

EDITORIAL

The Society celebrated its fortieth anniversary this year with a special ‘September’ (i.e. annual scientific, as opposed to business) meeting held in TCD. The anniversary was further marked at the AGM in December. Professor Pavel Exner, EMS President wrote as follows on 27 November:

it has not escaped the EMS attention that in about three weeks you are going to celebrate a rounded anniversary of your society’s first meeting, and I want to congratulate you to this birthday gathering on behalf of your larger mathematical family. Forty is a good age, you are older than the EMS itself but much younger than some of its constituents, in short, a golden age when one is typically full of strength. Ireland has its firm place on the mathematical map of Europe to mention just one example, everybody remembers the plaque on the Broom Bridge and I have no doubts that inspiration will keep coming from your country also in the future.

The 2017 Annual Scientific Meeting will be held at Sligo Institute of Technology, and the 2018 meeting will be held at UCD.

A very large conference on Mathematics Education will take place shortly, at Croke Park from 1–5 February. It is the Tenth Congress of the European Society for Research in Mathematics Education (CERME), hosted by the Institute of Education, Dublin City University, and organised by Maurice O’Reilly and and Therese Dooley, See <http://cerme10.org/>.

Members may be interested in a recent statement issued by the American Statistical Association that attempts to fight back against the widespread abuse of p -values in applied work involving statistics. See <http://www.amstat.org/asa/files/pdfs/P-ValueStatement.pdf> and <http://amstat.tandfonline.com/doi/abs/10.1080/00031305.2016.1154108>.

About 10% of the membership have given notice that they no longer require the hard-copy version of the Bulletin, and this has resulted in some cost-saving on printing and postage. If you are

content with online-only access to the Bulletin, please notify the Treasurer at <mailto:\subscriptions.ims@gmail.com>. Note that the online version includes all graphics and images in full colour, whereas these are printed in grayscale in the hard copy.

A couple of our exchange partners have terminated the exchange of hard-copy periodicals and continue on an electronic-only basis. These include the University of Bari, the Mediterranean Journal of Mathematics, and the Austrian Mathematical Society. Members who wish to access electronic exchange resources should contact Anthony Waldron at admin@maths.nuim.ie. Mr Waldron kindly manages exchange correspondence for us.

The Annals of Irish Mathematics continue at full steam. Colm Mulcahy has produced a second calendar, and blogs monthly. See <http://www.mathsireland.ie/> for these and other matters related to the enormously-successful annual Maths Week.

Links for Postgraduate Study

The following are the links provided by Irish Schools for prospective research students in Mathematics:

DCU: (Olaf Menkens)

http://www.dcu.ie/info/staff_member.php?id_no=2659

DIT: <mailto://chris.hills@dit.ie>

NUIG: <mailto://james.cruickshank@nuigalway.ie>

NUIM: <http://www.maths.nuim.ie/pghowtoapply>

QUB: http://www.qub.ac.uk/puremaths/Funded_PG_2016.html

TCD: <http://www.maths.tcd.ie/postgraduate/>

UCD: <mailto://nuria.garcia@ucd.ie>

UU: <http://www.compeng.ulster.ac.uk/rgs/>

The remaining schools with Ph.D. programmes in Mathematics are invited to send their preferred link to the editor, a url that works. All links are live, and hence may be accessed by a click, in the electronic edition of this Bulletin¹.

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¹<http://www.irishmathsoc.org/bulletin/>

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DIT: (Chris Hills)

<mailto://chris.hills@dit.ie>

NUIG:

<mailto://james.cruickshank@nuigalway.ie>

NUIM:

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NOTICES FROM THE SOCIETY

Officers and Committee Members 2016

President	Dr M. Mackey	University College Dublin
Vice-President	Prof S. Buckley	Maynooth University
Secretary	Dr D. Malone	Maynooth University
Treasurer	Prof G. Pfeiffer	NUI Galway

Dr P. Barry, Prof J. Gleeson, Dr B. Kreussler, Dr R. Levene, Dr M. Mac an Airchinnigh, Dr M. Mathieu, Dr A. Mustata, Dr J. O'Shea

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.

Local Representatives

Belfast	QUB	Dr M. Mathieu
Carlow	IT	Dr D. Ó Sé
Cork	IT	Dr D. Flannery
	UCC	Dr S. Wills
Dublin	DIAS	Prof T. Dorlas
	DIT	Dr D. Mackey
	DCU	Dr M. Clancy
	SPD	Dr S. Breen
	TCD	Prof R. Timoney
	UCD	Dr R. Higgs
Dundalk	IT	Mr Seamus Bellew
Galway	UCG	Dr J. Cruickshank
Limerick	MIC	Dr B. Kreussler
	UL	Mr G. Lessells

Maynooth	NUI	Prof S. Buckley
Tallaght	IT	Dr C. Stack
Tralee	IT	Dr B. Guilfoyle
Waterford	IT	Dr P. Kirwan

Applying for I.M.S. Membership

(1) The Irish Mathematical Society has reciprocity agreements with the American Mathematical Society, the Deutsche Mathematiker Vereinigung, the Irish Mathematics Teachers Association, the Moscow Mathematical Society, the New Zealand Mathematical Society and the Real Sociedad Matemática Española.

(2) The current subscription fees are given below:

Institutional member	€160
Ordinary member	€25
Student member	€12.50
DMV, I.M.T.A., NZMS or RSME reciprocity member	€12.50
AMS reciprocity member	\$15

The subscription fees listed above should be paid in euro by means of a cheque drawn on a bank in the Irish Republic, a Eurocheque, or an international money-order.

(3) The subscription fee for ordinary membership can also be paid in a currency other than euro using a cheque drawn on a foreign bank according to the following schedule:

If paid in United States currency then the subscription fee is US\$ 30.00.

If paid in sterling then the subscription is £20.00.

If paid in any other currency then the subscription fee is the amount in that currency equivalent to US\$ 30.00.

The amounts given in the table above have been set for the current year to allow for bank charges and possible changes in exchange rates.

(4) Any member with a bank account in the Irish Republic may pay his or her subscription by a bank standing order using the form supplied by the Society.

- (5) Any ordinary member who has reached the age of 65 years and has been a fully paid up member for the previous five years may pay at the student membership rate of subscription.
- (6) Subscriptions normally fall due on 1 February each year.
- (7) Cheques should be made payable to the Irish Mathematical Society. If a Eurocheque is used then the card number should be written on the back of the cheque.
- (8) Any application for membership must be presented to the Committee of the I.M.S. before it can be accepted. This Committee meets twice each year.
- (9) Please send the completed application form with one year's subscription to:

The Treasurer, IMS
School of Mathematics, Statistics and Applied Mathematics
National University of Ireland
Galway
Ireland

Deceased Members

It is with regret that we report the deaths of members:

Msgr Joseph A. Spelman formerly Professor of Mathematical Physics at St Patrick's College, Maynooth, and parish priest of Collooney, Co. Sligo. died 23 June 2016. He was an Honorary Member of the Society.

Michael A. Hayes, MRIA, Emeritus Professor of Mathematical Physics at UCD, died 2 January 2017

E-mail address: subscriptions.ims@gmail.com

Eoin Coleman (Oren Kolman) 1959 - 2015

Eoin Coleman, who died on December 4th 2015, was well known to many members of the IMS, either personally or via the interesting articles he published in this Bulletin over a number of years, [1,2,3,4,5,7,8]. Eoin had converted to Judaism some 35 years ago and later was widely known in mathematical circles as Oren Kolman. I will refer to him as Eoin/Oren throughout.

Eoin/Oren attended St. Gerards School in Bray and then Glenstal Abbey before commencing his university career in UCD. He was a gifted child, excelling at whatever he turned his hand to. Academically he was interested in mathematics and languages but was also an accomplished pianist, particularly liking Rachmaninov. His sister Orla recounts that he learned to type at the age of 3! In 1980 he graduated with a first-class honours BA in Mathematics and Philosophy and subsequently completed in 1986 a first-class honours MA in Mathematics under the supervision of Seán Dineen on the topic “Ultraproducts of Banach Spaces”; he had spent some of the intervening years travelling and working in a Kibbutz in Israel.

In 1987 Eoin/Oren held a teaching assistantship at the Hebrew University of Jerusalem working with the renowned logician Saharon Shelah with whom he went on to complete several research papers [6,9,10,12]. Although he learned a lot about set theory and model theory, Eoin/Oren was not totally happy with his experience there and decided not to submit a doctoral thesis even though encouraged to do so. For this and other personal reasons he decided to return to Dublin in 1991 and taught at both UCD and DIT. He was an active participant in my seminar on Abelian Groups during that time.

Eoin/Oren’s other great passion was music and in the late 1990s he decided to take a “break” from mathematics and study music. He moved to Kings College, London where he completed the degrees M.Mus(Historical Musicology) in 1997 and Ph.D(Musicology) in 2003. I deliberately put the word “break” in quotes since somehow he continued to publish interesting mathematical research while studying music - see [8,9,10,11,12]. He also produced at least one paper relating to his work in music, [13].

After this musical excursion, Eoin/Oren spent some time working in Banking and Economics in France; he entertained many of us with his stories of his discussions with Nobel Prize Winners in Economics during this time!

Eoin/Oren then decided to return full time to mathematics and, despite having a larger publication list than many post-doctoral colleagues, he started afresh doing research with Pierre Matet at the University of Caen. His thesis “Logical Aspects of Slender Groups” was a masterly and deep analysis of the model-theoretic, set-theoretic and logical properties of slender groups. (A brief introduction to this class of groups may be found in his paper in this Bulletin,[8].) He received his doctorate in 2009 and I have been informed that the doctoral committee thought it worthy of a habilitation thesis, but it was not technically possible to make such an award. (I believe there was a possibility that his thesis would appear as a book but I have been unable to find out what happened in this regard.)

Although I had known Eoin/Oren since his time in DIT (1991-94), it was during the period from 2005 onwards that I got to know him well. We collaborated on two research papers, [14,16] and it was a joy to work with him. He was precise and demanded full rigorous arguments for all details. During this period he had a number of teaching positions: University of East Anglia, University of Bedfordshire and a part-time position in Cambridge. His interaction with students in these teaching posts was tremendous. Let me quote from a colleague of his at East Anglia:

The students are raving about him, he is inventive, interesting and enthusiastic. He gets things done that others cannot. He is brilliant, both as a teacher of say first year geometry and as a presenter at a research seminar. At admissions days he gives talks to prospective students about interesting mathematics and these talks are really excellently prepared and received.

My last interaction with him was a few weeks before his sudden death when he asked me to act as a referee for him in his application for senior membership of Hughes College, Cambridge; following interview, his application was successful and he was delighted to have a firmer connection to Cambridge. His Part III lectures on Topics in Set Theory had been very well received there the previous year.

On a personal level, Eoin/Oren was wonderful company: a first-rate intellect, well informed about so many issues, a fluent speaker of many languages and an interested, and interesting, hill walker. He was intensely private and chose a somewhat solitary life, close to nature planting trees and honouring Jewish traditions. He had an enduring love affair with France, having had for many years a home in Calvedos and more recently in the Auvergne; he travelled to France every few weeks. His wry sense of humour was always a pleasure to encounter. In many ways he was a real renaissance man and will be sadly missed by so many in the mathematical community world-wide.

Ar dheis Dé go raibh a anam dílis.

Acknowledgements. The author wishes to acknowledge the help and input of Orla Coleman and Pauline Mellon.

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- [6] (with S. Shelah) Categoricity of theories in $L_{\kappa\omega}$, when κ is a measurable cardinal. I. Fund. Math. 151 (1996), no. 3, 209240.
- [7] Toronto spaces, minimality, and a theorem of Sierpiski. Irish Math. Soc. Bull. No. 38 (1997), 5365.
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- [9] (with S. Shelah) A result related to the problem CN of Fremlin. J. Appl. Anal. 4 (1998), no. 2, 161165.
- [10] (with S. Shelah) Almost disjoint pure subgroups of the Baer-Specker group. Abelian groups and modules (Dublin, 1998), 225230, Trends Math., Birkhuser, Basel, 1999.
- [11] Almost disjoint families: an application to linear algebra. Electron. J. Linear Algebra 7 (2000), 4152.

- [12] (with S. Shelah) Infinitary axiomatizability of slender and cotorsion-free groups. *Bull. Belg. Math. Soc. Simon Stevin* 7 (2000), no. 4, 623629.
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- [16] (with R. Göbel and B. Goldsmith) On modules which are self-slender. *Houston J. Math.* 35 (2009), no. 3, 725736.
- [17] A note on omitting types in propositional logic. *Armen. J. Math.* 7 (2015), no. 1, 15.
- [18] (with B. Wald) M -slenderness, to appear in *Israel J. Math.*

Brendan Goldsmith (DIT).

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President's Report 2016

The IMS committee had one change for 2016, with D. Malone (MU) being elected as secretary following R. Quinlan's greatly valued term in that office.

The year marks the 40th anniversary of the society and there are two 1976 meetings that we celebrate which might loosely be described as conception and birth. On April 19th, 1976, in room WR20 at Trinity College, a gathering of mathematicians proposed the formation of the Irish Mathematical Society, and began the work required to ensure this idea came to fruition. On December 20th of that year, the first meeting proper of the society took place where the constitution was adopted, the membership list recorded and officers and committee were elected.

In recognition, our annual "September" meeting was held in April and, like the spawning salmon, returned to Trinity College for the event. We thank Trinity, and in particular local organisers R. Timoney and V. Dotsenko, for the invitation and a most enjoyable meeting. While we took advantage of the gathering to have an ordinary business meeting of the society, the Annual General Meeting must occur post-July and so was held on the second of the anniversaries, December 20th, at UCD, at which this report was presented.

In April, I attended the European Mathematical Society presidents' meeting which was held this year in Budapest during the first weekend of April. It is a useful forum, in the first instance, to hear and comment on reports from the president of the EMS and, in the second, to have the opportunity to discuss with counterparts in the many national societies. The theme for open discussion on the Sunday was mathematical education where attendants had an opportunity to discuss and compare experiences of mathematical teaching at second and third level. A report of the meeting is available.

Moving beyond Europe, the national adhering body to the International Mathematical Union (IMU) is the Royal Irish Academy where mathematical affairs are dealt with by the Physical, Chemical and Mathematical Sciences committee. While there is often overlap between the Academy and IMS committee membership, this was

not a formal state of affairs. It is sensible that the two committees do not work in isolation and entirely appropriate that the society membership should have a line of representation to the PCMS committee. Following a letter from the society inviting closer working relations, it was agreed that the IMS president, or nominee thereof, will in future be an invited member of the PCMS committee.

The society funded five meetings in 2016:

- (1) Numeracy: A Critical Skill in Adult Education. The 23rd Annual International Conference of the ALM., Jul 3-6, Maynooth University.
- (2) Young Functional Analysts Workshop, Apr 6-8, Queens University Belfast.
- (3) Groups in Galway 2016, May 20-21, NUIG.
- (4) Irish Geometry Conference 2016, May 6-7, TCD.
- (5) 10th Annual Irish Workshop on Mathematics Learning and Support Centres, May 29, NUIG.

I thank our treasurer, Goetz Pfeiffer, for most efficient handling of the application process. We receive more applications than we can provide funding for and choosing which meetings to benefit is a difficult and onerous task for the committee.

I can report that the IMS now engages in an activity known as "tweeting" under the pseudonym @irishmathsoc. The good stuff is still to be found in the *Bulletin*, as I trust it will be for the next forty years.

M. Mackey

December, 2016

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Minutes of the Irish Mathematical Society
Annual General Meeting
University College Dublin, 20 December 2016

Present: P. Barry, C. Boyd, S. Buckley, L. Creedon, S. Dineen, E. Gill, J. Gleeson, J.M. Golden, B. Goldsmith, M. Golitsyna, R. Higgs, T. Hurley, K. Hutchinson, T. Laffey, G. Lessells, R. Levene, P. Lynch, M. Mac an Airchinnigh, D. Malone, P. Mellon, B. Murphy, M. Newell, G. Ó Calhaín, A.G. O’Farrell, J. O’Shea, M. O’Reilly, G. Pfeiffer, R. Ryan, H. Smigoc, N. Snigireva, C. Stack, R.M. Timoney.

Apologies: C. Hills, B. Kreussler, O. Mason, M. Mathieu, C. Mulcahy, E. Oldham, R. Quinlan.

The meeting began at 15:40.

(1) **Minutes**

Minutes of the last meeting were accepted.

(2) **Matters Arising**

- The next meeting has been confirmed for the 31st August and 1st September, thanks to Leo.
- The location for the 2018 scientific meeting will probably follow the tradition of being hosted by the outgoing president, so it will most likely be held in UCD.

(3) **Correspondence**

A letter from the EMS with birthday wishes for the society was read. Other correspondence concerned elections, and will be dealt with under that agenda item.

(4) **40 Year Members**

The society provided a small token to those who had been original members of the society and were still members today.

T. Hurley presented his records of early society meetings to the society. M. Newell noted that he had once been drafted as treasurer and accidentally discovered that some members were happy to pay multiple times per year!

(5) **Membership Applications**

Membership applications had been approved for A. Cronin (UCD), C. Larkin (Clonmel), R. Osburn (UCD), N. Panteleidis (WIT), A. O’Brien (NUIG), D. Degrijse (NUIG) and H. Render (UCD).

(6) President's report

The president gave a brief summary of his report, full details of which can be found in the Bulletin. The big item was the 40th Anniversary of the society. Another important item was the positive correspondence with the Royal Irish Academy's Physical, Chemical and Mathematical Sciences Committee.

R.M. Timoney and V. Dotsenko were thanked for hosting the September meeting of the society. The president had also attended the EMS meeting and found it useful for keeping in contact with other European Societies. Other conferences supported by the society were also listed.

The society now also tweets!

C. Stack thanked the president for his work in recognising the long-term members of the society.

(7) Treasurer's report

The treasurer's reports were accepted, with thanks to the Treasurer.

(8) Bulletin

A. O'Farrell thanked the various groups that helped with production, including the Editorial board, G. Lessells (printing) and R.M. Timoney (website). The Bulletin particularly welcomes articles from members and those with an Irish connection. He encouraged the submission of Thesis Reports and articles of all types.

(9) Educational Subcommittee

- This subcommittee was established approximately two years ago, thanks to efforts by C. Stack and M. Mac an Airchinnigh.
- The committee is discussing 'Item 4(a)', which concerns what can be done at third level to smooth the transition for those coming from second level. There is a proposal to draw the attention of those at third level to encourage the provision of a smooth transition, and provide a short web page with links to material that documents current practice at second level. After a discussion about who should be targeted and what information should be linked to, this proposal was accepted.
- The European Mathematics Education Conference is being held in Dublin in February, organised by a group in

DCU. This is the largest event on third level mathematics education.

(10) **Elections**

- The society rotates its secretary and treasurer in even years, and its president and vice-president in odd years, so there is an election for president and vice-president at this meeting.
- Nominations had been received for S. Buckley and C. Stack for president. A nomination had also been received for P. Mellon for vice-president and for D. Mackey for a general committee position, if one arose. No further nominations were received from the floor.
- Brendan Murphy, of UCD statistics, was nominated to act as returning officer for the election.
- S. Buckley and C. Stack both addressed the floor for $3 + \epsilon$ minutes. After a ballot, S. Buckley was elected as president.
- The uncontested positions were made unanimously.
- Thanks were extended to the committee for running the election and to all candidates for putting themselves forward.
- Old Committee: M. Mackey (President), S. Buckley (Vice-President), D. Malone (Secretary), G. Pfeiffer (Treasurer), P. Barry, J. Gleeson, B. Kreussler, R. Levene, M. Mac an Airchinnigh, M. Mathieu, A. Mustata, J. O'Shea.
- New Committee: S. Buckley (President), P. Mellon (Vice-President), D. Malone (Secretary), G. Pfeiffer (Treasurer), P. Barry, J. Gleeson, B. Kreussler, R. Levene, M. Mac an Airchinnigh, D. Mackey, A. Mustata, J. O'Shea.

(11) **AOB**

- The Fergus Gaines Cup will be presented at P. Lynch's talk this evening.
- E. Gill noted that C. Mulcahy has made a 2017 Irish Mathematics calendar. Copies are being distributed.
- C. Stack expressed concern about low levels of engagement at meetings and asked how it could be addressed by the society.
- M. Mackey extended thanks to E. Gill and C. Mulcahy for all the work they do to promote mathematics.

- R. Higgs extended a vote of thanks to the outgoing president.

Report by David Malone, Secretary.
david.malone@nuim.ie

IMS Annual Meeting Trinity College Dublin

APRIL 15–16, 2016

To mark the 40th anniversary of the foundation of the Society on April 14th 1976 (in TCD), the annual meeting for 2016 was held in April in Trinity again, with the support of the School of Mathematics of TCD. The meeting lasted all day Friday April 15th and finished in time for a late lunch Saturday 15th.

The meeting was opened by the Head of School, Professor Sinéad Ryan, in the Maxwell Theatre of the Hamilton building and was attended by about 45 members and guests. In order to clear the building by 5pm for the Trinity Ball, a sandwich lunch was provided on Friday to allow for a more compressed lunch break. There were also the usual coffee breaks and a conference dinner at a local restaurant Friday evening.

A General Meeting of the Society was held at 13:00 Saturday and a committee meeting at 9am.

These are the nine speakers and the titles of their talks, together with the abstracts they supplied.

David Conlon (Oxford). *Inequalities in graphs* [10am Friday]

Suppose that a graph G contains a certain number of copies of a graph H . What, if anything, does this tell us about the number of copies of another graph K in G ? In this talk, we will explore a number of questions of this variety, touching upon extremal graph theory, semidefinite programming and Hilbert's 10th problem along the way.

