángel Plaza of Universidad de Las Palmas de Gran Canaria, Spain, Henry Ricardo, the North Kildare Mathematics Problem Club, and the proposer, Prithwjit De of the Homi Bhabha Centre for Science Education, Mumbai, India. We present Henry Ricardo's solution. The other solutions were similar, with some differences in how the trigonometric integral was evaluated.

*Problem 79.2.* Let  $f$  be a function that is continuous on the interval  $[0, \pi/2]$  and that satisfies  $f(x) + f(\pi/2 - x) = 1$  for each  $x$  in  $[0, \pi/2]$ . Evaluate the integral

$$I = \int_0^{\pi/2} \frac{f(x)}{(\sin^3 x + \cos^3 x)^2} dx.$$

*Solution 79.2.* By substituting  $u = \pi/2 - x$ , we see that

$$I = \int_0^{\pi/2} \frac{f(\pi/2 - u)}{(\cos^3 u + \sin^3 u)^2} du = \int_0^{\pi/2} \frac{1 - f(u)}{(\cos^3 u + \sin^3 u)^2} du.$$

Hence

$$I = \frac{1}{2} \int_0^{\pi/2} \frac{1}{(\cos^3 u + \sin^3 u)^2} du.$$

Next let  $t = \tan u$ . Then

$$I = \frac{1}{2} \int_0^{\infty} \frac{t^4 + 2t^2 + 1}{1 + 2t^3 + t^6} dt.$$

Expanding the integrand using partial fractions, we obtain

$$I = \frac{5}{18} \int_0^{\infty} \frac{dt}{t^2 - t + 1} + \frac{4}{18} \int_0^{\infty} \frac{dt}{(t + 1)^2} + \frac{1}{6} \int_0^{\infty} \frac{t}{(t^2 - t + 1)^2} dt.$$

These are standard integrals, which can easily be evaluated to give

$$I = \left( \frac{10}{81} \pi \sqrt{3} \right) + \frac{2}{9} + \left( \frac{1}{9} + \frac{2}{81} \pi \sqrt{3} \right) = \frac{4}{27} \pi \sqrt{3} + \frac{1}{3}. \quad \square$$

The third problem was solved by Yagub Aliyev of ADA University, Azerbaijan, Henry Ricardo, Ángel Plaza, and the North Kildare Mathematics Problem Club. All solutions used either induction or quoted known identities involving sums of powers. The solution we present is similar to that submitted by Ángel Plaza.

*Problem 79.3.* Prove that, for any positive integer  $n$ ,

$$(1^5 + \dots + n^5) + (1^7 + \dots + n^7) = 2(1 + \dots + n)^4.$$

*Solution 79.3.* In brief, let  $L_n$  and  $R_n$  denote the expressions on the left-hand side and right-hand side, respectively, for  $n = 0, 1, 2, \dots$ . Then

$$L_{n+1} - L_n = (n + 1)^5 + (n + 1)^7 = (n + 1)^5(n^2 + 2n + 2).$$

Another calculation shows that  $R_{n+1} - R_n = (n + 1)^5(n^2 + 2n + 2)$ . Since  $L_0 = R_0 = 2$ , we deduce that

$$L_m = 2 + \sum_{n=0}^{m-1} (L_{n+1} - L_n) = 2 + \sum_{n=0}^{m-1} (R_{n+1} - R_n) = R_m,$$

for  $m = 1, 2, \dots$ , as required.  $\square$

We invite readers to submit problems and solutions. Please email submissions to [imsproblems@gmail.com](mailto:imsproblems@gmail.com) in any format (we prefer Latex). Submissions for the summer Bulletin should arrive before the end of April, and submissions for the winter Bulletin should arrive by October. The solution to a problem is published two issues after the issue in which the problem first appeared. Please include solutions to any problems you submit, if you have them.

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