Graham Ellis (NUIG). *Computing with 2×2 matrices* [14:55 Friday]

For certain groups G of 2×2 matrices, we consider the problem of constructing, on a computer, a contractible space X with a G -action that has finite (or even trivial) point stabilizers. Homological properties of G can be extracted from such a space and, at least in principle, can be used to compute certain automorphic forms.

Stephen Gardiner (UCD). *Universal approximation in analysis* [12:10 Friday]

Many different avenues of research in mathematical analysis have led to the discovery of objects which possess universal approximation properties. I will describe a range of such results, old and new, and then focus on recent insights about the Taylor series of a holomorphic function.

Derek Kitson (Lancaster). *Combinatorial characterisations for rigid symmetric frameworks* [11:15 Saturday]

In the last few years considerable progress has been made in finding combinatorial characterisations of generically rigid frameworks in various geometric contexts. Motivation comes in part from the associated pebble-game algorithms which can test these combinatorial conditions. The presence of symmetry in a framework imposes additional combinatorial constraints but also allows for greater efficiency due to the smaller size of the associated gain graphs. In this talk I will illustrate some of the linear algebraic and graph theoretic methods involved, focusing mainly on 2-dimensional frameworks with reflectional symmetry and norms with a quadrilateral unit ball. This is based on recent joint work with Bernd Schulze (Lancaster).

Anca Mustata (UCC). *Combinatorial data encoding the intersection theory and the Gromov-Witten invariants of a variety with \mathbb{C}^\times action* [10am Saturday]

In algebraic geometry, some of the most user-friendly examples are toric varieties, which are often used in testing new conjectures or computational techniques. They are relatively simple spaces which come together with the action of a large torus $(\mathbb{C}^\times)^n$. As a consequence, their cohomology, Gromov-Witten and other geometric invariants are determined by combinatorial data generated by the action. A considerably larger set of examples are varieties equipped with actions by a 1-dimensional torus \mathbb{C}^\times . In joint work with Andrei Mustata we identify a set of combinatorial data which determine the intersection theory and the Gromov-Witten invariants of these types of varieties. This is work in progress.

Andreea Nicoara (TCD). *Connections between several complex variables and real algebraic geometry* [11:15 Friday]

While fundamental questions in complex analysis in several variables such as the Levi problem drove much of the development of complex algebraic geometry in the 20th century, it turns out real algebraic geometry is considerably more relevant to complex analysis these days. I will explore these connections between complex analysis and real algebraic geometry arising from the study of the most famous PDE in complex analysis, the $\bar{\partial}$ -equation on a domain in \mathbb{C}^n .

Ann O'Shea (NUIM). *Understanding Understanding* [16:10 Friday]

The main aim of most mathematics courses at third level is to develop students' conceptual understanding. However, it is not easy to define this type of understanding. In this talk, I will consider attempts at coming up with such a definition and how these attempts can help us to study how learning takes place. I will use data from joint research projects with researchers in Ireland, the UK and Sweden.

Rachel Quinlan (NUIG). *I almost wish I hadn't gone down that rabbit-hole...* [12:10 Saturday]

Our story starts in 1773, with an observation by Lagrange on 3 by 3 determinants. In 1833 the corresponding statement was proved by Jacobi for all square matrices, and it is now known as the Desnanot-Jacobi identity. It is the basis for Dodgson's Condensation Algorithm of 1866, which is a scheme for computing determinants by repeatedly replacing contiguous 2×2 submatrices by the corresponding minors. More than 100 years later, the insertion of an experimental tweak to the definition of a 2×2 determinant in the condensation formula led to the notion of a λ -determinant. This is an adaptation of the determinant in which distinct products of matrix entries are indexed not by permutations but by more general objects known as *alternating sign matrices (ASMs)*. This talk will present some of the surprising but compelling connections between ASMs and permutations, and mention some some apparent connections to other structures from enumerative combinatorics.

Stuart White (Glasgow). *Quasidiagonality and Amenability* [14:00 Friday]

Quasidiagonality of a family of bounded operator was introduced by Halmos in terms of simultaneous blockdiagonal approximations. Despite it's simple definition, it's somewhat mysterious, with a kind of topological flavour, and unexpected connections to other concepts. For example, in the late 80's Rosenberg observed a connection to amenability of groups; showing that if the left regular representation of a discrete group is quasidiagonal, then the group must be amenable, and conjectured the converse. I'll survey quasidiagonality and Rosenberg's conjecture without assuming a background in operator algebras, and if time allows discuss it's role in the classification of simple nuclear C^* -algebras.

Report by Vladimir Dotsenko (vdots@maths.tcd.ie) and Richard M. Timoney (richardt@maths.tcd.ie), Trinity College Dublin.

Reports of Sponsored Meetings

IRISH MATHS LEARNING SUPPORT NETWORK 10TH ANNUAL
WORKSHOP
27 MAY 2016, NUI GALWAY

The theme of the 10th Annual Workshop of the Irish Mathematics Learning Support Network (IMLSN) Workshop was ‘The key role of tutors of mathematics and statistics in Post-Secondary Education’. The aim of the event was to discuss

- a variety of aspects including professional development for tutors,
- the diverse teaching roles of tutors and the challenges they face,
- mathematics learning support centres as rich learning experiences for tutors, and
- tutoring as opportunity for postgraduate and undergraduate students to be part of the mathematics community.

47 delegates attended this workshop including tutors and lecturers involved in support of mathematics and statistics at third level, or in teaching and learning at third level education in general.

Keynote speakers on the day included Michael Grove (University of Birmingham) and Ciarán O’Sullivan (Institute of Technology Tallaght, Dublin). Michael Grove spoke in his talk (*The Strongest Link? Supporting the Teaching Assistant, Demonstrator, Marker, Advisor, Tutor, . . .*) about the development of tutor training in mathematics support in the U.K. Ciarán O’Sullivan (*Staff development in Mathematics Learning Support in Ireland: where are we now and where to next?*) described approaches to tutor training made by the Irish community of mathematics and statistics support practitioners and made suggestions of how tutors’ professional teaching development can be supported.

Six other talks were contributed at this one day conference, including the perspectives of tutors engaged in mathematics and statistics support, suggestions for teaching from experienced lecturers and an overview of the current situation of mathematics learning support in Ireland: *Niall McInerney and Kevin Brosnan*, Challenges for tertiary level mathematics tutors with no formal education training: The

experience of two practitioners, *Ted Hurley*, ‘Make it (Mathematics) stick!’, *Richard Walsh*, An analysis of pedagogy of mathematics support tutors, *Cesar Scrotchi*, Tutoring in a Maths Support Centre as an enrichment experience for tutoring large groups, *Maura Clancy*, Audit of mathematics learning support in Ireland in 2015 the key findings, and *Julie Crowley*, Online e-assessment tool *Numbas* as a tutorial tool.

The workshop provided delegates with an opportunity to share ideas and experiences in supporting tutors engaged in higher education mathematics and statistics support, and to further consolidate links between academics and support staff.

Abstract of the talks and presentation are available at the IMLSN website: <http://imlsn.own.ie/imlsn10nuigalway/>

The organizers are grateful for financial support from the School of Mathematics, Statistics and Applied Mathematics and the Irish Mathematical Society.

Report by Kirsten Pfeiffer, NUI Galway

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ADULT LEARNING MATHEMATICS 23RD ANNUAL
INTERNATIONAL CONFERENCE
3-6 JULY, 2016, MAYNOOTH UNIVERSITY

The three day conference focused on the following themes:

- Numeracy: A Critical Skill in Adult Education.
- The Language of Mathematics, and Language and Mathematics.
- Adults Learning Mathematics: Research, Practice and Policy.

Numeracy is acknowledged as a critical skill needed to succeed in our 21st century society. For example the Irish National Strategy for Literacy and Numeracy states (p.10): *Numeracy encompasses the ability to use mathematical understanding and skills to solve problems and meet the demands of day-to-day living in complex social settings. To have this ability, a person needs to be able to think and communicate quantitatively, to make sense of data, to have a spatial awareness, to understand patterns and sequences, and to recognise situations where mathematical reasoning can be applied to solve problems.*

The conference endeavoured to explore the importance of numeracy in Adult Education and ways of developing numeracy in adult learners.

Language: We considered the challenges of teaching mathematics when the language of instruction is not the first language of the student (or the teacher). In addition, we focussed on the language of mathematics and how it affects teaching and or learning.

Research, Practice and Policy: We discussed aspects of research, practice and policy in relation to Adults Learning Mathematics. We were especially interested in the range of difference practices and policies employed at different levels and in different countries and in research on the effectiveness of these practices and policies.

The Keynote Speakers were

- Inez Bailey, Director of NALA, the National Adult Literacy Agency.
- Professor Raymond Flood, Gresham College.
- Professor Núria Planas, Universitat Autònoma de Barcelona
- Professor John O'Donoghue, Professor Emeritus University of Limerick.
- Professor Katherine Safford-Ramus, Saint Peters University, the Jesuit College of New Jersey

There were 19 further short talks, 12 workshops and several posters. Over 90 delegates from 12 different countries also enjoyed the National Science Museum at Maynooth University and the collection of old Mathematical and Scientific texts at the Russell Library.

Further details on both the ALM, and on the conference, including abstracts for all presentations, and some of the presentation slides are available from the conference website

<http://www.alm-online.net/alm-23-maynooth/>

The conference was supported by the 3U Partnership, the Department of Mathematics and Statistics at Maynooth University, Fáilte Ireland, Maynooth University, the National Adult Literacy Agency

(NALA), the National Forum for the Enhancement of Teaching and Learning in Higher Education and the IMS.

Report by Ciarán Mac an Bháird and Ann O'Shea, Maynooth University

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YOUNG FUNCTIONAL ANALYSTS' WORKSHOP
4-8 APRIL, 2016, QUEEN'S UNIVERSITY, BELFAST

The Young Functional Analysts' Workshop (YFAW) is an annual meeting, organised by PhD students, for early-stage researchers (mainly PhD students and postdocs) in all areas of Functional Analysis. YFAW aims to bring together young people working in Functional Analysis to interact, share ideas and experiences, present their work to a sympathetic audience, and learn about new areas of Functional Analysis from participant talks and invited talks from experienced researchers. See <https://sites.google.com/site/yfawuk/>.

The invited talks were:

- Dr Yemon Choi (Lancaster University): Derivations on Fourier algebras of connected groups;
- Prof Cho-Ho Chu (Queen Mary University of London): Jordan algebras in analysis and geometry;
- Dr Tatiana Shulman (IM PAN Warsaw): Completely positive maps and zero-error in quantum information theory;
- Dr Aaron Tikuisis (University of Aberdeen): C^* -algebras: structure and classification;
- Prof Ivan Todorov (Queen's University Belfast): Schur multipliers.

The conference also featured 17 short participant talks covering many areas of Functional Analysis. There were a total of 24 participants from 10 institutions. The organisers are very grateful to the Irish Mathematical Society, the London Mathematical Society and several local companies for supporting the event. We would also like to express our thanks to the participants and invited speakers for their enthusiastic participation.

Report by Andrew McKee and Linda Mawhinney, Queen's University,
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GROUPS IN GALWAY 2016
20–21 MAY 2016, NUI GALWAY

Groups in Galway, an annual conference on group theory and related topics which has been running since 1978, was held at NUI Galway on 20–21 May. The conference had 41 participants and featured ten talks of speakers from Ireland, UK, continental Europe and Brazil. The following wide range of topics were covered: Lie groups, geometric group theory, fusion systems, semigroup theory, pro- p -groups, representation theory and other structural problems for infinite groups. The speakers and titles were:

- Collin Bleak (University of St Andrews):
On detecting solubility for finitely generated subgroups of the group $PL_o(I)$
- John Burns (NUIG):
Discrete Tori in Weyl groups and their applications
- Francesco de Giovanni (University of Naples):
The murdered cardinal: a countably recognizable crime
- Ellen Henke (University of Aberdeen):
Normal subsystems of fusion systems and partial normal subgroups of localities
- Mark Lawson (Heriot-Watt University):
Boolean full groups
- Nadia Mazza (Lancaster University):
On a pro- p group of upper triangular matrices
- Bob Oliver (Université Paris 13):
Automorphisms and extensions of fusion systems
- Shane O'Rourke (Cork Institute of Technology):
A combination theorem for affine tree-free groups
- Said Sidki (Universidade de Brasilia):
From the Alternating Groups to Orthogonal Groups over Laurent Polynomial Rings
- Peter Symonds (University of Manchester):
Endotrivial modules for infinite groups

Besides talks, there was also a poster competition for students and young researchers, and research expenses prizes were awarded according to conference participants's vote. Further details of the program, as well as some photographs from the event, can be found at

<http://www.maths.nuigalway.ie/conferences/gig16/>

The organizers, Ted Hurley and Sejong Park, are grateful to NUI Galway (Registrar's Office), SFI and the Irish Mathematical Society for financial support of the conference.

Report by Sejong Park, NUI Galway

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A Mean Value Inequality for Euler's Beta Function

HORST ALZER AND RICHARD B. PARIS

ABSTRACT. Let $B(x, y)$ be Euler's beta function. We prove that the inequalities

$$0 < \frac{B\left(\sqrt{xy}, \frac{x+y}{2}\right)}{B(x, y)} < 1$$

hold for all $x, y > 0$ with $x \neq y$. The given constant bounds are best possible. This result is extended to the case when the beta function in the numerator has arguments given by the weighted geometric and arithmetic means.

1. INTRODUCTION

The beta function, also known as the Eulerian integral of the first kind, is defined for positive real numbers x and y by

$$B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1} dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \quad (1)$$

where Γ denotes the classical gamma function. From the product representation of $1/\Gamma(x)$ [6, Eq. (1.1.9)], it follows that

$$B(x, y) = \frac{x+y}{xy} \prod_{n=1}^{\infty} \left(1 + \frac{xy}{n(x+y+n)}\right)^{-1}. \quad (2)$$

The beta function plays an important role in the theory of special functions and it also has remarkable applications in physics, stochastic processes and other fields. A collection of the main properties of $B(x, y)$ as well as interesting historical comments on this subject can be found, for instance, in [6].

In the recent past, several research papers have appeared providing various inequalities for the beta function and its relatives. We refer to [1-5], [7, 9, 10, 11] and the references cited therein. For example,

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for all $x, y > 0$ we have that $B(x, y)$ separates the geometric and arithmetic means of $B(x, x)$ and $B(y, y)$, that is,

$$\sqrt{B(x, x)B(y, y)} \leq B(x, y) \leq \frac{B(x, x) + B(y, y)}{2}.$$

The first inequality is given in [9], whereas a proof for the second one can be found in [3]. In this note we present a new mean value inequality for the ratio of two beta functions.

Theorem. *For all positive real numbers x and y with $x \neq y$ we have*

$$0 < \frac{B\left(\sqrt{xy}, \frac{x+y}{2}\right)}{B(x, y)} < 1. \quad (3)$$

Both constant bounds are sharp.

2. A LEMMA

In order to prove the right-hand side of (3) we apply a lemma which offers an upper bound for the ratio given in (3) in terms of geometric and arithmetic means.

Let $x, y > 0$ and $w \in (0, 1)$. The weighted geometric and arithmetic means are defined by

$$G_w(x, y) = x^w y^{1-w} \quad \text{and} \quad A_w(x, y) = wx + (1-w)y.$$

Moreover, we set

$$G = G_{1/2}(x, y) = \sqrt{xy} \quad \text{and} \quad A = A_{1/2}(x, y) = \frac{x+y}{2}.$$

Lemma. *Let $v, w \in (0, 1)$. The inequality*

$$\frac{B(G_v(x, y), A_w(x, y))}{B(x, y)} \leq \frac{1}{2} \left[\left(\frac{G_v(x, y)}{A_w(x, y)} \right)^2 + \frac{G_v(x, y)}{A_w(x, y)} \right] \quad (4)$$

holds for all $x, y > 0$ if and only if $v = w = 1/2$.

Proof. First, we assume that (4) is valid for all $x, y > 0$. Let

$$F_{v,w}(x, y) = 2 \frac{B(G_v(x, y), A_w(x, y))}{B(x, y)} \left[\left(\frac{G_v(x, y)}{A_w(x, y)} \right)^2 + \frac{G_v(x, y)}{A_w(x, y)} \right]^{-1}.$$

Then, we have for $x, y > 0$:

$$F_{v,w}(x, y) \leq 1 = F_{v,w}(y, y).$$

Use of

$$\frac{\partial}{\partial x} B(x, y) = B(x, y) [\psi(x) - \psi(x+y)],$$

where $\psi = \Gamma'/\Gamma$ is the logarithmic derivative of the gamma function, yields

$$0 = 2y \frac{\partial}{\partial x} F_{v,w}(x, y) \Big|_{x=y} = 3(w-v) + 2(v+w-1)y(\psi(y) - \psi(2y)). \quad (5)$$

We denote the expression on the right-hand side of (5) by $H_{v,w}(y)$. Since

$$\lim_{t \rightarrow 0^+} t\psi(t) = -1 \quad \text{and} \quad \psi(1) - \psi(2) = -1,$$

we obtain

$$\lim_{y \rightarrow 0^+} H_{v,w}(y) = 1 - 4v + 2w = 0$$

and

$$H_{v,w}(1) = 2 - 5v + w = 0.$$

This leads to $v = w = 1/2$.

Next, we prove (4) with $v = w = 1/2$. Application of (2) leads to

$$B(x, y) = \frac{2A}{G^2} \prod_{n=1}^{\infty} \left(1 + \frac{G^2}{n(2A+n)} \right)^{-1}$$

and

$$\frac{B(G, A)}{B(x, y)} = \frac{G(G+A)}{2A^2} \prod_{n=1}^{\infty} f_n \quad (6)$$

with

$$f_n = \left(1 + \frac{G^2}{n(2A+n)} \right) \left(1 + \frac{GA}{n(G+A+n)} \right)^{-1}. \quad (7)$$

Since $A - G \geq 0$, we obtain

$$f_n = 1 - \frac{G(A-G)(G+2A+n)}{(G+n)(A+n)(2A+n)} \leq 1 \quad \text{for } n \geq 1. \quad (8)$$

Therefore,

$$\frac{B(G, A)}{B(x, y)} \leq \frac{G(G+A)}{2A^2} = \frac{1}{2} \left[\left(\frac{G}{A} \right)^2 + \frac{G}{A} \right]. \quad (9)$$

This establishes (4) with $v = w = 1/2$. \square

Remark 2.1. If $x \neq y$, then $A - G > 0$, so that (8) gives $f_n < 1$ for $n \geq 1$. This implies that (9) holds with “<” instead of “≤”. Thus, if $v = w = 1/2$, then the sign of equality is valid in (4) if and only if $x = y$.

We are now in a position to establish our main result.

3. PROOF OF THEOREM

An application of (4) with $v = w = 1/2$ yields for $x, y > 0$ with $x \neq y$:

$$\frac{B(G, A)}{B(x, y)} < 1 - \frac{(A - G)(A + G/2)}{A^2} < 1.$$

It remains to show that the bounds 0 and 1 are best possible. We denote the ratio in (3) by $R(x, y)$. Then,

$$R(x, x) = 1. \quad (10)$$

Use of the recurrence formula $\Gamma(x + 1) = x\Gamma(x)$ and (1) gives

$$R(x, 1) = \sqrt{x} \frac{\Gamma(\sqrt{x} + 1)\Gamma((x + 1)/2)}{\Gamma(\sqrt{x} + (x + 1)/2)}.$$

It follows that

$$\lim_{x \rightarrow 0^+} R(x, 1) = 0. \quad (11)$$

From (10) and (11) we conclude that the constant upper and lower bounds given in (3) cannot be improved. \square

Remark 3.1. Inequality (4) with $v = w = 1/2$ reveals that

$$\frac{A}{G} \leq \frac{B(x, y)}{B(G, A)}$$

is valid for all $x, y > 0$. This is a converse of the well-known arithmetic mean - geometric mean inequality $A/G \geq 1$. Many additional inequalities for arithmetic and geometric means as well as for numerous other mean values are given in the monograph [8].

It is natural to ask whether the right-hand side of (3) is valid for geometric and arithmetic means with a weight different from $1/2$. The following remark reveals that if both means have the same weight, then the answer is “no”.

Remark 3.2. Let $w \in (0, 1)$. The inequality

$$B(G_w(x, y), A_w(x, y)) \leq B(x, y) \quad (12)$$

holds for all $x, y > 0$ if and only if $w = 1/2$. We define

$$I_w(x) = B(x, 1) - B(G_w(x, 1), A_w(x, 1)).$$

If (12) is valid for all $x, y > 0$, then we obtain

$$I_w(x) \geq 0 = I_w(1) \quad \text{and} \quad I'_w(1) = 2w - 1 = 0.$$

Thus, $w = 1/2$.

Remark 3.3. If $w \in (0, 1)$, then (12) holds for $x, y > 0$ satisfying $G^2 \leq G_w(x, y)A_w(x, y)$ or, equivalently, $G_w(y, x) \leq A_w(x, y)$. To see this we observe that an extension of (6) and (7) (we omit to display the x, y dependence of G_w and A_w) shows that

$$\frac{B(G_w, A_w)}{B(x, y)} = Q_w \prod_{n=1}^{\infty} g_n(w), \quad Q_w = \frac{G^2(G_w + A_w)}{2AG_wA_w}$$

where

$$\begin{aligned} g_n(w) &= \left(1 + \frac{G^2}{n(2A + n)}\right) \left(1 + \frac{G_wA_w}{n(G_w + A_w + n)}\right)^{-1} \\ &= 1 - \frac{n(G_wA_w - G^2) + 2AG_wA_w(1 - Q_w)}{(2A + n)(G_w + n)(A_w + n)}. \end{aligned}$$

Since

$$\begin{aligned} \frac{G^2(G_w + A_w)}{G_w} &= G^2 + G_{1-w}A_w \leq A_w(G_w + G_{1-w}) \\ &\leq A_w(A_w + A_{1-w}) = 2AA_w, \end{aligned}$$

where we have employed the arithmetic mean - geometric mean inequality $G_w \leq A_w$, we see that $Q_w \leq 1$ and $g_n(w) \leq 1$ for $n \geq 1$, whence the result follows.

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Finite Differences and Terminating Hypergeometric Series

WENCHANG CHU

ABSTRACT. By means of finite difference method, new proofs are presented for the binomial convolution formulae of Abel, Chu–Vandermonde and Hagen–Rothe. The same approach is illustrated also for the summation theorems of classical hypergeometric series due to Dixon, Pfaff–Saalschütz, Stanton and Minton (1970).

Finite differences are very useful in numerical mathematics. In this paper, we shall illustrate how to employ them to evaluate binomial sums and terminating hypergeometric series. The approach consists of the following three steps:

- First for a given a binomial identity, identifying a parameter x as a variable and expressing the binomial sum in terms of finite differences.
- Then evaluating the binomial sum for particular values of x with the help of properties of finite differences.
- Finally confirming the binomial identity via the fundamental theorem of algebra, i.e., two polynomials of degrees $\leq n$ are identical if they have the same values at $n + 1$ distinct points.

New proofs will be presented for the binomial convolution formulae of Abel, Chu–Vandermonde and Hagen–Rothe. As further examples of classical hypergeometric series, we examine also Pfaff–Saalschütz summation theorem, Dixon’s formula, Stanton’s extension [23] of

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Andrews' ${}_5F_4$ -sum [1] and Minton's seminal theorem [18] on the series with integer parameter differences.

Following Bailey [2, §2.1], we shall use, the notation below for the classical hypergeometric series

$${}_{1+p}F_p \left[\begin{matrix} a_0, a_1, a_2, \dots, a_p \\ b_1, b_2, \dots, b_p \end{matrix} \middle| z \right] = \sum_{k=0}^{\infty} \frac{(a_0)_k (a_1)_k (a_2)_k \cdots (a_p)_k}{k! (b_1)_k (b_2)_k \cdots (b_p)_k} z^k$$

where the shifted factorial is defined by

$$(\lambda)_0 = 1 \quad \text{and} \quad (\lambda)_n = \lambda(\lambda+1)\cdots(\lambda+n-1) \quad \text{for} \quad n = 1, 2, \dots$$

with its multi-parameter form being abbreviated as

$$\left[\begin{matrix} \alpha, \beta, \dots, \gamma \\ A, B, \dots, C \end{matrix} \right]_n = \frac{(\alpha)_n (\beta)_n \cdots (\gamma)_n}{(A)_n (B)_n \cdots (C)_n}.$$

Throughout the paper, our attention will focus only on the terminating series, i.e., one of the numerator parameters $\{a_i\}_{i=0}^p$ results in a nonpositive integer.

1. FINITE DIFFERENCES

The finite difference operator Δ with unit increment is defined by

$$\Delta^0 f(x) := f(x) \quad \text{and} \quad \Delta f(x) := f(1+x) - f(x).$$

For a natural number n , the differences of order n is given by

$$\Delta^n f(x) := \Delta \{ \Delta^{n-1} f(x) \}$$

which is expressed by the following Newton–Gregory formula (cf. [21, Chapter 1])

$$\Delta^n f(x) = \sum_{k=0}^n (-1)^{n+k} \binom{n}{k} f(x+k). \quad (1)$$

In particular, when $p_m(x)$ is a polynomial of degree $m \leq n$ with the leading coefficient c_m , the following properties are quite useful:

$$\Delta^n p_m(x) = n! c_n \chi(m=n) \quad \text{and} \quad \Delta^n \frac{p_m(x)}{x-\lambda} = (-1)^n \frac{n! p_m(\lambda)}{(x-\lambda)_{n+1}}$$

where χ stands for the usual logical function with $\chi(\text{true}) = 1$ and $\chi(\text{false}) = 0$.

The former equality is well-known. The latter can be justified easily as follows. First when $p_m(x) \equiv 1$, it is trivial to check it by the induction principle. Observing that $\frac{p_m(x)-p_m(\lambda)}{x-\lambda}$ is a polynomial of degree $m-1$ with the n th differences equal to zero, we have immediately

$$\Delta^n \frac{p_m(x)}{x-\lambda} = \Delta^n \frac{p_m(\lambda)}{x-\lambda} = (-1)^n \frac{n! p_m(\lambda)}{(x-\lambda)_{n+1}}.$$

In addition, we shall use $\Delta_c^n f(x) = \Delta^n f(x)|_{x=c}$ for the differences starting at $x=c$.

2. CHU-VANDERMONDE CONVOLUTION

As a warm-up, we illustrate the method first by showing the Chu-Vandermonde convolution formula (cf. Bailey [2, §1.3])

$${}_2F_1 \left[\begin{matrix} -n, x \\ y \end{matrix} \middle| 1 \right] = \frac{(y-x)_n}{(y)_n} \quad (2)$$

which is often stated equivalently as the following binomial identity:

$$\sum_{k=0}^n \binom{x}{k} \binom{y}{n-k} = \binom{x+y}{n}.$$

Rewrite (2) equivalently as

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \frac{(x)_k}{(y)_k} = \frac{(y-x)_n}{(y)_n}. \quad (3)$$

Denote by $P(x)$ the above binomial sum, which is a polynomial of degree n in x . Keeping in mind of the relation

$$\frac{(y+m)_k}{(y)_k} = \frac{(y+k)_m}{(y)_m}$$

we can reformulate $P(x)$ at $x = y + m$ as

$$\begin{aligned} P(y+m) &= \sum_{k=0}^n (-1)^k \binom{n}{k} \frac{(y+m)_k}{(y)_k} \\ &= \sum_{k=0}^n (-1)^k \binom{n}{k} \frac{(y+k)_m}{(y)_m}. \end{aligned}$$

Observing that the binomial sum just displayed results in the n th differences of the polynomial $\frac{(x+y)_m}{(y)_m}$ with degree m , we deduce that

$$P(y+m) = (-1)^n \frac{n!}{(y)_n} \chi(m=n) \quad \text{for } 0 \leq m \leq n.$$

Therefore the polynomial $P(x)$ has the same values at the $n+1$ distinct points $\{y+m\}_{m=0}^n$ as $(y-x)_n/(y)_n$ with the same degree n . According to the fundamental theorem of algebra, they are identical. This proves (3) and so the Chu–Vandermonde identity (2).

3. PFAFF–SAALSCHÜTZ SUMMATION THEOREM

In classical hypergeometric series, the Pfaff–Saalschütz summation theorem is fundamental (cf. Bailey [2, §2.2] and Chu [4])

$${}_3F_2 \left[\begin{matrix} -n, & x, & y \\ 1+z, & x+y-z-n \end{matrix} \middle| 1 \right] = \frac{(1+z-x)_n (1+z-y)_n}{(1+z)_n (1+z-x-y)_n} \quad (4)$$

which can be reproduced, as the following binomial sum (cf. Gould [14, Entry 17.3; P.71]):

$$\sum_{k=0}^n \binom{n}{k} \frac{\binom{x}{k} \binom{y}{k}}{\binom{x+y+z+n}{k} \binom{z+k}{k}} = \frac{\binom{x+z+n}{n} \binom{y+z+n}{n}}{\binom{x+y+z+n}{n} \binom{z+n}{n}}.$$

Firstly, rewrite the equality (4) equivalently as

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \frac{(x)_k (y)_k}{(1+z)_k (x+y-z-n)_k} = \frac{(1+z-x)_n (1+z-y)_n}{(1+z)_n (1+z-x-y)_n}. \quad (5)$$

For the polynomial given by $(1+z-x-y)_n = (-1)^n (x+y-z-n)_n$, if multiplying by this across the last equation, we would get an identity between two polynomials of degree n in x . In order to prove it, it suffices to check the equality (5) for $n+1$ distinct values of x .

Let $R(x)$ be the sum displayed in (5). In view of the relation

$$\frac{(z+m)_k (y)_k}{(1+z)_k (y+m-n)_k} = \frac{(1+z+k)_{m-1} (1-y-k)_{n-m}}{(1+z)_{m-1} (1-y)_{n-m}}$$

we have the following expression

$$\begin{aligned} R(z+m) &= \sum_{k=0}^n (-1)^k \binom{n}{k} \frac{(z+m)_k (y)_k}{(1+z)_k (y+m-n)_k} \\ &= \sum_{k=0}^n (-1)^k \binom{n}{k} \frac{(1+z+k)_{m-1} (1-y-k)_{n-m}}{(1+z)_{m-1} (1-y)_{n-m}} \end{aligned}$$

which vanishes for $m = 1, 2, \dots, n$ because it results in the n th differences of the following polynomial $(1+x+z)_{m-1} (1-x-y)_{n-m}$ of degree $n-1$.

Taking into account of $R(0) = 1$ besides, we conclude that equality (5) is valid for the $n+1$ distinct values $\{0\} \cup \{z+m\}_{m=0}^{n-1}$ of x . This confirms (5) and so the Pfaff–Saalschütz summation formula (4).

It should be pointed out that the proof presented here resembles much the one found recently by Gessel [13], but with the difference that our proof is based on the polynomial $R(x)$ of degree n while Gessel's on another polynomial of degree $2n$ together with its symmetric property.

4. CONVOLUTION FORMULAE OF HAGEN–ROTHER

More general convolutions of binomial coefficients are evaluated by Hagen and Rothe (cf. Comtet [11, §3.1] and Mohanty [19, §4.2])

$$\sum_{k=0}^n \frac{x}{x+ky} \binom{x+ky}{k} \binom{z-ky}{n-k} = \binom{x+z}{n}, \quad (6a)$$

$$\sum_{k=0}^n \frac{x}{x+ky} \binom{x+ky}{k} \frac{z-ny}{z-ky} \binom{z-ky}{n-k} = \frac{x+z-ny}{x+z} \binom{x+z}{n}. \quad (6b)$$

There are many different proofs. Some of them can be found in [8, 10, 15, 24]. Denote by $\mathcal{P}(x)$ the binomial sum in (6a), which is obviously a polynomial of degree n . Its value at $x = m - z$ can be

manipulated as

$$\begin{aligned}
\mathcal{P}(m-z) &= \sum_{k=0}^n \frac{m-z}{m-z+ky} \binom{m-z+ky}{k} \binom{z-ky}{n-k} \\
&= \frac{m-z}{n!} \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} (1+m-k+ky-z)_{k-1} (ky-z)_{n-k} \\
&= \frac{m-z}{n!} \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} (ky-z)_m (1+m-k+ky-z)_{n-m-1}.
\end{aligned}$$

For $0 \leq m < n$, we assert that $\mathcal{P}(m-z)$ vanishes because it results in the n th differences of the following polynomial

$$(xy-z)_m (1+m-x-z+xy)_{n-m-1} \quad \text{of degree } n-1.$$

When $m = n$, we can evaluate

$$\begin{aligned}
\mathcal{P}(n-z) &= \frac{n-z}{n!} \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} \frac{(ky-z)_n}{n-k-z+ky} \\
&= \frac{n-z}{n!} \Delta_0^n \frac{(xy-z)_n}{n-x-z+xy} \\
&= \frac{n-z}{n!(y-1)} \Delta_0^n \frac{\left(\frac{z-ny}{y-1}\right)_n}{x-\frac{z-n}{y-1}} \\
&= (-1)^n \frac{n-z}{y-1} \frac{\left(\frac{z-ny}{y-1}\right)_n}{\left(-\frac{z-n}{y-1}\right)_{n+1}} = 1.
\end{aligned}$$

Therefore, $\mathcal{P}(x)$ is a polynomial with the same values at the $n+1$ distinct points $\{m-z\}_{m=0}^n$ as another polynomial $\binom{x+z}{n}$ of degree n . This shows that both polynomials are identical which proves (6a).

Analogously, let $\mathcal{Q}(x)$ be the binomial sum in (6b), which is again a polynomial of degree n . Its value at $x = m - z$ reads as

$$\begin{aligned}\mathcal{Q}(m - z) &= \sum_{k=0}^n \frac{m - z}{m - z + ky} \binom{m - z + ky}{k} \frac{z - ny}{z - ky} \binom{z - ky}{n - k} \\ &= \frac{(z-m)(z-ny)}{n!} \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} (1+m-k+ky-z)_{k-1} (1+ky-z)_{n-k-1} \\ &= \frac{(z-m)(z-ny)}{n!} \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} (1+ky-z)_{m-1} (1+m-k+ky-z)_{n-m-1}.\end{aligned}$$

For $1 \leq m < n$, it is clear that $\mathcal{Q}(m - z)$ vanishes because it results in the n th differences of the following polynomial

$$(1 - z + xy)_{m-1} (1 + m - x - z + xy)_{n-m-1} \quad \text{of degree } n - 2.$$

In addition, we can evaluate

$$\begin{aligned}\mathcal{Q}(0 - z) &= \frac{z(z - ny)}{n!} \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} \frac{(1 - k - z + ky)_{n-1}}{ky - z} \\ &= \frac{z(z - ny)}{n!} \Delta_0^n \frac{(1 - x + xy - z)_{n-1}}{xy - z} \\ &= \frac{z(z - ny)}{n!y} \Delta^n \frac{(1 - z/y)_{n-1}}{x - z/y} \Big|_{x=0} \\ &= (-1)^n \frac{z(z - ny)}{y} \frac{(1 - z/y)_{n-1}}{(-z/y)_{n+1}} = (-1)^n y\end{aligned}$$

and

$$\begin{aligned}\mathcal{Q}(n - z) &= \frac{(z - n)(z - ny)}{n!} \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} \frac{(1 + ky - z)_{n-1}}{n - k + ky - z} \\ &= \frac{(z - n)(z - ny)}{n!} \Delta_0^n \frac{(1 + xy - z)_{n-1}}{n - x + xy - z} \\ &= \frac{(z - n)(z - ny)}{n!(y - 1)} \Delta^n \frac{(1 + \frac{z-ny}{y-1})_{n-1}}{x - \frac{z-n}{y-1}} \Big|_{x=0} \\ &= (-1)^n \frac{(z - n)(z - ny)}{y - 1} \frac{(1 + \frac{z-ny}{y-1})_{n-1}}{(-\frac{z-n}{y-1})_{n+1}} = 1 - y.\end{aligned}$$

Therefore, $\mathcal{Q}(x)$ is a polynomial with the same values at the $n + 1$ distinct points $\{m - z\}_{m=0}^n$ as another polynomial $\frac{x+z-ny}{x+z} \binom{x+z}{n}$ of degree n . This implies that both polynomials are identical which proves (6b).

5. DIXON'S TERMINATING SUMMATION FORMULA

One of the terminating forms of Dixon's summation theorem is (cf. Bailey [3])

$${}_3F_2 \left[\begin{matrix} -n, & x, & y \\ 1-x-n, & 1-y-n \end{matrix} \middle| 1 \right] = \frac{(1+\ell)_\ell (x+y+\ell)_\ell}{(x+\ell)_\ell (y+\ell)_\ell} \chi(n=2\ell). \quad (7)$$

A well-known particular case of it is the following alternating sum of cubic binomial coefficients (cf. Gould [14, Entry 6.6; P.51]):

$$\sum_{k=0}^n (-1)^k \binom{n}{k}^3 = (-1)^\ell \frac{(3\ell)!}{(\ell!)^3} \chi(n=2\ell).$$

For a real number x , denote by $\lfloor x \rfloor$ its integer part. Rewrite (7) equivalently as

$$\begin{aligned} \sum_{k=0}^n (-1)^k \binom{n}{k} \frac{(x)_k (y)_k}{(1-x-n)_k (1-y-n)_k} \\ = \frac{(1+\ell)_\ell (x+y+\ell)_\ell}{(x+\ell)_\ell (y+\ell)_\ell} \chi(n=2\ell). \end{aligned} \quad (8)$$

Multiplying this equation by $(x+n-\lfloor \frac{n}{2} \rfloor)_{\lfloor \frac{n}{2} \rfloor}$, we would get a polynomial identity of degree $\leq \lfloor \frac{n}{2} \rfloor$. This can be justified by combining the relation

$$\frac{(x)_k (x+n-\lfloor \frac{n}{2} \rfloor)_{\lfloor \frac{n}{2} \rfloor}}{(1-x-n)_k} = (-1)^k \frac{(x)_k (x+n-\lfloor \frac{n}{2} \rfloor)_{\lfloor \frac{n}{2} \rfloor}}{(x+n-k)_k}$$

with

$$\frac{(x)_k (x+n-\lfloor \frac{n}{2} \rfloor)_{\lfloor \frac{n}{2} \rfloor}}{(x+n-k)_k} = \begin{cases} (x)_k (x+n-k)_{\lfloor \frac{n}{2} \rfloor - k}, & k \leq \lfloor \frac{n}{2} \rfloor; \\ (x)_k / (x+n-k)_{k-\lfloor \frac{n}{2} \rfloor}, & k > \lfloor \frac{n}{2} \rfloor. \end{cases}$$

In order to prove (8), we need only to validate it for $1 + \lfloor \frac{n}{2} \rfloor$ distinct values of x . Let $S(x)$ be the finite sum displayed in (8). In view of

the equation

$$\frac{(y)_k(1-y+m-n)_k}{(y-m)_k(1-y-n)_k} = \frac{(y-m+k)_m(1-y-n+k)_m}{(y-m)_m(1-y-n)_m}$$

we have the following expression

$$\begin{aligned} S(1-y+m-n) &= \sum_{k=0}^n (-1)^k \binom{n}{k} \frac{(y)_k(1-y+m-n)_k}{(y-m)_k(1-y-n)_k} \\ &= \sum_{k=0}^n (-1)^k \binom{n}{k} \frac{(y-m+k)_m(1-y-n+k)_m}{(y-m)_m(1-y-n)_m} \end{aligned}$$

which vanishes for $0 \leq m < n/2$ because it results in the n th differences of the polynomial $(x+y-m)_m(1+x-y-n)_m$ of degree $2m < n$. When n is odd, we are done because (8) is valid for the $1 + \lfloor \frac{n}{2} \rfloor$ distinct values $\{1-y+m-n\}_{m=0}^{\lfloor \frac{n}{2} \rfloor}$ of x .

When $n = 2\ell$ is even, we have found that the polynomial $S(x)$ has ℓ zeros $\{1-y+m-2\ell\}_{m=0}^{\ell-1}$. In addition, we have to compute, for $m = \ell$, the following extreme value

$$\begin{aligned} S(1-y-\ell) &= \sum_{k=0}^{2\ell} (-1)^k \binom{2\ell}{k} \frac{(y+k-\ell)_\ell(1-y-2\ell+k)_\ell}{(y-\ell)_\ell(1-y-2\ell)_\ell} \\ &= \frac{(2\ell)!}{(y-\ell)_\ell(1-y-2\ell)_\ell} = \frac{(2\ell)!}{(1-y)_\ell(y+\ell)_\ell} \end{aligned}$$

which coincides with the right member of equation (8) specified with $x = 1-y-\ell$. Therefore, we have validated the equality (8) for the $1 + n/2$ distinct values $\{1-y+m-n/2\}_{m=0}^{n/2}$ of x , also when n is even.

This completes the proof of (8) and also the terminating summation formula (7).

6. STANTON'S EXTENSION OF ANDREWS' ${}_5F_4$ -SUM

Recently, Gessel [13] found an ingenious proof for the following summation formula

$${}_5F_4 \left[\begin{matrix} -1-2n, & 1+x+n, & x, & z, & \frac{1}{2}+x-z \\ & \frac{x-n}{2}, & \frac{1+x-n}{2}, & 2z, & 1+2x-2z \end{matrix} \middle| 1 \right] \equiv 0. \quad (9)$$

It was discovered by Andrews [1, Eq 1.6] in determinant evaluation connected to plane partitions. For different proofs of (9), refer to [9, 12, 23, 25].

Following Gessel's approach, we present a similar proof for the extended formula below which is due to Stanton [23, Eq.A.2] (cf. Chu [9, Eq.2.22] also):

$$\begin{aligned} & {}_6F_5 \left[\begin{matrix} -1 - 2n, & 1 + \lambda, & x + n, & x, & z, & \frac{1}{2} + x - z \\ & \lambda, & \frac{x-n}{2}, & \frac{1+x-n}{2}, & 2z, & 1 + 2x - 2z \end{matrix} \middle| 1 \right] \\ &= \frac{\lambda - x - n}{\lambda(1 + x + 3n)} \left[\begin{matrix} \frac{3}{2}, & 1 + x - 2z, & 2z - x \\ 1 - x, & \frac{1}{2} + z, & 1 + x - z \end{matrix} \right]_n. \end{aligned} \quad (10)$$

Rewrite the last formula as a binomial equality

$$\begin{aligned} & \sum_{k=0}^{2n+1} (-1)^k \binom{2n+1}{k} \frac{\lambda + k}{\lambda} \frac{(x)_k (x+n)_k}{\left(\frac{x-n}{2}\right)_k \left(\frac{1+x-n}{2}\right)_k} \frac{(z)_k \left(\frac{1}{2} + x - z\right)_k}{(2z)_k (1 + 2x - 2z)_k} \\ &= \frac{\lambda - x - n}{\lambda(1 + x + 3n)} \left[\begin{matrix} \frac{3}{2}, & 1 + x - 2z, & 2z - x \\ 1 - x, & \frac{1}{2} + z, & 1 + x - z \end{matrix} \right]_n. \end{aligned} \quad (11)$$

Multiplying across the last equation by $\left(\frac{1}{2} + z\right)_n (1 + x - z)_n$, we would get a polynomial identity of degree $2n$ in z , if we can show that the following expression results in a polynomial of degree $2n$ in z :

$$\frac{(z)_k \left(\frac{1}{2} + z\right)_n}{(2z)_k} \times \frac{\left(\frac{1}{2} + x - z\right)_k (1 + x - z)_n}{(1 + 2x - 2z)_k}.$$

Because the second fraction becomes the first one under the substitution $z \rightarrow \frac{1}{2} + x - z$, it is sufficient to prove that the first fraction is a polynomial of degree n in z . This is indeed the case in view of the following expression:

$$\frac{(z)_k \left(\frac{1}{2} + z\right)_n}{(2z)_k} = \begin{cases} \frac{(z)_k \left(\frac{1}{2} + z\right)_k \left(\frac{1}{2} + z + k\right)_{n-k}}{(2z)_k} = \frac{(2z)_{2k} \left(\frac{1}{2} + z + k\right)_{n-k}}{4^k (2z)_k}, & k \leq n; \\ \frac{(z)_n (z+n)_{k-n} \left(\frac{1}{2} + z\right)_n}{(2z)_k} = \frac{(2z)_{2n} (z+n)_{k-n}}{4^n (2z)_k}, & k > n. \end{cases}$$

In order to prove the identity (11), it is enough to validate it for $2n + 1$ distinct values of z . Denote by $T(z)$ the binomial sum

displayed on the left of (11). We are going to evaluate

$$T\left(\frac{x-m}{2}\right) = \sum_{k=0}^{2n+1} (-1)^k \binom{2n+1}{k} \frac{\lambda+k}{\lambda} P(k) Q(k)$$

where

$$P(k) := \frac{\left(\frac{x-m}{2}\right)_k \left(\frac{1+x+m}{2}\right)_k}{\left(\frac{x-n}{2}\right)_k \left(\frac{1+x-n}{2}\right)_k} \quad \text{and} \quad Q(k) := \frac{(x)_k (x+n)_k}{(x-m)_k (1+x+m)_k}.$$

According to the expressions

$$P(k) = \begin{cases} \frac{\left(\frac{x-n}{2} + k\right)_{\frac{n-m}{2}} \left(\frac{1+x-n}{2} + k\right)_{\frac{m+n}{2}}}{\left(\frac{x-n}{2}\right)_{\frac{n-m}{2}} \left(\frac{1+x-n}{2}\right)_{\frac{m+n}{2}}}, & m = n \pmod{2}; \\ \frac{\left(\frac{x-n}{2} + k\right)_{\frac{m+n+1}{2}} \left(\frac{1+x-n}{2} + k\right)_{\frac{n-m-1}{2}}}{\left(\frac{x-n}{2}\right)_{\frac{m+n+1}{2}} \left(\frac{1+x-n}{2}\right)_{\frac{n-m-1}{2}}}, & m \neq n \pmod{2}; \end{cases}$$

and

$$Q(k) = \begin{cases} \frac{(x-m+k)_m (1+x+m+k)_{n-m-1}}{(x-m)_m (1+x+m)_{n-m-1}}, & m \geq 0; \\ \frac{(x-m+k)_{m+n} (1+x+m+k)_{-m-1}}{(x-m)_{m+n} (1+x+m)_{-m-1}}, & m < 0; \end{cases}$$

we can see that for $-n \leq m < n$, both $P(k)$ and $Q(k)$ are polynomials of k with degrees n and $n-1$, respectively. Therefore $T\left(\frac{x-m}{2}\right)$ vanishes for $-n \leq m < n$ because it is essentially the $(2n+1)$ th differences of a polynomial of degree $2n$. Furthermore, we can also compute the following extreme value:

$$\begin{aligned} T\left(\frac{x-n}{2}\right) &= \sum_{k=0}^{2n+1} (-1)^k \binom{2n+1}{k} \frac{\lambda+k}{\lambda} \frac{(x)_k (x+n)_k}{(x-n)_k (1+x+n)_k} \frac{\left(\frac{1+x+n}{2}\right)_k}{\left(\frac{1+x-n}{2}\right)_k} \\ &= \sum_{k=0}^{2n+1} (-1)^k \binom{2n+1}{k} \frac{(x+n)(\lambda+k)}{\lambda(x-n)_n} \frac{(x-n+k)_n \left(\frac{1+x-n}{2} + k\right)_n}{(k+x+n) \left(\frac{1+x-n}{2}\right)_n} \\ &= \frac{(x+n)(\lambda-x-n)(2n+1)!(-2n)_n \left(\frac{1-x-3n}{2}\right)_n}{\lambda(x-n)_n (x+n)_{2n+2} \left(\frac{1+x-n}{2}\right)_n} \\ &= \frac{\lambda-x-n}{\lambda(1+x+3n)} \left[1 - x, \frac{\frac{3}{2}, 1+n, -n}{\frac{1+x-n}{2}, 2+x+n} \right]_n \end{aligned}$$

which coincides with the right member (11) at $z = \frac{x-n}{2}$. In conclusion, we have checked the equality (11) for the $2n+1$ distinct values $z = \frac{x-m}{2}$ with $-n \leq m \leq n$. This completes the proof of (11) and so Stanton's summation formula (10).

7. CONVOLUTION IDENTITIES OF ABEL

Instead of the fundamental theorem of algebra on polynomials, the Lagrange interpolation can also be utilized to justify the final passage in the proving process of binomial identities for the examples hitherto illustrated.

In this section, we prove, by means of Taylor polynomials, the following deep generalization for the binomial theorem discovered by Abel (cf. Graham *et al* [16, §5.4], Riordan [20, §1.5] and [6, 10, 22] for example):

$$x \sum_{k=0}^n \binom{n}{k} (x+ky)^{k-1} (z-ky)^{n-k} = (x+z)^n, \quad (12a)$$

$$x \sum_{k=0}^n \binom{n}{k} (x+ky)^{k-1} \frac{z-ny}{z-ky} (z-ky)^{n-k} = \frac{x+z-ny}{x+z} (x+z)^n. \quad (12b)$$

Denote by $P(z)$ the binomial sum in (12a). Its m th derivative at $z = -x$ gives

$$P^{(m)}(-x) = m!x \binom{n}{m} \sum_{k=0}^{n-m} (-1)^{n-m-k} \binom{n-m}{k} (x+ky)^{n-m-1}.$$

For $0 \leq m < n$, the last sum results in the $(n-m)$ th differences of a polynomial of degree $n-m-1$. Therefore $P^{(m)}(-x)$ is equal to zero for $0 \leq m < n$ and $P^{(n)}(-x) = n!$.

Observing further that $P(z)$ is a polynomial of degree n with its derivatives of orders $\{m\}_{m=0}^n$ at $z = -x$ equal to those of another polynomial $(x+z)^n$. Hence both polynomials are identical, which gives Abel's first identity (12a).

Analogously, let $Q(z)$ be the binomial sum in (12b). Then it is not hard to compute its m th derivative at $z = -x$ by

$$\begin{aligned} Q^{(m)}(-x) &= m!x(x - my + ny) \binom{n}{m} \\ &\times \sum_{k=0}^{n-m} (-1)^{n-m-k} \binom{n-m}{k} (x + ky)^{n-m-2} \end{aligned}$$

which vanishes for $0 \leq m \leq n - 2$ because the last sum results in the $(n - m)$ th differences of a polynomial of degree $n - m - 2$.

Taking into account of $Q^{(n)}(-x) = n!$ and $Q^{(n-1)}(-x) = n!(-y)$, we conclude that the polynomial $Q(z)$ have the same derivatives of orders $\{m\}_{m=0}^n$ at $z = -x$ as those of another polynomial $(x + z - ny)(x + z)^{n-1}$. Hence both polynomials are identical, which confirms Abel's second identity (12b).

8. MINTON'S SUMMATION THEOREM

Finally, we examine a seminal result of Minton [18] in classical hypergeometric series. It reads as the following summation theorem

$${}_{\ell+2}F_{\ell+1} \left[\begin{matrix} -n, & \lambda, & \{a_i + m_i\}_{i=1}^{\ell} \\ 1 + \lambda, & \{a_i\}_{i=1}^{\ell} \end{matrix} \middle| 1 \right] = \frac{n!}{(1 + \lambda)_n} \prod_{i=1}^{\ell} \frac{(a_i - \lambda)_{m_i}}{(a_i)_{m_i}} \quad (13)$$

provided that m_i and n are nonnegative integers with $n \geq \sum_{i=1}^{\ell} m_i$. Different proofs and extensions of this formula can be found in Karlsson [17] and Chu [5, 7]. However, we believe that the proof given here is the simplest.

According to the relation

$$\frac{(a_i + m_i)_k}{(a_i)_k} = \frac{(a_i + k)_{m_i}}{(a_i)_{m_i}}$$

we may express (13) equivalently as the following equality

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \frac{\prod_{i=1}^{\ell} (a_i + k)_{m_i}}{\lambda + k} = \frac{n!}{(\lambda)_{n+1}} \prod_{i=1}^{\ell} (a_i - \lambda)_{m_i}. \quad (14)$$

Writing the last binomial sum in terms of finite differences, we can evaluate it immediately as

$$(-1)^n \Delta^n \frac{\prod_{i=1}^{\ell} (a_i + x)_{m_i}}{\lambda + x} \Big|_{x=0} = \frac{n!}{(\lambda)_{n+1}} \prod_{i=1}^{\ell} (a_i - \lambda)_{m_i}$$

which confirms (14) and so Minton's summation formula (13).

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IRELAND'S PARTICIPATION IN THE 57TH INTERNATIONAL MATHEMATICAL OLYMPIAD

BERND KREUSSLER

From 6th until 16th July 2016, the 57th International Mathematical Olympiad took place in Hong Kong. This was the second time that an IMO was hosted in Hong Kong. A total of 602 students (71 of whom were girls) participated from 109 countries. In four categories (number of countries, number of contestants, number of participating girls, percentage of girls) previous IMO records were broken. The following countries sent a team for the first time to the IMO: Iraq, Jamaica, Kenya, Laos, Madagascar, Myanmar and Egypt.

The Irish delegation consisted of six students (see Table 1), the Team Leader, Bernd Kreussler (MIC Limerick) and the Deputy Leader, Anca Mustață (UCC).

Name	School	Year
Antonia Huang	Mount Anville Secondary School, Dublin 14	4 th
Robert Sparkes	Wesley College, Ballinteer, Dublin 16	6 th
Cillian Doherty	Coláiste Eoin, Booterstown, Co. Dublin	5 th
Ioana Grigoras	Mount Mercy College, Model Farm Road, Cork	6 th
Liam Toebes	Carrigaline Community School, Co. Cork	6 th
Anna Mustață	Bishopstown Community School, Cork	4 th

TABLE 1. The Irish contestants at the 57th IMO

1. TEAM SELECTION AND PREPARATION

Each year in November, the Irish Mathematical Olympiad starts with Round 1, a contest that is held in schools during a regular class period. In 2015 almost 14,000 students, mostly in their senior cycle, from about 290 second level schools participated in Round 1. Teachers were encouraged to hand out invitations to mathematics enrichment classes to their best performing students.

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At five different locations all over Ireland (UCC, UCD, NUIG, UL and MU), mathematical enrichment programmes are offered to mathematically talented students, usually in their senior cycle of secondary school. These classes run each year from December/January until April and are offered by volunteer academic mathematicians from these universities or nearby third-level institutions.

Rarely, students who participate for the first time in the mathematics enrichment programme qualify for the Irish IMO team – Cillian Doherty is one of these rare exceptions. Usually, those who make it to the team come back after their first enrichment year to get more advanced training.

In order to activate the full potential of these returning students – like in previous years – an Irish Maths Olympiad Squad was formed. It consisted of the 13 best performing students at IrMO 2015, who were eligible to participate in IMO 2016. Between IrMO and the restart of the enrichment classes, for this group of students the following extra training activities were offered: two training camps (one in June and one at the end of August), participation in the Iranian Geometry Olympiad (September), a remote training which runs from September to December and participation in Round 1 of the British Mathematical Olympiad (November).

The centrally organised remote training, which was offered for the first time in 2013, is now an established pillar of the preparation of the Irish IMO team. At the beginning of each of the four months from September to December, two sets of three problems were emailed to the participating students. They emailed back their solutions before the end of the month to the sender of the problems, who gave feedback on their attempts as soon as possible. The eight trainers involved in 2015 were: Mark Flanagan, Eugene Gath, Bernd Kreussler, Gordon Lessells, John Murray, Anca Mustață, Andrei Mustață and Rachel Quinlan.

An important component of the training for maths olympiads is to expose the students to olympiad-type exams. It is now an established tradition in all five enrichment centres to hold a local contest in February or March. In addition, this year a number of students from Ireland was invited to participate in the British Mathematical Olympiad Round 1 (27 November 2015) and Round 2 (28 January 2016). I would like to thank UKMT, in particular Geoff Smith, for giving our students this opportunity.

For the first time in 2015, some members of the Irish Maths Olympiad Squad participated in the Iranian Geometry Olympiad, which took place on 3 September 2015. Participants of about 20 countries sat the olympiad exam in their home counties. Each participating country received solutions with marking schemes from the organisers and was responsible to grade the exam papers of their own students. The results together with scans of the solutions of the four best students could then be sent by email to the organisers. The exam problems were fairly tough; there was no especially 'easy' problem on the paper. The maximally possible score was 40 points, the Bronze medal cut-off at IGO was 14 points and the best score of an Irish participant was 8 points. More information can be found on www.igo-official.ir.

The selection contest for the Irish IMO team is the Irish Mathematical Olympiad (IrMO), which was held for the 29th time on Saturday, 23rd April 2016. The IrMO contest consists of two 3-hour papers on one day with five problems on each paper. The participants of the IrMO, who normally also attend the enrichment classes, sat the exam simultaneously in one of the five centres. This year, a total of 88 students took part in the final Round of IrMO. The top performer is awarded the Fergus Gaines cup; this year this was Antonia Huang. The best six students (listed in order in Table 1) were invited to represent Ireland at the IMO in Hong Kong.

In addition to the training camps mentioned above and an IMO-team camp at UCC before departure, immediately before the IMO a four-day joint training camp with the team from Trinidad and Tobago was held in Hong Kong. The sessions were conducted by the two Deputy Leaders Anca Mustață and Jagdesh Ramnanan.

2. THE DAYS IN HONG KONG

The team (including Leader and Deputy Leader) arrived around 10pm on Monday, the 4th of July in Hong Kong. A bus ride of more than 90 minutes took us to the Holiday Inn Express Hotel in Kowloon East, where the team would carry out their intensive pre-IMO training camp in collaboration with the team from Trinidad and Tobago.

During the camp the students enjoyed the excellent free facilities of some of the local public libraries. On three of the days they took separate mock exams that were similar to IMO exams in duration

and difficulty. On each day, different members of the team had particular success in solving the mock exam problems, which indicated that each team member was capable of a good performance at the IMO on their best day.

On Tuesday at noon I was transferred to the Harbour Grand Hotel Kowloon where the Jury resided until the day of the second exam.

The Jury of the IMO, which is composed of the Team Leaders of the participating countries and a Chairperson who is appointed by the organisers, is the prime decision making body for all IMO matters. Its most important task is choosing the six contest problems out of a shortlist of 32 problems provided by a problem selection committee, also appointed by the host country. This year's official Chairperson of the Jury was Prof. Kar-Ping Shum. He was already Chair of the Jury at the IMO in Hong Kong in 1994. He received the Paul Erdős Award 2016 of the World Federation of National Mathematics Competitions for devoting himself for more than 30 years to the promotion of mathematics, mathematics education and mathematics competitions in Hong Kong and all over Southeast Asia.

The Jury sessions this year were conducted by Andy Loo on behalf of the Chairman. With his excellent communication skills, the Silver Medallist at the IMO in Argentina 2012 made the Jury sessions a pleasant experience. A novelty introduced by the organisers was the use of electronic voting devices during the Jury meetings. This sped up the usually very lengthy process of selecting the six contest problems, but also made all votes secret. In situations where a clear majority was expected to vote in favour of a certain motion, Andy Loo used voice votes (“Those in favour of the motion say ‘Aye’, . . . , those against say ‘No’, I think the Ayes have it, the Ayes have it.”). This procedure was even faster than the use of the electronic voting devices. If the voice vote didn't end with an obvious majority for one option, an electronic vote would be conducted.

Before the process of problem selection was begun, the Jury decided if they want to continue the practice of recent years to have one problem from each of the four areas (algebra, combinatorics, geometry and number theory) included in problems 1, 2, 4 and 5. A majority of almost two thirds voted in favour of continuing this practice. Two problems, one from algebra and one from combinatorics, had to be removed from the short list because similar problems with similar solutions were used in recent competitions in Bulgaria and

Russia. From the remaining 30 shortlist problems, the six contest problems were selected in an efficient way during a number of Jury meetings on Friday, 8 July. During the remaining Jury meetings, translations and marking schemes were approved. The creativity of the leaders when translating the exam problems into their native languages becomes evident in Problem 6: about 36 different names were used for the person who claps his or her hands.

During a couple of joint meetings of the Jury with the IMO Advisory Board, important changes to the general regulations were discussed and approved. The most important change concerns eligibility rules for contestants. So far, it was required that Contestants are not formally enrolled at a university. The newly adopted regulations require instead that Contestants must have been normally enrolled in full-time primary or secondary education. Also, the reference date for the age limit is no longer the day of the second Contest paper, but now is the first of July. The new regulations will be phased in within the next two years.

On Saturday, the 9th of July, the Irish team moved from the Holiday Inn Express to student accommodation on the campus of HKUST and the IMO got under way. The opening ceremony took place at Queen Elizabeth Stadium on Sunday afternoon. During the traditional parade, the teams appeared in order of the first participation of their countries at the IMO. In addition to the usual speeches, there were performances of several pieces of music written especially for this event, such as the IMO 2016 Theme Song “In Love We Are One”.

The two exams took place on the 11th and 12th of July, starting at 9 o'clock each morning. During the first 30 minutes, the students were allowed to ask questions if they had difficulties in understanding the formulation of a contest problem. The Q&A session on the first day of the contest, where 85 questions were asked by students from 44 countries, was completed at 10.51 am. On the second day, 95 questions from students of 50 countries were answered by 10.45 am. Initial discussions took place about possibilities to streamline the Q&A sessions at future IMOs, for example by dealing with routine questions in a standardised way and by having designated people, involving coordinators, for each of the three questions who follow the process closely.

The student's scripts were available on the evening of the first exam day. Skimming through their work it seemed that Ioana could probably get full marks for Problem 1. Because all our team members were aware of spiral similarity, they could secure at least one mark for the only geometry problem on this year's paper. The work of our students for Problems 2 and 3 didn't look that promising. After joining the contestants at HKUST, Anca and I went into the detailed study of our student's scripts. Anca's excellent knowledge in geometry proved to be crucial for securing all the 23 marks our students deserved for Problem 1.

During the coordination days, the students were entertained with excursions to a variety of interesting places in Hong Kong. On the first day they took advantage of Antonia's familiarity with the place to visit the second tallest building in Hong Kong, the Two International Finance Centre, from where they could view the city from the top. On the second day they went on a bus excursion to The Peak, a unique high point on Hong Kong Island offering spectacular views of the city. The shape of the Peak Tower blended with the letter π forms a major component of the logo of this year's IMO. The students also visited a traditional market and a school.

The final Jury meeting, at which the medal cut-offs were decided, took place on the evening of Thursday, 14th July. The closing ceremony followed by the IMO Dinner was held on Friday evening at the Hong Kong Convention and Exhibition Centre. The journey back home started for our team on Saturday very early in the morning.

3. THE PROBLEMS

The two exams took place on the 11th and 12th of July, starting at 9 o'clock each morning. On each day, $4\frac{1}{2}$ hours were available to solve three problems.

Problem 1. Triangle BCF has a right angle at B . Let A be the point on line CF such that $FA = FB$ and F lies between A and C . Point D is chosen such that $DA = DC$ and AC is the bisector of $\angle DAB$. Point E is chosen such that $EA = ED$ and AD is the bisector of $\angle EAC$. Let M be the midpoint of CF . Let X be the point such that $AMXE$ is a parallelogram (where $AM \parallel EX$ and $AE \parallel MX$). Prove that lines BD , FX , and ME are concurrent. (Belgium)

Problem 2. Find all positive integers n for which each cell of an $n \times n$ table can be filled with one of the letters I , M and O in such a way that:

- in each row and each column, one third of the entries are I , one third are M and one third are O ; and
- in any diagonal, if the number of entries on the diagonal is a multiple of three, then one third of the entries are I , one third are M and one third are O .

Note: The rows and columns of an $n \times n$ table are each labelled 1 to n in a natural order. Thus each cell corresponds to a pair of positive integers (i, j) with $1 \leq i, j \leq n$. For $n > 1$, the table has $4n - 2$ diagonals of two types. A diagonal of the first type consists of all cells (i, j) for which $i + j$ is a constant, and a diagonal of the second type consists of all cells (i, j) for which $i - j$ is a constant. (Australia)

Problem 3. Let $P = A_1A_2 \dots A_k$ be a convex polygon in the plane. The vertices A_1, A_2, \dots, A_k have integral coordinates and lie on a circle. Let S be the area of P . An odd positive integer n is given such that the squares of the side lengths of P are integers divisible by n . Prove that $2S$ is an integer divisible by n . (Russia)

Problem 4. A set of positive integers is called *fragrant* if it contains at least two elements and each of its elements has a prime factor in common with at least one of the other elements. Let $P(n) = n^2 + n + 1$. What is the least possible value of the positive integer b such that there exists a non-negative integer a for which the set

$$\{P(a+1), P(a+2), \dots, P(a+b)\}$$

is fragrant?

(Luxembourg)

Problem 5. The equation

$$(x-1)(x-2) \cdots (x-2016) = (x-1)(x-2) \cdots (x-2016)$$

is written on the board, with 2016 linear factors on each side. What is the least possible value of k for which it is possible to erase exactly k of these 4032 linear factors so that at least one factor remains on each side and the resulting equation has no real solutions? (Russia)

Problem 6. There are $n \geq 2$ line segments in the plane such that every two segments cross, and no three segments meet at a point. Geoff has to choose an endpoint of each segment and place a frog on it, facing the other endpoint. Then he will clap his hands $n - 1$ times. Every time he claps, each frog will immediately jump forward to the next intersection point on its segment. Frogs never change the direction of their jumps. Geoff wishes to place the frogs in such a way that no two of them will ever occupy the same intersection point at the same time.

- (a) Prove that Geoff can always fulfil his wish if n is odd.
 (b) Prove that Geoff can never fulfil his wish if n is even.

(Czech Republic)

4. THE RESULTS

The Jury tries to choose the problems in such a way that Problems 1 and 4 are easier than Problems 2 and 5. Problems 3 and 6 are usually designed to be the hardest problems. That this goal was met this year is reflected in the scores achieved by the contestants on the problems (see Table 2).

The medal cut-offs were as follows: 29 points needed for a Gold medal (44 students), 22 for Silver (101 students) and 16 for Bronze (135 students). A further 162 students received Honourable Mentions – a record number. Overall, 35.2 % of the possible points were scored by the contestants, which is in line with the IMOs 2008–2012,

	P1	P2	P3	P4	P5	P6
0	52	277	548	132	353	474
1	63	65	25	22	36	31
2	32	99	14	26	55	9
3	9	30	0	10	21	39
4	6	7	0	26	50	4
5	35	8	2	15	2	4
6	14	9	3	24	4	4
7	391	107	10	347	81	27
average	5.272	2.033	0.251	4.744	1.678	0.806

TABLE 2. The number of contestants achieving each possible number of points on Problems 1–6

Name	P1	P2	P3	P4	P5	P6	total	ranking
Robert Sparkes	5	0	0	7	3	0	15	281
Anna Mustata	4	0	0	7	3	0	14	312
Ioana Grigoras	7	0	0	2	0	0	9	409
Cillian Doherty	3	2	0	1	0	0	6	469
Liam Toebes	2	0	0	3	0	0	5	481
Antonia Huang	2	0	0	0	0	0	2	515

TABLE 3. The results of the Irish contestants

but lower than the average scores in 2013 and 2014, and 4.3 points higher than the historically low average of last year.

Table 3 shows the results of the Irish contestants. Writing a complete solution to a problem during the exam is a difficult task at a competition of this level, and is rewarded by the award of an Honourable Mention. The three Honourable Mentions awarded to the Irish contestants this year consolidate a recent trend: 2016 represents the fifth year in a row with at least two Honourable Mentions for the Irish team.

The figures in Table 4 have the following meaning. The first figure after the problem number indicates the percentage of all points scored out of the maximum possible. The second number is the same for the Irish team and the last column indicates the Irish average score as a percentage of the overall average. The last column of this table shows that the Irish Team is approaching a competitive level at Problems 1 (geometry) and 4 (number theory). Improvements in this direction have been seen in recent years; now this seems to be a sustained trend.

Problem	topic	all countries	Ireland	relative
1	geometry	75.3	54.8	72.7
2	combinatorics	29.0	4.8	16.4
3	number theory	3.6	0.0	0.0
4	number theory	67.8	47.6	70.3
5	algebra	24.0	14.3	59.6
6	combinatorics	11.5	0.0	0.0
all		35.2	20.2	57.5

TABLE 4. Relative results of the Irish team for each problem

Although the IMO is a competition for individuals only, it is interesting to compare the total scores of the participating countries. This year's top teams were from USA (214 points), Republic of Korea (207 points) and China (204 points). Ireland, with 51 points in total achieved the 75th place. This is the fifth highest team score and the fifth best relative ranking of an Irish team at the IMO.

This year, six student achieved the perfect score of 42 points. The detailed results can be found on the official IMO website <http://www.imo-official.org>.

5. OUTLOOK

The next countries to host the IMO will be

2017	Brazil	12–24 July
2018	Romania	
2019	United Kingdom	
2020	Russia	
2021	USA	

6. CONCLUSIONS

This year's results of the Irish IMO team are in line with performance in recent years. When comparing Ireland with other countries, it is more meaningful to consider relative ranks than looking at absolute ranks, because the number of participating countries has increased over the years. This year, 31.48% of the participating teams scored less than the Irish team. After 2005 (the year in which Fiachra Knox achieved a Silver Medal) the Irish team achieved a higher relative rank only twice: in 2007, when Stephen Dolan got a Bronze Medal and in 2014, when all six students received Honourable Mentions.

Since Ireland's first participation in 1988, the Irish teams won eight medals and 37 Honourable Mentions, 18 of these since 2012. This underscores the increased ability level of recent students which is supported by increased training activities. This year, Robert Sparkes only narrowly missed a Bronze Medal.

It should be mentioned that Ireland's involvement in the European Girls' Mathematical Olympiad (EGMO) certainly had a positive impact on the training and performance of the IMO team members. This year, for the first time ever, there were three girls among the

top six performers at the Irish Mathematical Olympiad. They came with a lot of international experience: Anna has four EGMO participations under her belt and returned home with a Silver Medal from EGMO 2015 and 2016; Ioana, who participated three times at the EGMO, achieved a Bronze Medal at EGMO 2016; and Antonia, who currently holds the Fergus Gaines cup, has participated twice at EGMO. Two of these three, Anna and Antonia, will be eligible to participate at the IMO for two more years.

To sustain the positive development in the performance of the Irish Team at the IMO we have seen in recent years, more needs to be done to increase the ability and confidence of our students to solve an easy IMO problem in each of the four subject areas (algebra, combinatorics, geometry and number theory).

From successful past Irish IMO contestants we know that a crucial prerequisite for achieving an award at an IMO is the ability to work independently through new training materials and the desire to work intensely on difficult problems in their own time. One of the aims of the remote training, which runs from September until December, is to support students in developing the ability to work on their own. The score boards of the remote training in recent years indicate that only those who qualified for the Irish IMO team had responded regularly to the monthly problems. A challenge for the near future will be to increase the number of those students who engage fully with the remote training.

Experience from a large number of international and national mathematical problem-solving competitions suggest that students who get involved in such activities at an earlier age have a much higher probability to succeed at a high level. The earlier students start to engage independently in mathematical problem-solving activities, the more profoundly their problem solving skills can be developed.

With this in mind, it becomes clear how valuable any initiative is that aims at getting students in their Junior Cycle or students in Primary School involved in mathematical problem solving activities. A notable example is the Maths Circles initiative which was set up for Junior Cycle students in second level schools in the Cork area in 2013. As a follow-up, the maths enrichment centre at UCC now runs Junior Maths Enrichment Classes for students in second and third year.

Currently there is a bid for an SFI grant which aims at extending the Maths Circles initiative nationwide. One could hope that this initiative helps to motivate teachers to support problem-solving activities at a local level so that early-stage mathematical problem solving activities would become more widespread. Such activities for younger students would greatly enhance the general mathematical education of school-level students. Only with a broad base of young students with mathematical problem-solving skills, it will be possible in the long term to lead the best students to an internationally competitive level.

Also worth mentioning in this context is the PRISM (Problem Solving for Post-Primary Schools) competition, which is a multiple-choice contest designed to involve the majority of pupils in mathematical problem solving; it has a paper for Junior Cycle students and one for Senior Cycle students. This contest is organised since 2006 by mathematicians from NUI Galway.

While our students are well equipped to solve problems at the level of the Irish Maths Olympiad, they have less experience in attempting problems at IMO level. This can be disheartening for students who, at the IMO contest, find themselves unable to comfortably deal with the difficulty level as well as aspects of time management within the exam. Students from other countries have more experience in sitting exams of the difficulty and format of the IMO. We have started to build such experience into our training programmes, mainly at some of the training camps for the Irish Maths Olympiad Squad and at the joint camp with the team from Trinidad and Tobago. Ways should be explored in which IMO style exams could be made part of the team selection process.

A number of IMO teams regularly organise joint training camps that take place immediately before the start of the IMO. Joint sessions with other teams strengthen international relationships among mathematically gifted students and enrich the training of all participating teams. The joint training in Hong Kong with the team from Trinidad and Tobago was very successful and everybody agreed that similar camps should be held in future years, provided that sufficient funding is available. Prior to the IMO in Brazil 2017, such a camp could help the Irish contestants to adjust to the different time zone and the tropical climate.

To be able to fund such camps, to continue with all the other training activities mentioned in this report, to send a full team of six students and to restart the practice to send an Official Observer to any of the next IMOs, efforts have to be increased to secure funding.

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Also instrumental to funding the Irish IMO participation this year was the generous donation received by the Irish Mathematical Trust from Eoghan Flanagan, who was himself a member of the Irish IMO team in 1993 and 1994. The pre-IMO training camp in Hong Kong would not have been possible without Eoghan's generous sponsorship.

The foundation for the success of the contestants is the work with the students done in the Enrichment Programmes at the five universities. This work is carried out for free by volunteers in their spare time. Thanks go to this year's trainers at the five Irish centres:

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- NUIG: Daron Anderson, James Cruickshank, Graham Ellis, Niall Madden, Götz Pfeiffer, Rachel Quinlan and Jerome Sheahan.
- UL: Mark Burke, Ronan Flatley, Mary Frawley, Eugene Gath, Bernd Kreussler, Jim Leahy and Gordon Lessells.
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Bends in the Plane with Variable Curvature

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ABSTRACT. Explicit formulae for planar variable curvature bends are constructed using Euler’s method of natural equations. The bend paths are expressed in terms of special functions. It is shown that the length of the different bend types varies linearly with increasing radius and that the curvature of variable curvature bends can be expressed as a multiple of the curvature of a circle.

1. INTRODUCTION

Two points in the plane can be connected via the construction of a circle of radius R subtending an angle θ at the circle’s origin. The curvature κ of this curve is a constant value over its length and is given by the inverse of its radius $\kappa = 1/R$. However, there are applications for which variable curvature paths between two points in the plane are necessary. One such example can be found in the field of photonic integrated circuit design where the use of variable curvature optical waveguide bends has led to a significant reduction in optical propagation losses [1]. Use of variable curvature paths has also led to more compact designs for photonic devices [2, 3] and they are also finding applications in the realm of autonomous vehicles, e.g. in designing paths for obstacle avoidance [4].

This paper will present explicit formulae for the parameterisation of three alternative bend paths in the plane: a linear curvature bend, a trapezoidal curvature bend and a quadratic curvature bend. The bend paths are constructed using Euler’s method of natural equations [5]. The resulting formulae can be expressed in terms of the Fresnel sine and cosine integrals in the case of the linear and trapezoidal bends, and in terms of Gauss’ hypergeometric function in the case of quadratic curvature bends [6].

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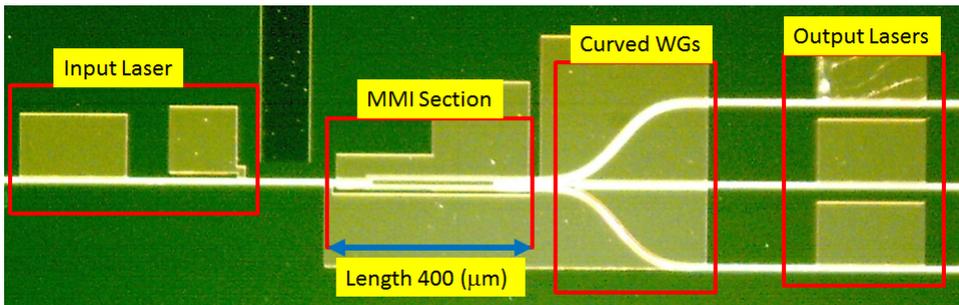


FIGURE 1. Photonic Integrated Circuit (PIC) containing curved optical waveguides [8].

The reason such formulae are necessary is that they can be used to simplify numerical simulations in photonic integrated circuits (PICs). To estimate the loss in optical waveguides one simulates the propagation of a wave in that structure. If the waveguide has no curvature along its pathlength propagation can be achieved using standard beam propagation techniques [7]. However, if the waveguide path is curved, which is often the case in compact PICs, see Figure 1 for an example [8], then it may be necessary to develop a numerical propagation scheme in an alternative coordinate system, this can be quite difficult to implement. A more straightforward approach is to include the curvature variation along the pathlength by adapting the standard beam propagation algorithm using conformal mapping techniques [9]. The curvature variation along the waveguide pathlength can be updated during simulations using the analytical formulae in this paper, for full details see [1]. Another reason explicit formulae for variable curvature bends are needed is that they can be deployed in lithographic mask layout software [10, 11] to define the geometry of photonic devices prior to fabrication.

Euler's method of natural equations is described in section 2, followed by the construction of constant, linear, trapezoidal and quadratic curvature bend paths in sections 3, 4, 5 and 6 respectively. To ensure that the variable curvature bends can be used in a practical setting, i.e. the variable curvature path should be able to replace a constant curvature path without changing the path endpoint locations, an algorithm for scaling variable curvature bends to the correct endpoint locations is provided in section 7. A bend of radius $R = 500$ and $\theta = \pi/3$ is constructed according to the different curvature schemes, the resulting bend profile is discussed in section 8.

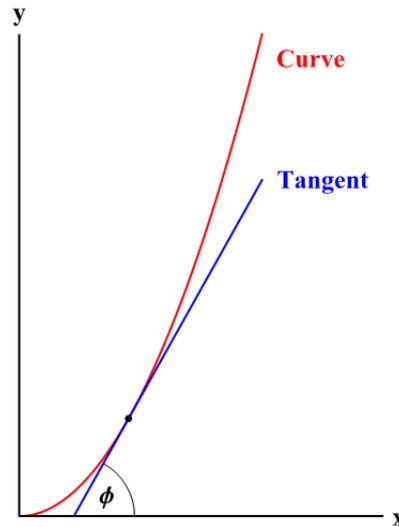


FIGURE 2. Illustration of the bending angle associated with a curve in the plane

2. EULER'S METHOD OF NATURAL EQUATIONS

The curvature κ versus path-length s profile of a curve in the plane describes the curvature at each point along that curve. A parametric representation of that curve can be constructed from its curvature using Euler's method of natural equations [5]. Euler's method requires the evaluation of three integrals; the first integral yields the bending angle as a function of path-length, the second pair of integrals yield a parametric representation of a curve that has the prescribed curvature along its path-length.

If $\kappa(s)$ is integrated along the length of the curve the result is the bending angle for that curve. The bending angle is the angle that a tangent drawn to any point on a curve makes with the tangent to the curve at the point s_0 ¹, see Figure 2. It is denoted by $\phi(s)$ and defined by

$$\phi(s) = \int_{s_0}^s \kappa(u) du \quad (1)$$

To determine the parametric representation of the bend consider a short length ds along the bend. ds can be determined from horizontal and vertical progressions along the bend via $ds^2 = dx^2 + dy^2$,

¹In this work s_0 is assumed to be the origin.

where dx and dy are given by

$$dx = \cos \phi ds \quad (2)$$

$$dy = \sin \phi ds \quad (3)$$

Summation over all of the dx and dy along the bend provides the parametric representation of the bend. The horizontal coordinates are given by

$$x(s) = \int_{s_0}^s \cos \phi(u) du \quad (4)$$

the vertical coordinates are given by

$$y(s) = \int_{s_0}^s \sin \phi(u) du \quad (5)$$

Equations (1), (4) and (5) are used to construct parametric representations of curves having curvature $\kappa(s)$ assuming the curve starts at the origin.

3. CONSTANT CURVATURE

Two points in the plane can be connected by a circle of radius R , with angle θ at its centre. We call this curve the equivalent circle. The equivalent circle has constant curvature (CC) $\kappa_{CC} = 1/R$, see Figure 3 for illustration of the CC curvature profile

$$\kappa_{cc}(s) = \frac{1}{R}, \quad 0 \leq s \leq L_{cc} \quad (6)$$

The bending angle for this curve is found by integrating (6) according to (1). The result, upon integration, shows that for a CC bend the bending angle varies linearly along its length.

$$\phi_{cc}(s) = \frac{s}{R}, \quad 0 \leq s \leq L_{cc} \quad (7)$$

The parameterisation of CC curve is obtained with the application of (4) and (5) to equation (7) to yield $(x_{cc}(s), y_{cc}(s))$ valid on $[0, L_{cc}]$.

$$x_{cc}(s) = R \sin \left(\frac{s}{R} \right) \quad (8)$$

$$y_{cc}(s) = R \left(1 - \cos \left(\frac{s}{R} \right) \right) \quad (9)$$

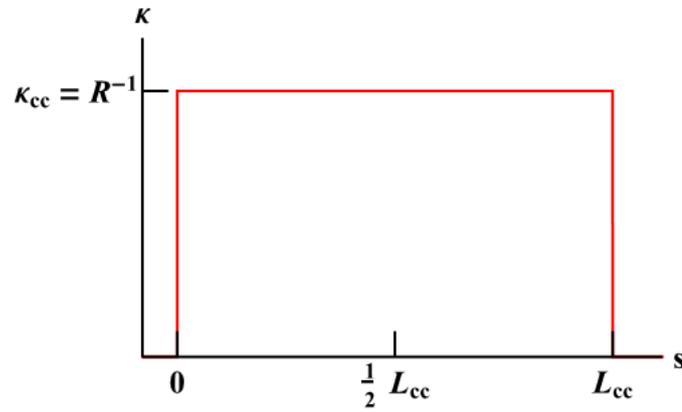


FIGURE 3. Curvature profile associated with a constant curvature bend

4. LINEAR CURVATURE

The next case to be considered is a curve with a linear curvature (LC) profile. The curvature varies as

$$\kappa_{lc}(s) = \begin{cases} \alpha_l s, & 0 \leq s \leq \frac{L_{lc}}{2} \\ \alpha_l (L_{lc} - s), & \frac{L_{lc}}{2} < s \leq L_{lc} \end{cases} \quad (10)$$

This “tent”-like profile ensures that the curve is symmetric about its midpoint, see Figure 4.

The slope α_l is chosen to ensure that the LC bend will turn through the same angle as an equivalent CC bend, this is called the equal angle condition. If a CC bend must turn through some angle θ , then α_l is determined by solving

$$\int_0^{L_{lc}/2} \alpha_l u \, du = \frac{\theta}{2} \Rightarrow \frac{1}{8} \alpha_l L_{lc}^2 = \frac{\theta}{2} \quad (11)$$

Assuming $L_{lc} = L_{cc} = R\theta$, then $\alpha_l = 4/R L_{lc}^2$. With this definition for α_l the peak curvature of the linear curvature bend is roughly twice that of an equivalent circle, $\kappa_{lc}(L_{lc}/2) \approx 2/R$, where it is required that $\kappa_{lc}(s)$ be a continuous function of path-length.

²It will be seen in Section 7 how the lengths for bends having different curvature profiles are determined

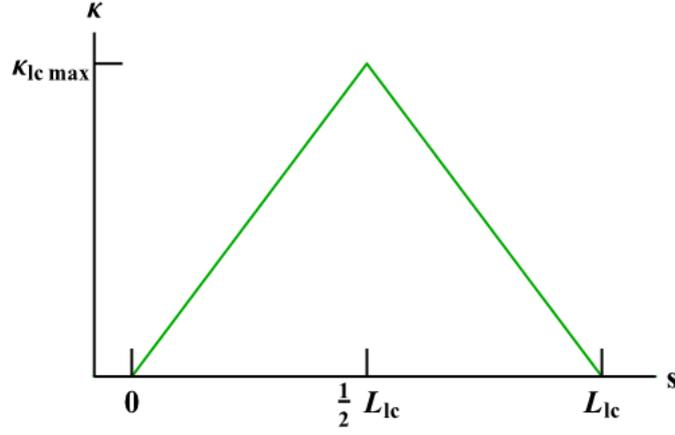


FIGURE 4. Curvature profile associated with a linear curvature bend

The bending angle for the LC bend is determined by evaluating (1) with $\kappa(s)$ defined by equation (11).

$$\phi_{lc}(s) = \begin{cases} \int_0^s \alpha_l u \, du, & 0 \leq s \leq \frac{L_{lc}}{2} \\ \int_{L_{lc}/2}^s \alpha_l (L_{lc} - u) \, du, & \frac{L_{lc}}{2} < s \leq L_{lc} \end{cases} \quad (12)$$

To ensure continuity of $\phi_{lc}(s)$ at the midpoint of the bend the value $\phi_{lc(-)}(L_{lc}/2)$, i.e. $\phi_{lc}(L_{lc}/2)$ evaluated on $0 \leq s \leq L_{lc}/2$, is added to the second branch of the bend. The bending angle for a linear curvature bend is then given by

$$\phi_{lc}(s) = \begin{cases} \frac{2s^2}{RL_{lc}}, & 0 \leq s \leq \frac{L_{lc}}{2} \\ \frac{2s^2}{RL_{lc}} - \frac{4s}{R} + \frac{L_{lc}}{R}, & \frac{L_{lc}}{2} < s \leq L_{lc} \end{cases} \quad (13)$$

$\phi_{lc}(s)$ is zero at the start of the waveguide, continuous at the midpoint and equals θ at the end.

The parameterisation of the LC bend can be computed from the integrals

$$x_{lc}(s) = \begin{cases} \int_0^s \cos\left(\frac{2u^2}{RL_{lc}}\right) du, & 0 \leq s \leq \frac{L_{lc}}{2} \\ \int_{L_{lc}/2}^s \cos\left(\frac{2u^2}{RL_{lc}} - \frac{4u}{R} + \frac{L_{lc}}{R}\right) du, & \frac{L_{lc}}{2} < s \leq L_{lc} \end{cases} \quad (14)$$

$$y_{lc}(s) = \begin{cases} \int_0^s \sin\left(\frac{2u^2}{RL_{lc}}\right) du, & 0 \leq s \leq \frac{L_{lc}}{2} \\ \int_{L_{lc}/2}^s \sin\left(\frac{2u^2}{RL_{lc}} - \frac{4u}{R} + \frac{L_{lc}}{R}\right) du, & \frac{L_{lc}}{2} < s \leq L_{lc} \end{cases} \quad (15)$$

The integrals (14) and (15) can be computed in terms of the Fresnel cosine and sine integrals, specifically using formulae (7.3.1), (7.3.2), (7.4.38) and (7.4.39) of [6]. Upon evaluation of (14) it is seen that the horizontal coordinates of a LC bend are determined by

$$x_{lc}(s) = c_1 \begin{cases} C\left(\frac{s}{c_1}\right), & 0 \leq s \leq \frac{L_{lc}}{2} \\ P_{lc}^{(x)}(s), & \frac{L_{lc}}{2} < s \leq L_{lc} \end{cases} \quad (16)$$

where

$$P_{lc}^{(x)}(s) = \cos\left(\frac{L_{lc}}{R}\right) \left(C\left(\frac{s - L_{lc}}{c_1}\right) + C(c_2) \right) + \sin\left(\frac{L_{lc}}{R}\right) \left(S\left(\frac{s - L_{lc}}{c_1}\right) + S(c_2) \right) + C(c_2) \quad (17)$$

Similarly, the parameterisation of the vertical coordinates is given by

$$y_{lc}(s) = c_1 \begin{cases} S\left(\frac{s}{c_1}\right), & 0 \leq s \leq \frac{L_{lc}}{2} \\ P_{lc}^{(y)}(s), & \frac{L_{lc}}{2} < s \leq L_{lc} \end{cases} \quad (18)$$

where

$$P_{lc}^{(y)}(s) = \sin\left(\frac{L_{lc}}{R}\right) \left(C\left(\frac{s - L_{lc}}{c_1}\right) + C(c_2) \right) - \cos\left(\frac{L_{lc}}{R}\right) \left(S\left(\frac{s - L_{lc}}{c_1}\right) + S(c_2) \right) + S(c_2) \quad (19)$$

In (16) - (19) $C(\cdot)$, $S(\cdot)$ represent the Fresnel cosine and sine integrals respectively and the following constants are used

$$c_1 = \frac{1}{2}\sqrt{\pi R L_{lc}}, \quad c_2 = \sqrt{\frac{L_{lc}}{\pi R}}$$

$C(\cdot)$, $S(\cdot)$ are numerically evaluated for real arguments using the C routine *frenel* provided in [12]. The functions for $x_{lc}(s)$ and $y_{lc}(s)$ are discontinuous at $s = \frac{L_{lc}}{2}$ when the integrals are initially evaluated. To ensure continuity of $x_{lc}(s)$ and $y_{lc}(s)$ at $s = \frac{L_{lc}}{2}$ the value of the limit of the function to the left of $s = \frac{L_{lc}}{2}$ must be added to the function on the right of $s = \frac{L_{lc}}{2}$ for each of $x_{lc}(s)$ and $y_{lc}(s)$, this ensures that the linear curvature bend is a continuous function of path-length. Equations (16) and (18) have already been adjusted to ensure continuity at $s = L_{lc}/2$.

5. TRAPEZOIDAL CURVATURE

The trapezoidal curvature (TC) bend has a three-part curvature profile defined by equation (20) see Figure 5.

$$\kappa_{tc}(s) = \begin{cases} \alpha_t s, & 0 \leq s \leq \sigma \\ \kappa_t, & \sigma < s \leq \nu \\ \alpha_t (L_{tc} - s), & \nu < s \leq L_{tc} \end{cases} \quad (20)$$

where σ defines the length of the linear portion of the bend and ν is defined by σ . The parameters α_t , κ_t , σ and ν must be chosen so that the curvature profile is continuous and that the area under the curve equals the bend angle for an equivalent circle.

The locations of the points σ and ν define the length of the linear curvature portion of the curve. To start with, choose $0 < \sigma < \frac{L_{tc}}{2}$. If $\sigma = 0$ the CC profile is recovered, if $\sigma = L_{tc}/2$ the LC profile is recovered. The curvature must satisfy $\kappa_{tc}(0) = \kappa_{tc}(L_{tc}) = 0$ and $\kappa_{tc}(\sigma) = \kappa_{tc}(\nu) = \kappa_t$. To ensure continuity at $s = \sigma$ the limits from the left and the right must be calculated. By defining $\sigma = \frac{\kappa_t}{\alpha_t}$ the curvature profile (20) is continuous at $s = \sigma$. This defines the

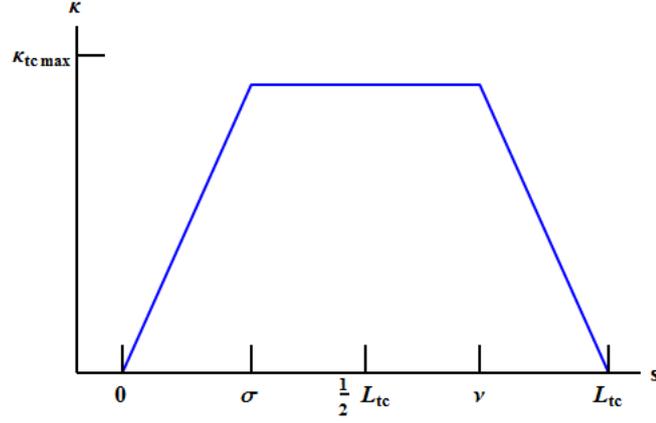


FIGURE 5. Curvature profile associated with a trapezoidal curvature bend

parameter ν to be $\nu = L_{tc} - \sigma$. With σ defined the curvature profile is also continuous at $s = \nu$. The curvature of the equivalent circle is given by $\kappa_{cc} = \frac{1}{R}$. As was seen in section 4 the maximum curvature of a linear curvature waveguide is $\kappa_{lc max} = \frac{2}{R}$, here the maximum value of the trapezoidal curvature bend must lie above that of the equivalent circle and below that of the linear waveguide, therefore $\kappa_t = \frac{\gamma}{R}$, by choosing $\gamma \in (1, 2)$ the curvature profile will maintain its trapezoidal shape³.

The slope of the linear curvature region α_t needs to be determined. This is done by invoking the equal bend-angle condition. By symmetry, only the first half of the bend need be considered. The trapezoidal curvature bend will turn through the correct angle if the condition

$$\int_0^{L_{tc}/2} \kappa_{tc}(s) ds = \frac{\theta}{2} \quad (21)$$

is valid. Substitution of (20) in (21) results in

$$\begin{aligned} \frac{\theta}{2} &= \int_0^{\sigma} \alpha_t s ds + \int_{\sigma}^{L_{tc}/2} \kappa_t ds \\ &= \frac{1}{2} \alpha_t \sigma^2 + \frac{1}{2} \kappa_t L_{tc} - \kappa_t \sigma \end{aligned}$$

Working through the algebra results in the following expression for the slope of the linear curvature region

$$\alpha_t = \frac{\kappa_t^2}{\kappa_t L_{tc} - \theta} \quad (22)$$

³If $\gamma = 1$ then $\kappa_{tc}(s) \equiv \kappa_{cc}(s)$, $\gamma = 2$ then $\kappa_{tc}(s) \equiv \kappa_{lc}(s)$

At this point only the curvature scaling parameter γ remains unknown⁴. To determine γ the fraction of the total bend length to be given over to linear curvature must be decided. Assume that the linear portion is some fraction f of the total length of the bend, $\sigma = fL_{tc}$, where $0 < f < \frac{1}{2}$. Taking α_t as it's defined by (22), σ can be written in terms of L_{tc} and γ .

$$\sigma = \frac{\kappa_t}{\alpha_t} = L_{tc} \left(1 - \frac{1}{\gamma} \right) \quad (23)$$

Since this must equal $\sigma = fL_{tc}$ the curvature scaling parameter can be defined in terms of the fraction of the bend whose curvature is linear.

$$\gamma = \frac{1}{1-f} \quad (24)$$

Choosing f between 0 and 1/2 will ensure the profile maintains its trapezoidal shape. For the moment a value of $f = 1/4$ is chosen, this means that 50% of each TC bend has linear curvature, and 50% has constant curvature and also that for TC bends $\gamma = 4/3$.

Now that the curvature profile parameters are defined, the bending angle, and hence the parameterisation of the trapezoidal curvature bend can be computed. Computation of the bending angle requires the evaluation of three integrals, and confirmation of continuity at the points σ and ν . The integrals that define the bending angle are

$$\phi_{tc}(s) = \begin{cases} \int_0^s \alpha_t u \, du, & 0 \leq s \leq \sigma \\ \int_\sigma^s \kappa_t \, du, & \sigma < s \leq \nu \\ \int_\nu^s \alpha_t (L_{tc} - u) \, du, & \nu < s \leq L_{tc} \end{cases} \quad (25)$$

The result upon integration is a discontinuous function of path-length, but it can be made continuous at $s = \sigma$ by adding $\phi_{lc(-)}(\sigma) = \frac{\kappa_t^2}{2\alpha_t}$ to the portion defined on $\sigma \leq s \leq \nu$. Similarly, at $s = \nu$ add $\phi_{lc(-)}(\nu) = \kappa_t L_{tc} - \frac{3\kappa_t^2}{2\alpha_t}$ to the portion defined on $\nu \leq s \leq L_{tc}$. The resulting function satisfies $\phi_{tc}(0) = 0$, $\phi_{tc}(L_{tc}) = \theta$ and is

⁴Leaving aside the fact that $L_{tc} \neq L_{cc}$, which we will come to in Section 7

continuous at $s = \sigma$ and $s = \nu$.

$$\phi_{tc}(s) = \begin{cases} \frac{1}{2} \alpha_t s^2, & 0 \leq s \leq \sigma \\ \kappa_t \left(s - \frac{\kappa_t}{\alpha_t} \right) + \frac{\kappa_t^2}{2 \alpha_t}, & \sigma < s \leq \nu \\ F_{tc}(s), & \nu < s \leq L_{tc} \end{cases} \quad (26)$$

where

$$F_{tc}(s) = \frac{1}{2 \alpha_t} (\kappa_t^2 - \alpha_t^2 (L_{tc} - s)^2) + \kappa_t L_{tc} - \frac{3 \kappa_t^2}{2 \alpha_t} \quad (27)$$

The parameterisation of the trapezoidal curvature bend can be computed by substituting (26) into equations (4) and (5). The integrals on $[0, \sigma]$ can be evaluated in terms of the Fresnel cosine and sine integrals, see (7.3.1), (7.3.2) in [6]. The integrals on $(\sigma, \nu]$ can be evaluated exactly because the argument of the cosine and sine function in each case is a linear function of u . The integrals $(\nu, L_{tc}]$ are evaluated using (7.4.38), (7.4.39) in [6]. After ensuring that the parameterisation is continuous at positions $s = \sigma$ and $s = \nu$ the horizontal coordinates of the TC bend are provided by

$$x_{tc}(s) = \begin{cases} d_1 C \left(\frac{s}{d_1} \right), & 0 \leq s \leq \sigma \\ P_{tc}^{(x)}(s), & \sigma < s \leq \nu \\ Q_{tc}^{(x)}(s), & \nu < s \leq L_{tc} \end{cases} \quad (28)$$

where

$$P_{tc}^{(x)}(s) = \frac{2}{\kappa_t} \sin \left(\frac{1}{2} \left(\kappa_t s - \frac{\kappa_t^2}{\alpha_t} \right) \right) \cos \left(\frac{1}{2} \kappa_t s \right) + d_1 C(d_2) \quad (29)$$

$$\begin{aligned} Q_{tc}^{(x)}(s) = & d_1 \left\{ \cos(d_3) \left[C \left(\frac{s - L_{tc}}{d_1} \right) + C(d_2) \right] \right. \\ & \left. + \sin(d_3) \left[S \left(\frac{s - L_{tc}}{d_1} \right) + S(d_2) \right] \right\} \\ & + \frac{1}{\kappa_t} \sin(d_4) - \frac{1}{\kappa_t} \sin \left(\frac{\kappa_t^2}{2 \alpha_t} \right) + d_1 C(d_2) \end{aligned} \quad (30)$$

and the vertical coordinates are given by

$$y_{tc}(s) = \begin{cases} d_1 S\left(\frac{s}{d_1}\right), & 0 \leq s \leq \sigma \\ P_{tc}^{(y)}(s), & \sigma < s \leq \nu \\ Q_{tc}^{(y)}(s), & \nu < s \leq L_{tc} \end{cases} \quad (31)$$

where

$$P_{tc}^{(y)}(s) = \frac{2}{\kappa_t} \sin\left(\frac{1}{2}\left(\kappa_t s - \frac{\kappa_t^2}{\alpha_t}\right)\right) \sin\left(\frac{1}{2}\kappa_t s\right) + d_1 S(d_2) \quad (32)$$

$$Q_{tc}^{(y)}(s) = d_1 \left\{ \sin(d_3) \left[C\left(\frac{s - L_{tc}}{d_1}\right) + C(d_2) \right] - \cos(d_3) \left[S\left(\frac{s - L_{tc}}{d_1}\right) + S(d_2) \right] \right\} - \frac{1}{\kappa_t} \cos(d_4) + \frac{1}{\kappa_t} \cos\left(\frac{\kappa_t^2}{2\alpha_t}\right) + d_1 S(d_2) \quad (33)$$

In equations (28) - (33) the following constants are used

$$d_1 = \sqrt{\frac{\pi}{\alpha_t}}, \quad d_2 = \frac{\kappa_t}{\sqrt{\pi \alpha_t}}, \quad d_3 = \kappa_t L_{tc} - \frac{\kappa_t^2}{\alpha_t}, \quad d_4 = \kappa_t L_{tc} - \frac{3\kappa_t^2}{2\alpha_t}$$

6. QUADRATIC CURVATURE

The quadratic curvature (QC) bend has the following curvature profile, see Figure 6.

$$\kappa_{qc}(s) = \alpha_q (L_{qc} s - s^2) \quad (34)$$

$\kappa_{qc}(s)$ satisfies the equal bending angle constraint, i.e. the total area under $\kappa(s)$ equals θ , when the parameter α_q is defined by

$$\alpha_q = \frac{6\theta}{L_{qc}^3} \quad (35)$$

L_{qc} is the length of the bend with quadratic curvature profile. The bending angle formula is found to be

$$\phi_{qc}(s) = \int_0^s \kappa_{qc}(u) du = \alpha_q \left(\frac{L_{qc} s^2}{2} - \frac{s^3}{3} \right) \quad (36)$$

Equation (36) satisfies the constraint that $\phi_{qc}(L_{qc}) = \theta$, α_q is defined in (35).

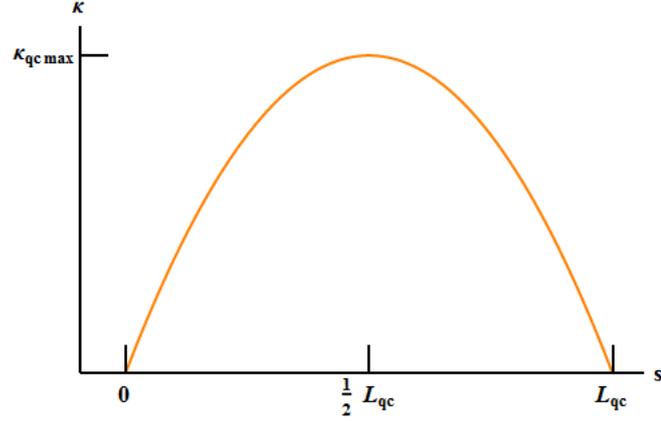


FIGURE 6. Curvature profile associated with a quadratic curvature bend

The parameterisations of the horizontal and vertical coordinates of the quadratic curvature bend can be determined by substitution of (36) into equations (4) and (5). The integrals that result from the application of equations (4) and (5) with (36) cannot be computed in terms of elementary functions. Hence, it is necessary to make an approximation. By replacing the cosine and sine functions by their Taylor series approximations it becomes possible to construct power series approximations to the necessary integrals. The resulting formulae are

$$x_{qc}(s) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} \alpha_q^{2k} \int_0^s \left(\frac{L_{qc} u^2}{2} - \frac{u^3}{3} \right)^{2k} du \quad (37)$$

$$y_{qc}(s) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \alpha_q^{2k+1} \int_0^s \left(\frac{L_{qc} u^2}{2} - \frac{u^3}{3} \right)^{2k+1} du \quad (38)$$

To evaluate the integrals in (37) and (38) proceed by defining the integral

$$I_m = \int_0^s u^{2m} \left(\frac{L_{qc}}{2} - \frac{u}{3} \right)^m du \quad (39)$$

To evaluate (39) make the substitution $u = t/\mu \Rightarrow du = (1/\mu) dt$, this changes the limits of integration in (39) from $[0, s]$ to $[0, \mu s]$, where μ is defined by

$$\mu = \frac{2}{3 L_{qc}} \quad (40)$$

The integral I_m can be written as

$$I_m = \frac{3^{2m+1} L_{qc}^{3m+1}}{2^{3m+1}} \int_0^{\mu s} t^{2m} (1-t)^m dt \quad (41)$$

The integral in (41) is the incomplete beta function $B_{\mu s}(2m+1, m+1)$ defined by formula (6.6.1) in [6]. It is possible to express the incomplete beta function in terms of Gauss' hypergeometric function using the transformation (6.6.8) in [6]

$$B_x(a, b) = \frac{x^a}{a} {}_2F_1(a, 1-b; a+1; x) \quad (42)$$

where ${}_2F_1(\cdot, \cdot; \cdot; \cdot)$ is defined by formula (15.1.1) in [6]. Integral (39) with $m = 2k$, after application of the transformation (42), is given by

$$I_{2k} = \left(\frac{L_{qc}}{2}\right)^{2k} \left(\frac{s^{4k+1}}{4k+1}\right) {}_2F_1(4k+1, -2k; 4k+2; \mu s) \quad (43)$$

Similarly, integral (39) with $m = 2k+1$, after application of the transformation (42), is given by

$$I_{2k+1} = \left(\frac{L_{qc}}{2}\right)^{2k+1} \left(\frac{s^{4k+3}}{4k+3}\right) {}_2F_1(4k+3, -2k-1; 4k+4; \mu s) \quad (44)$$

Using (43) and (44) the coordinates for the QC bend can be computed. The result is a sum over a set of hypergeometric functions. The necessary formulae are

$$x_{qc}(s) = \sum_{k=0}^N \frac{(-1)^k}{(2k)!} \alpha_q^{2k} I_{2k} \quad (45)$$

$$y_{qc}(s) = \sum_{k=0}^N \frac{(-1)^k}{(2k+1)!} \alpha_q^{2k+1} I_{2k+1} \quad (46)$$

The series (45) and (46) give accurate results when the sums are truncated after $N = 10$ terms. Gauss' hypergeometric function ${}_2F_1(\cdot, \cdot; \cdot; \cdot)$ is numerically evaluated for complex arguments using the C routine *hypser* provided in [12].

7. BEND CONSTRUCTION ALGORITHM

Given an input bend radius R , bend angle θ and a curvature profile, equations (1) - (5) can be evaluated to return N positions, (x_i, y_i) , $1 \leq i \leq N$, that represent the central path of a curve of

assumed length $L = R\theta$. The first point on the curve is the origin, $(x_1, y_1) = (0, 0)$, the last point (x_N, y_N) is generally unknown. If a variable curvature curve is to replace the CC bend then the first and last points of each bend must be the same, otherwise different bends will end at different positions. All bends can be constructed from the same starting point, so the final position of the equivalent bend is used as a control point, this will ensure that all bends start and finish at the same position. The location of the control point, labelled (x_c, y_c) , will cause the length of each bend to be determined, since the coordinates along the bend will all be scaled to ensure that $(x_N, y_N) = (x_c, y_c)$ for each of the different curvature schemes.

The algorithm for computing the coordinates of a variable curvature bend that must replace a CC bend described by an equivalent circle is described by Algorithm 1. The algorithm proceeds by computing the path followed by an equivalent circle, using the routine *define_eqc_coords()*. This provides the location of the control point (x_c, y_c) . The routine *define_bend_coords()* computes the positions of the centre of a bend with a specified curvature profile using (1) - (5) initially assuming a length $L_{bend} = R\theta$. Once the coordinates of the new bend are known the endpoint control test is applied by the routine *re_scale_coords()*. The routine compares (x_N, y_N) from the computed bend positions with the known endpoint from the equivalent circle (x_c, y_c) . Scaling parameters for the horizontal and vertical coordinates are defined in *re_scale_coords()*. For the horizontal coordinates the scaling parameter is $x_s = x_c/x_N$, for the vertical coordinates use $y_s = y_c/y_N$. If $x_s = y_s = 1$, the algorithm is complete because the initial and final positions of the variable curvature bend and the equivalent circle match. If x_s and y_s are not both equal to one, the horizontal coordinates are scaled by x_s , and the vertical coordinates are scaled by y_s . The length of the bend is then computed using

$$L_{bend} = \int ds = \int \sqrt{dx^2 + dy^2} \quad (47)$$

Since $x_s \neq y_s \neq 1$ the loop starts again by computing the bend coordinates assuming the newly computed bend length L_{bend} . The scaling is also repeated, and another bend length is computed from the new set of coordinates. This process is repeated until the bend

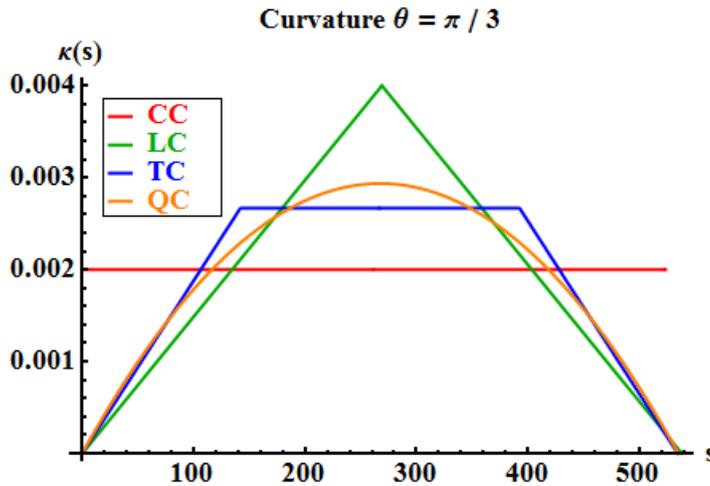


FIGURE 7. Curvature profiles for the constructed CC, LC, TC and QC curves. $\kappa_{cc} = 0.002$, $\kappa_{lc}^{max} = 0.004$, $\kappa_{tc}^{max} = 0.0027$, $\kappa_{qc}^{max} = 0.0029$.

length has converged to within a specified tolerance, ϵ . Upon convergence of the bend length the horizontal and vertical scaling parameters will be very close to unity and the result will be a set of coordinates that describe a curve that has a specified curvature profile. A C++ implementation of the code required to generate the various curves is provided and can be found at [13].

Once the bend-length is known, curvature and bend-angle profiles can be computed from the analytical formulae for a particular bend type, or numerically from the bend coordinate data.

8. RESULTS

A bend of radius $R = 500$ turning through an angle of $\theta = \pi/3$ was computed. The resulting curvature profile is shown in Figure 7. The profiles for the quadratic and trapezoidal bends are very similar. The maximum curvature in the quadratic case, $\kappa_{qc}^{max} = 0.0029$, is less than the maximum in the linear curvature case, $\kappa_{lc}^{max} = 0.004$, but greater than that in the trapezoidal curvature case, $\kappa_{tc}^{max} = 0.0027$. The bend angle profile for the constructed bend, shown in Figure 8, shows similarities between the quadratic and trapezoidal bends. The actual path of the constructed bend in the plane is shown in Figure 9, where the similarities between the trapezoidal and quadratic curvature bend paths can be observed.

Algorithm 1 Algorithm for computing the coordinates of a variable curvature bend that replaces an equivalent constant curvature bend

```

1: {Input bend radius, bend angle, bend type}
Require:  $R \leftarrow R_{bend}$ ,  $T \leftarrow \theta_{bend}$ ,  $BT \leftarrow type$ 
2:
3: {Define the coordinates that make up the equivalent circle}
4: {This step corresponds to evaluating (8) and (9) for a circle of
   radius  $R$ , and bend angle  $\theta$ }
5: define_eqc_coords()
6:
7: {Proceed with variable curvature bend calculation}
8:  $L_{bend} \leftarrow RT$ ,  $L_{bendold} \leftarrow 0.0$ 
9:  $n_{iter} \leftarrow 1$ ,  $max_{iter} \leftarrow 30$ 
10: while  $n_{iter} < max_{iter}$  do
11:   {Initialise the convergence condition}
12:    $L_{bendold} \leftarrow L_{bend}$ 
13:
14:   {Evaluate the appropriate integrals depending on the value
   of  $BT$ }
15:   define_bend_coords()
16:
17:   {Rescale the coordinate positions if necessary}
18:   re_scale_coords()
19:
20:   {Compute the bend length from (47)}
21:    $L_{bend} \leftarrow 0.0$ 
22:   for  $i = 2$  to  $N$  do
23:      $L_{bend} \leftarrow L_{bend} + ((X[i] - X[i - 1])^2 + (Y[i] - Y[i - 1])^2)^{1/2}$ 
24:   end for
25:
26:   {Apply convergence test}
27:   if  $|L_{bend} - L_{bendold}| < \epsilon$  then
28:     print Algorithm has converged
29:   else
30:      $n_{iter} \leftarrow n_{iter} + 1$ 
31:   end if
32:
33: end while
34:
35: {Output the positions of the centre of the bend}
36: return  $X[]$ ,  $Y[]$ 

```

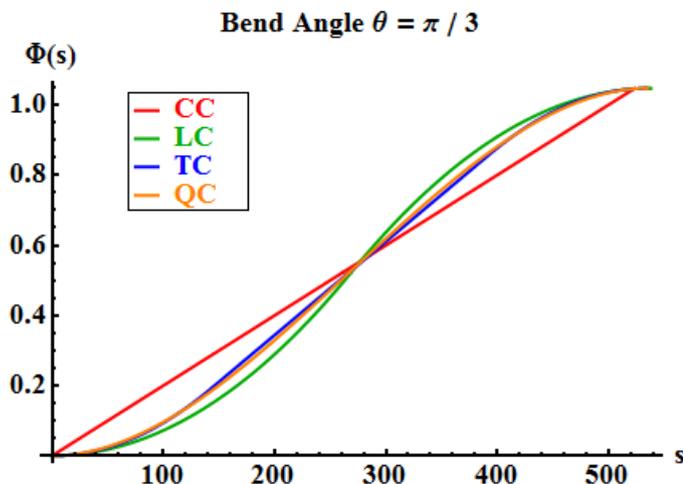


FIGURE 8. Bend angle profiles for the constructed CC, LC, TC and QC curves. All bends start at $\theta = 0$ and end at $\theta = \pi/3$.

If bends of different radii are plotted it is seen that the lengths of the different bend types increase linearly. For $\theta = \pi/3$ it is observed that $L_{lc} = 1.02777 L_{cc}$, $L_{tc} = 1.02106 L_{cc}$ and $L_{qc} = 1.02182 L_{cc}$, where $L_{lc} = R\theta$. This tells us that as the bend radius increases the LC bend will be longer than the other bend types and that TC and QC bends will have similar lengths. Bends at different radii have a maximum curvature that is proportional to the inverse radius, see Figure 10. In fact the data shows that for different bend radii $\kappa_{lc}^{max} = 2 \kappa_{cc}$, $\kappa_{tc}^{max} = \frac{4}{3} \kappa_{cc}$ and $\kappa_{qc}^{max} = \frac{527}{359} \kappa_{cc}$, where $\kappa_{cc} = 1/R$. The reader will observe that for the TC bend $\kappa_{tc}^{max} = \gamma \kappa_{cc}$, where γ is given by (24) on page 70, hence it should be possible to construct a TC bend whose maximum curvature approaches that of a CC bend if we let $\gamma \rightarrow 1$, this is done by decreasing the fraction of the TC bend whose curvature is linear, i.e. let f take a value closer to zero to get a TC bend with lower curvature.

9. CONCLUSION

Explicit formulae for variable curvature curves in the plane were constructed using Euler's method of natural equations. Curves whose curvature varies linearly were found to be represented by the Fresnel sine and cosine integrals, curves whose curvature varies quadratically were found to be expressible in terms of Gauss' hypergeometric function. The constructed curves are continuous, and

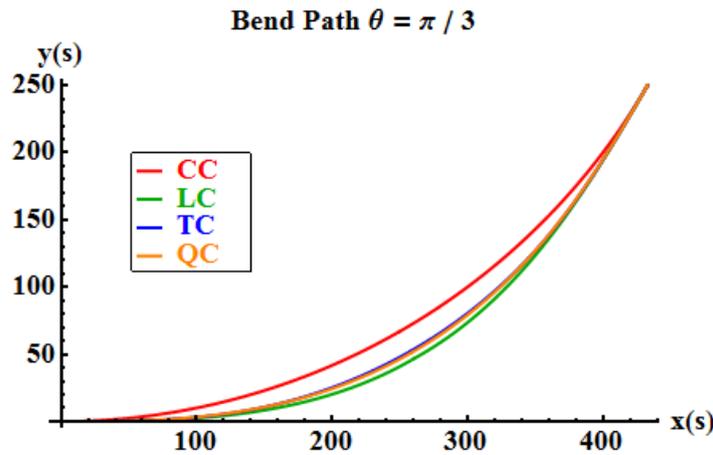


FIGURE 9. Bend paths for the constructed CC, LC, TC and QC curves. All bends start and finish at the same position. The variable curvature bends have slightly longer path-lengths $L_{cc} = 523.6$, $L_{lc} = 538.1$, $L_{tc} = 534.6$, $L_{qc} = 535.0$.

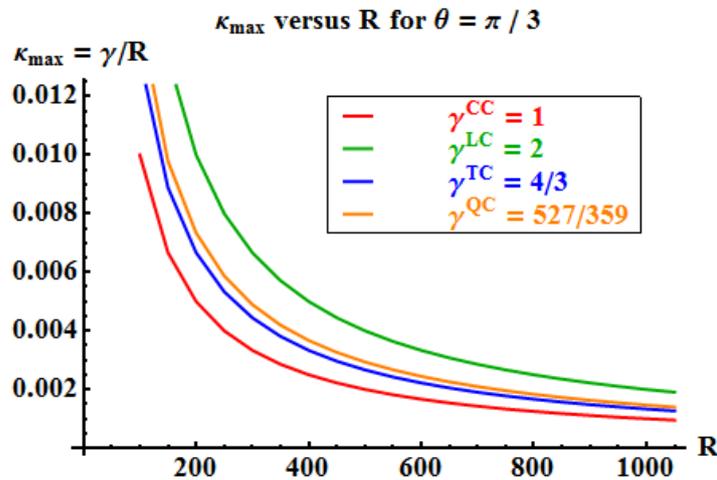


FIGURE 10. Variation of κ_{max} with R for $\theta = \pi/3$.

when the coordinates of the variable curvature curves are scaled appropriately their endpoints match those of a circle of radius R with θ at the origin. The shape of the trapezoidal curvature bend is very similar to that of a quadratic curvature bend. A linear relationship between bend path length has been observed for each bend type. The maximum curvature of the different bend types can be expressed as a multiple of the curvature of the CC bend.

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TORSION AND GROUND STATE MAXIMA: CLOSE BUT NOT THE SAME

BRIAN A. BENSON, RICHARD S. LAUGESEN, MICHAEL MINION,
AND BARTŁOMIEJ A. SIUDEJA

ABSTRACT. Could the location of the maximum point for a positive solution of a semilinear Poisson equation on a convex domain be independent of the form of the nonlinearity? Cima and Derrick found certain evidence for this surprising conjecture.

We construct counterexamples on the half-disk, by working with the torsion function and first Dirichlet eigenfunction. On an isosceles right triangle the conjecture fails again. Yet the conjecture has merit, since the maxima of the torsion function and eigenfunction are unexpectedly close together. It is an open problem to quantify this closeness in terms of the domain and the nonlinearity.

1. Introduction

Suppose the Poisson equation

$$\begin{cases} -\Delta u = f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

has a positive solution on the bounded convex plane domain Ω . Here the nonlinearity f is assumed to be Lipschitz and *restoring*, which means $f(z) > 0$ when $z > 0$. Cima and Derrick [2, 3] have conjectured that the location of the maximum point of u is independent of the form of the nonlinearity f .

This conjecture sounds impossible, since the graph of the solution must vary with the nonlinearity. Numerical computations by Cima and co-authors give surprising support for the conjecture, though, and Figure 1 provides further food for thought by considering a triangular domain and plotting the level curves and maximum point

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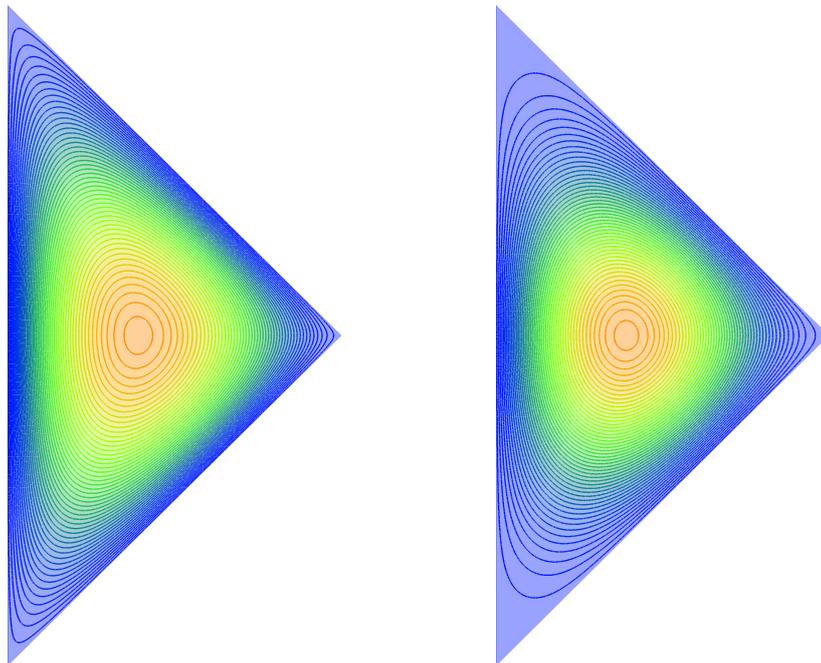


FIGURE 1. Level curves and the maximum point on a triangular domain, for solutions of two different Poisson type equations: the torsion function (left) and the first eigenfunction (right).

for the choices $f(z) = 1$ and $f(z) = \lambda z$. The corresponding linear Poisson equations describe the *torsion function* and the *ground state of the Laplacian* (see below). Our solutions were computed numerically by the finite element method on a mesh with approximately 10^6 triangles. The maximum points for the two solutions in Figure 1 appear to coincide, even though the level curves differ markedly near the boundary.

We disprove the conjecture on a half-disk in section 2, and again on the right isosceles triangle in section 3. Interestingly, the conjecture is remarkably close to being true in these counterexamples, with the maximum points occurring in almost but not quite the same location. We cannot explain this unexpected closeness.

A fascinating open problem is to bound the difference in location of the maximum points of two semilinear Poisson equations in terms of the difference between their nonlinearity functions and geometric information on the shape of the domain. Also, note that for both the half-disk and right isosceles triangle, our results show that the

maximum point of the torsion function lies to the left of the maximum for the ground state (when oriented as in Figure 1), which perhaps hints at a general principle for a class of convex domains.

Notation. The *torsion* or *landscape* function is the unique solution of the Poisson equation

$$\begin{cases} -\Delta u = 1 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Here we have chosen $f(z) = 1$. Clearly u is positive inside the domain, by the maximum principle.

The *Dirichlet ground state* or *first Dirichlet eigenfunction of the Laplacian* is the unique positive solution of

$$\begin{cases} -\Delta v = \lambda v & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\lambda > 0$ is the first eigenvalue of the Laplacian on the domain under Dirichlet boundary conditions. Here we have chosen $f(z) = \lambda z$.

2. The half-disk

The maximum points for the torsion function and ground state can lie so close together that one cannot distinguish them by the naked eye, as the following Proposition reveals. Yet the two points are not the same.

Proposition 2.1. *Take $\Omega = \{(x, y) : x > 0, x^2 + y^2 < 1\}$ to be the right half-disk. On this domain the torsion function u attains its maximum at approximately $(0.48022, 0)$ while the ground state v attains its maximum at approximately $(0.48051, 0)$. Here the x -coordinates have been rounded to 5 decimal places.*

Proof. (i) The ground state is given in polar coordinates by

$$v(r, \theta) = J_1(j_{1,1}r) \cos \theta$$

where J_1 is the first Bessel function and $j_{1,1} \approx 3.831706$ is its first positive zero. Clearly the maximum is attained on the x -axis, where $\theta = 0$, and the function is plotted along this line in Figure 2. By setting $J_1'(j_{1,1}r) = 0$ and solving, we find $r = j'_{1,1}/j_{1,1} \approx 0.48051$, rounded to five decimal places, where $j'_{1,1} \approx 1.841184$ is the first zero of J_1' .

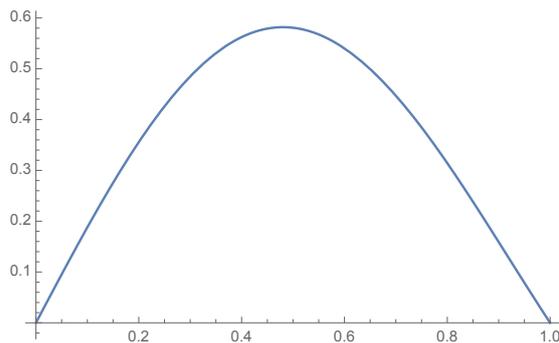


FIGURE 2. The radial part of the ground state on the right half-disk: $v(r, 0) = J_1(j_{1,1}r)$.

(ii) The torsion function is more complicated [5, Section 4.6.2], and is given by

$$u(x, y) = \frac{1}{4\pi} \left[-2\pi x^2 - 2x \left((x^2 + y^2)^{-1} - 1 \right) + \left(2 + (x^2 - y^2) \left((x^2 + y^2)^{-2} + 1 \right) \right) \arctan \frac{2x}{1 - (x^2 + y^2)} + xy \left((x^2 + y^2)^{-2} - 1 \right) \log \frac{x^2 + (1+y)^2}{x^2 + (1-y)^2} \right].$$

One verifies the Dirichlet boundary condition on the right half-disk by examining four cases: (i) $u = 0$ if $x = 0$ and $0 < |y| < 1$, (ii) $u \rightarrow 0$ as $(x, y) \rightarrow (0, 0)$, (iii) $u \rightarrow 0$ as $(x, y) \rightarrow (0, \pm 1)$, and (iv) $u \rightarrow 0$ as $(x, y) \rightarrow (x_1, y_1)$ with $x_1 > 0$ and $x_1^2 + y_1^2 = 1$.

To check u satisfies the Poisson equation $-\Delta u = 1$, a lengthy direct calculation suffices.

We claim u attains its maximum at a point on the horizontal axis. For this, first notice u is even about the x -axis by definition, meaning $u(x, y) = u(x, -y)$. Hence the harmonic function u_y equals zero on the x -axis for $0 < x < 1$. Further, $u_y \leq 0$ at points on the unit circle lying in the open first quadrant, since $u > 0$ in the right half-disk and $u = 0$ on the boundary. Also, one can compute that $u_y(x, y)$ approaches 0 as $(x, y) \rightarrow (0, 0)$ or $(x, y) \rightarrow (1, 0)$ or $(x, y) \rightarrow (0, 1)$ from within the first quadrant of the unit disk. Lastly u_y vanishes on the y -axis for $0 < y < 1$ (since $u = 0$ there). Hence we conclude from the maximum principle that $u_y \leq 0$ in the first quadrant of the unit disk, and so u attains its maximum somewhere on the x -axis.

On the x -axis we have

$$u(x, 0) = \frac{1}{4\pi} \left[-2\pi x^2 - 2x^{-1} + 2x + (2 + x^{-2} + x^2) \arctan \frac{2x}{1-x^2} \right]$$

for $0 < x < 1$. Clearly $u(0, 0) = u(1, 0) = 0$, and

$$u_x(x, 0) = \frac{1}{\pi x^3} \left[x + x^3 - \pi x^4 + \frac{1}{2}(x^4 - 1) \arctan \frac{2x}{1-x^2} \right].$$

One can show by taking another derivative and applying elementary estimates that $u(x, 0)$ is concave. Calculations show $u_x(x, 0)$ is positive at $x = 0.480219$ and negative at $x = 0.480220$, and so the maximum of u lies between these two points, that is, at $x \approx 0.48022$ to 5 decimal places. \square

3. The right isosceles triangle

Proposition 3.1. *Take $\Omega = \{(x, y) : 0 < x < 1, |y| < 1 - x\}$, which is an isosceles right triangle. On this domain the torsion function u attains its maximum at approximately $(0.39168, 0)$ while the ground state v attains its maximum at approximately $(0.39183, 0)$. Here the x -coordinates have been rounded to 5 decimal places.*

Proof. (i) Rotate the triangle by 45 degrees clockwise about the origin and scale up by a factor of $\pi/\sqrt{2}$, then translate by $\pi/2$ to the right and upwards, so that the triangle becomes

$$T = \{(x, y) : 0 < y < x < \pi\}.$$

This new triangle has ground state

$$v(x, y) = \sin x \sin 2y - \sin 2x \sin y = 2 \sin x \sin y (\cos y - \cos x) > 0$$

with eigenvalue $1^2 + 2^2 = 5$. One checks easily that $v = 0$ on the boundary of T , where $y = 0$ or $x = \pi$ or $y = x$. To find the maximum point, set $v_x = 0$ and $v_y = 0$ and deduce $\cos 2x = \cos x \cos y = \cos 2y$. Therefore the maximum lies on the line of symmetry $y = \pi - x$ of the triangle T . A little calculus shows that $v(x, \pi - x)$ attains its maximum when $x = \arcsin(1/\sqrt{3}) + \pi/2$. Hence the ground state of the original triangle attains its maximum at $((2/\pi) \arcsin(1/\sqrt{3}), 0) \approx (0.39183, 0)$ to 5 decimal places.

(ii) The torsion function on the triangle T is

$$u(x, y) = -\frac{1}{4}(x - y)^2 + \sum_{n=1}^{\infty} \frac{n^2 \pi^2 - 2(1 - (-1)^n)}{2\pi n^3 \sinh n\pi} \left[\begin{aligned} &\sinh nx \sin ny - \sin nx \sinh ny \\ &+ \sin n(\pi - x) \sinh n(\pi - y) \\ &- \sinh n(\pi - x) \sin n(\pi - y) \end{aligned} \right],$$

as we now explain. Observe that $-\Delta u = 1$ because the infinite series is a harmonic function, and $u = 0$ on the boundary of T by simple calculations with Fourier series when $0 < x < \pi, y = 0$, and when $x = \pi, 0 < y < \pi$; also $u = 0$ on the hypotenuse where $y = x$.

The torsion function is known to attain its maximum somewhere on the line of symmetry $y = \pi - x$, either by general symmetry results [2, 3] or else by arguing as in the proof of Proposition 2.1 part (ii). On that line of symmetry we evaluate

$$u(x, \pi - x) = -(x - \pi/2)^2 - \sum_{n=1}^{\infty} \frac{n^2 \pi^2 - 2(1 - (-1)^n)}{\pi n^3 \sinh n\pi} [(-1)^n \sinh nx + \sinh n(\pi - x)] \sin nx.$$

The series converges exponentially on each closed subinterval of $(0, \pi)$, and so we may differentiate term-by-term to find

$$\begin{aligned} \frac{d}{dx} u(x, \pi - x) &= -2(x - \pi/2) \\ &- \sum_{n=1}^{\infty} \frac{n^2 \pi^2 - 2(1 - (-1)^n)}{\pi n^2 \sinh n\pi} \left\{ [(-1)^n \cosh nx - \cosh n(\pi - x)] \sin nx \right. \\ &\quad \left. + [(-1)^n \sinh nx + \sinh n(\pi - x)] \cos nx \right\}, \end{aligned} \tag{1}$$

where once again the series converges exponentially on closed subintervals of $(0, \pi)$.

The absolute value of the n -th term in series (1) is bounded by

$$\frac{\pi(e^{nx} + e^{n(\pi-x)})}{\sinh(n\pi)} < 3\pi(e^{-n(\pi-x)} + e^{-nx}),$$

as we see by bounding the sin and cos terms with 1, adding the sinh and cosh terms having the same arguments, and using that $\sinh(n\pi) > e^{n\pi}/3$ for $n \geq 1$. Hence the infinite series (1) is bounded term-by-term by 3π times the sum of two geometric series having ratios $e^{-(\pi-x)}$ and e^{-x} .

The derivative of u along the line of symmetry is positive at $x = 2.1860525$ and negative at $x = 2.1860530$, as one finds by evaluating the first 20 terms of the series in (1) and then estimating the remainder with the geometric series as above. Hence u has a local maximum at $x \approx 2.186053$ to 6 decimal places. This local maximum is a global maximum because \sqrt{u} is concave (see [1, Example 1.1] or [4]). Translating to the left and downwards by $\pi/2$ and then scaling down by a factor of $\sqrt{2}/\pi$ and rotating counterclockwise by

45 degrees, we find the torsion function on the original triangle has a maximum at

$$x \approx \frac{2}{\pi}(2.186053 - \pi/2) = 0.39168$$

to 5 decimal places. □

4. Concluding remarks

The counterexamples in this paper concern Poisson's equation for $f(z) = 1$ and $f(z) = \lambda z$. One can find a whole family of counterexamples using $f(z) = a + bz$, where $a > 0$ and $0 < b \leq \lambda$. Note the maximum point depends on b but not a , as one checks by rescaling the solution u to u/a . To study this maximum point as b varies, one starts with the eigenfunctions of $-\Delta - b$ on the half-disk or right isosceles triangle and notes that the eigenfunctions are the same as for $-\Delta$, just with eigenvalues shifted by b . The corresponding torsion function can be computed in terms of an eigenfunction expansion, and then the position of the maximum point can be carefully numerically located. We leave such investigations to the interested reader.

Finally, while our counterexamples involve linear Poisson equations, our choices of f could presumably be perturbed to obtain genuinely nonlinear counterexamples.

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Leila Schneps (editor): Alexandre Grothendieck: a mathematical portrait, International Press, 2014. ISBN:978-1-57146-282-4, USD 85.00, 307+viii+6 pp.

REVIEWED BY BERND KREUSSLER

Alexandre Grothendieck, born on 28 March 1928 in Berlin, died on 13 November 2014 in the French Pyrenees. He was one of the greatest scientists of the twentieth century who has influenced significantly the development of a number of fields in pure mathematics: functional analysis, algebraic geometry, arithmetic geometry, category theory, logic, homological algebra and related areas. Grothendieck's contribution to the problem of space is considered to be of the same depth as Einstein's; his originality was to deepen the idea of a geometric point, see [4].

He was the greatest master of extracting the most essential features of a particular topic in a very general and abstract way so that solving an old problem often became a simple exercise. His method to solve problems was described by himself with the following allegory: instead of cracking a big nut by brute force one could immerse the nut in a softening fluid until the nut opens just by itself.

His personal life was as exceptional as his mathematical achievements. In 1966 he was awarded the Fields Medal for his fundamental contributions to algebraic geometry. In 1970, at the age of 42, he suddenly resigned from his prestigious position at the IHES, withdrawing himself from the mathematical community and later, in the 1990s, from social life completely.

The cubist portrait by Pablo Picasso, that was chosen as the front cover image of this book, reflects the style of this mathematical portrait of Grothendieck: his life and work is described on different levels and from many different angles by people who knew him personally, some of them very well, and who have included personal memories, anecdotes and explanations about Grothendieck's personality, his ideas and visions.

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The plan for this book took shape in August 2008 at a conference in the French Alps where former friends, colleagues, students, and collaborators of Alexandre Grothendieck met and discussed how one could explain to future generations the extraordinary and special nature of the contributions of this mathematical genius.

This book contains thirteen articles, each devoted to a particular aspect of Grothendieck's work. The authors did not concentrate on specific mathematical content of his monumental work but instead on the style how he did mathematics and on the impact he had on mathematics. To facilitate a better appreciation of Grothendieck's achievements, the authors also outlined the situation before the grand master started to work on a particular subject. This way the fundamental simplicity and the extraordinary power of his ideas become apparent.

A more detailed outline of the contents of the thirteen articles follows. At the end, the interested reader of this review is pointed to further literature about Alexandre Grothendieck.

JOE DIESTEL: Grothendieck and Banach space theory

The subject of this article is Grothendieck's work between 1953 and 1956 in Functional Analysis, especially on topological tensor products, nuclear spaces and the local theory of Banach spaces. It becomes obvious that he was a theory builder par excellence already at this early stage in his mathematical career just after receiving his Ph.D. in Nancy in 1953. The article concludes with a short account of the status of the problems posed by Grothendieck in this area.

MAX KAROUBI: L'influence d'Alexandre Grothendieck en K -théorie

This note deals with one of Grothendieck's revolutionary ideas: the introduction of the K -group of algebraic vector bundles, originally introduced for the proof of the Riemann-Roch Theorem. In Grothendieck's formulation, the Riemann-Roch formula for a morphism expresses the deviation from commutativity of a certain natural diagram that involves the Chern character map which links the K -group to ordinary (purely topological) cohomology. His ideas initiated the development of higher K -theory and of topological K -theory which has connections to Functional Analysis. From the personal memories the author has included in this article one learns how generous Grothendieck was with his ideas and advice.

MICHEL RAYNAUD: Grothendieck et la théorie des schémas

The notion of an algebraic scheme is at the core of Grothendieck's revolutionary re-foundation of algebraic geometry. An essential and influential novelty is the functorial point of view, which allows the description of a scheme by its 'functor of points'. As a consequence, commutative rings instead of fields are now in a unifying manner at the heart of algebraic geometry. It also suggests the shift of focus from objects to morphisms which played an essential role in Grothendieck's proof of the Riemann-Roch Theorem, his first outstanding masterpiece in algebraic geometry. This note is a kind of condensed survey of the essentials of EGA and (part of) SGA.

STEVEN L. KLEIMAN: The Picard scheme

Grothendieck's theory of the relative Picard scheme was an early impressive success of his new approach to algebraic geometry: the functorial point of view, focus on morphisms instead on objects (known as the relative point of view), the importance of considering nilpotent elements etc. In this article, the author describes the substance and spirit of Grothendieck's theory in an informal way. After an illuminating exposition of the historical context, he highlights the simplicity, natural generality and originality of Grothendieck's approach.

DAVID MUMFORD: My introduction to schemes and functors

In this short note, Fields medallist David Mumford gives a very personal description of how Grothendieck influenced his own view of algebraic geometry in 1958 at Harvard University. Essential for Mumford, who was interested at that time in moduli spaces of curves and vector bundles, was that the moduli space idea of Riemann and Picard could be made precise using functors. For him the most convincing aspect of Grothendieck's theory of schemes was that it allows to consider infinitesimal deformations, used in an intuitive way by Enriques, as actual families of schemes over one-point bases that are spectra of Artin rings. At the end, the author gives a very brief summary how Grothendieck transformed algebraic geometry.

CARLOS T. SIMPSON: Descent

The notion of descent, putting together a global object out of local pieces and gluing data, is ubiquitous in Grothendieck's work. It is crucially present in the theory of sheaves, leads to Grothendieck topologies and is essential in the theory of stacks. In topos theory

the idea of descent is pushed to a new level: the abstract collection of gluing data is the only true reality, no need for the existence of glued objects. After a very brief introduction to gluing, the author puts his main emphasis on modern developments and possible future directions of descent theory. This includes higher categories, higher stacks, higher non-abelian cohomology and derived stacks. These developments were strongly influenced by Grothendieck's 595-page manuscript *À la Poursuite des Champs* (Pursuing Stacks) which he wrote in 1983.

JACOB P. MURRE: On Grothendieck's work on the fundamental group

This lecture gives the reader an impression of the power and beauty of Grothendieck's method. He primarily looked for naturalness, not generality, he aimed at simplifying situations by extracting the key features so that eventually the solution to the problem falls out easily. His theory of the algebraic fundamental group unified classical Galois Theory of fields and the topological theory of the fundamental group. It led to a deep understanding of the algebraic fundamental group of an algebraic curve in positive characteristic, which was out of reach before the introduction of schemes. The lecture concludes with some remarks on further developments, including Grothendieck's famous manuscript *Esquisse d'un Programme* from 1984.

ROBIN HARTSHORNE: An apprenticeship

In this essay, the author describes vividly his experience with Grothendieck when he wrote *Residues and Duality*, published in 1966. The story starts in 1963, when Grothendieck was busy with fundamental works (EGA, SGA) but nevertheless prepared a 250-page manuscript for Hartshorne to conduct a seminar about his theory of duality. After many rounds of corrections by Grothendieck to the drafts written by Hartshorne, the now published version was accepted by Grothendieck as the best possible at the time, but both agreed that the theory had not yet reached a satisfactory state.

LUC ILLUSIE: Grothendieck et la cohomologie étale

This article deals with Grothendieck's concept of étale cohomology and the impact it had on arithmetic geometry. What inspired these ideas, how the Weil conjectures motivated this development and which obstacles had to be overcome is thoroughly explained.

LEILA SCHNEPS: The Grothendieck-Serre correspondence

The exchange of letters between Serre and Grothendieck started at the beginning of 1955 and continued until 1969. A bilingual version of the Grothendieck-Serre correspondence was published in [5]. The aim of the present article is to give a short explanation of the main results and notions discussed in these letters, whereby giving a first impression of the nature of [5] as a ‘living maths book’. The explanations are enriched with information about the personalities and the lives of these two outstanding mathematicians. In the closing chapter of this article, letters are discussed that were exchanged after Grothendieck started in 1986 the distribution of his monumental autobiographical work *Récoltes et Semailles* (Reaping and Sowing).

FRANS OORT: Did earlier thoughts inspire Grothendieck?

The author raises the question if Grothendieck’s brilliant ideas had simply occurred to him out of the blue or whether they have their roots in earlier works. He analyses this question in the context of three topics: the fundamental group, Grothendieck topologies and anabelian geometry. In each case he describes the situation before Grothendieck entered the scene and then carves out the extraordinary contribution of Grothendieck. The author of this article advocates for writing a scientific biography of Alexandre Grothendieck in a similar style. For each aspect of his work, he suggests to include a discussion of possible roots, then describe the leap Grothendieck made from those roots to general ideas and finally investigate the impact his ideas had on the development of this branch of mathematics.

PIERRE CARTIER: A country of which nothing is known but the name: Grothendieck and ‘motives’

In this note, the interactions between Grothendieck’s outstanding scientific work and his extraordinary personality are discussed in a way in which it is accessible to a broad audience. The author, who was a very close friend of Grothendieck, tries to stay as rational as possible in his analysis of the work and biography of Grothendieck before he lets *Récoltes et Semailles* illuminate the situation ‘from within’. He included a discussion of Grothendieck’s sufferings, spirituality and obsession in his later years. When Grothendieck’s story is compared with Botzmann’s and Cantor’s, a striking difference is that his scientific work was immediately and enthusiastically accepted by the scientific community. His work is unique in that his

fantasies and obsessions are not erased from them; he also delivered to us what he believed to be the meaning of his mathematical work. Included in this note is a very readable and brief overview of the scientific work of Grothendieck which covers, in a less detailed way, most of the content of the other articles in this book.

YURI I. MANIN: Forgotten motives: the varieties of scientific experience

The author of this short essay describes his experience when he visited Grothendieck in 1967 for five or six weeks. The central topic of this private tutoring was Grothendieck's newly emerging project of motives including the so-called standard conjectures. The content of this essay spans from the early history of motives to recent developments of this active field of research – Grothendieck's standard conjectures are still unproven. In addition to personal anecdotes, the author gives an intuitive description of Grothendieck's idea of motives and highlights the close relationship the theory of motives has to homological algebra and to mathematical physics.

The book ends with six pages of photographs of the contributing authors, some of them taken before 1970.

The algebraic geometer will find in each article interesting aspects of Grothendieck's work and life. Readers who are less familiar with algebraic geometry may enjoy most the articles of R. Hartshorne, L. Schneps, F. Oort, P. Cartier and Y. Manin as well as the less technical and more anecdotal parts of the other articles. Even without a detailed understanding of the mathematical ideas, through these articles it is possible to get a feeling of the mathematical atmosphere around Grothendieck and to appreciate the personality of this outstanding mathematician.

The book under review is not the only text that deals with the work and life of A. Grothendieck, but currently the best available mathematical portrait. Interesting other texts include the English translation [13] of Part 1 of Winfried Scharlau's four volume biographical project, the articles [6], [11], [10], [3] and from the Notices of the AMS: [8], [9], [12], [7], [1], [2]. The curious reader is referred to the Grothendieck Circle website www.grothendieckcircle.org where one finds some of Grothendieck's later texts, links to many other fine articles about the work and life of Grothendieck as well as an interesting collection of photographs.

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REVIEWED BY COLM MULCAHY

Persi Diaconis once aptly remarked, “Pick up anything Martin Gardner wrote, you’ll smile and learn something.” Based on the evidence found within the covers of this delightful compendium, it’s possible that the same can be said of Peter Lynch. The book is based on his long-running blog and *Irish Times* column of the same name, and one can’t help wondering why excerpts from some of 100 gems found here haven’t also made it onto radio. RTE’s long-running *Sunday Miscellany*, if it were to broaden its horizons, would be a good fit. That show’s avowed goal it to present radio essays that “capture our times, passions and curiosities.” Surely, in the year mo twogro¹, that should include more maths exposition of this calibre.

The author of this tome is a UCD mathematical science graduate with a PhD in dynamic meteorology from TCD under Ray Bates. Peter joined the meteorological service in 1971, eventually rising to the rank of Deputy Director, having also served as Head of the Research and Training Division there. In 2004, he switched gears, and threw in his lot with UCD’s School of Mathematical Sciences, as Met Eireann Professor of Meteorology and Director of the Meteorology & Climate Centre. A decade before the book under review appeared, he published *The Emergence of Numerical Weather Prediction: Richardson’s Dream* (Cambridge University Press).

That’s Maths, then, is a book by a seasoned applied mathematician. A casual reader might be surprised to learn that, however, based on reading the diverse essays presented in these pages. Peter’s writing displays a fine appreciation for both the elegance and beauty that can characterise the best mathematics, and the power of abstraction in the subject. In the Preface, he writes, “The articles are accessible to anyone who has studied mathematics at secondary school. Mathematics can be enormously interesting and inspiring,

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¹i.e., 1200 in base 12, or 2016 in base 10; see Peter’s “Dozenal Digits” chapter.

but its beauty and utility are often hidden.” The level of mathematics expected of his readers varies from the purposely equation-free musings of the *Irish Times* articles, to the more sophisticated pieces published only in his blog. A hint of whose writings Peter’s can be compared to can be found in a recent interview at mathsireland.ie [1], where he cites E. T. Bell, Martin Gardner, Dave Richeson, Glen van Brummelen and Bill Dunham as writers he has admired or learnt from. Unlike, say Gardner—who was generally less interested in mere application—Peter seems to be equally intrigued by utility, surprise, elegance and beauty.

The self-contained chapters are generally two to three pages long, occasionally four or five. This makes them ideal for quick dips: flip to a random page, and within a few minutes one has learnt something interesting. He crams a lot of information into each page, very effectively and with great style, and often leaves one wanting more. The 2-pager pieces always induced disappointment for this reader when the second page was turned and a new piece began unexpectedly. One rarely sees the end coming, which, upon reflection, is related to the fact that Peter sometimes dispenses with summary concluding paragraphs.

Since he writes for the general public, he covers many classics from down through the ages, ranging from Pythagorean triples and Platonic solids, to Bayes’ rule and Cantor’s breakthroughs on infinity, right up to RSA cryptography and fractals. Even when gathering and displaying these chestnuts—as so many have done before him—Peter manages to do so in a fresh, engaging way. The same applies to his vignettes on remarkable personalities from the history of mathematics, such as Kovalevskaya and Ramanujan. Since his interests and passions are broad, he throws in mathematical delights such as Bézout’s Theorem, Bézier curves, and the Bailey-Borwein-Plouffe formula, as well as applied material like population growth, epidemic spread, CAT scans, musical instrument tuning and the Black-Scholes equation.

As a meteorologist with intimate knowledge of the geometry and physics associated with spheres, Peter has several related articles and insights concerning weather, astronomy, engineering, transatlantic cable laying and GPS. He rescues spherical geometry from the obscurity to which history has unfairly banished it, though oddly, both on pages 14 & 66, the spherical triangles depicted are

“traditional Napier-style but anatomically incorrect” ones. They are shown with one bulging and two pinched corners, whereas actual spherical triangles have three bulging corners. (Their hyperbolic counterparts have three pinched ones. Perhaps it was Peter’s spherical propensity that prevented him from pointing out that the Pythagorean theorem $\cos c = \cos a \cos b$ — which he rightly touts for spheres — has a predictable hyperbolic parallel.)

There are numerous local heroes found within these pages too, from Wicklow’s Robert Halpin, Sligo’s George Stokes, Clare’s Matthew O’Brien, Dublin’s John Graves, TCD’s Galbraith & Haughton, and Guinness’s William Gosset, to the atmospheric railway that once shuttled back and forth between Dun Laoghaire and Dalkey. By including them, and tying them to their geographical origins, Peter is also continuing some of the good work done by the late Mary Mulvihill in books such as her landmark *Ingenious Ireland* [2].

In “A Hole Through the Earth” (pages 104-106) we learn that under ideal conditions, paying attention only to gravity while ignoring air resistance and molten magma, an object dropped into a straight tunnel, burrowed through the earth, would emerge at the far end of the tunnel after a fixed period of time. That time, which amazingly is independent of the tunnel length (and hence the angle at which the tunnel was drilled), is about 42 minutes and 12 seconds. One is reminded of a mindboggler made famous by Gardner, the solution of which is that the volume remaining when a cylindrical hole six units long is drilled straight through the centre of a solid sphere is 36π cubic units, regardless of the radius of the sphere.

Unlike Gardner, who loved brainteasers and wrote dozens of puzzle books, Peter’s focus is on concise, original exposition. However, he does give pride of place to this counter-intuitive watermelon puzzle:

A farmer brings a load of watermelons to the market. Before he sets out, he measures the total weight and the percentage water content. He finds that the total weight is 100kg and the water content is 99%. The weather is hot, so his load loses some moisture *en route*. He checks the water content when he arrives at the market: it has dropped to 98%.

QUESTION: What is the total weight of the load on arrival at market?

In “Kelvin’s Wake” (pages 75-77), we are told that if a duck swims in the shallow pond at St Stephen’s Green, then the V in the water which bounds the dispersive waves the bird leaves in its wake forms an angle of about 40 degrees. The reason given is that half the angle in question is $\arcsin(1/3)$, which is about 0.3398 radians, or roughly 19.47 degrees. (The duck apex angle is hence a tad under 39 degrees.) Moreover, this angle is allegedly independent of the speed at which the duck moves. We are not told if this result also holds for ducks in Herbert Park or other metropolitan waterways, or indeed for those beyond the Pale. Nor is there any discussion of the converse: if it takes (to water) and wakes (at an angle of approximately 40 degrees) like a duck, then it is a duck.

For several reasons it would be helpful if the chapters were numbered². As it is, it’s often challenging to go back to find something which caught one’s attention on an earlier perusal. For instance, while there is an index, no wakes or ducks (dead or alive) seem to lurk therein.

These are minor quibbles. *That’s Maths* is a superb collection of thought-provoking essays—100 of them!—which every numerate or curious teen or adult in Ireland and elsewhere should devour.

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²The week the book appeared, Peter provided chapter numbers at his own blog <https://thatsmaths.com/2016/10/27/thats-maths-book-published>.

PROBLEMS

IAN SHORT

PROBLEMS

We have a selection of inequalities to prove this issue, each of a significantly different type. The first was contributed by Ángel Plaza of Universidad de Las Palmas de Gran Canaria, Spain.

Problem 78.1. Given positive real numbers a, b, c, u and v , prove that

$$\frac{a}{bu + cv} + \frac{b}{cu + av} + \frac{c}{au + bv} \geq \frac{3}{u + v}.$$

The next problem arose in a course taught at the Open University.

Problem 78.2. Let a_1, \dots, a_n be distinct complex numbers. Prove that

$$\left| 1 + \prod_{\substack{i=1 \\ i \neq j}}^n (a_j - a_i) \right| \geq 1$$

for at least one of the integers $j = 1, \dots, n$.

We finish with an inequality proposed by Finbarr Holland of University College Cork.

Problem 78.3. Suppose that the continuous function $f : [0, 1] \rightarrow \mathbb{R}$ is twice differentiable on $(0, 1)$ and the second derivative f'' is square integrable on $[0, 1]$. Suppose also that $f(0) + f(1) = 0$. Prove that

$$120 \left| \int_0^1 f(t) dt \right|^2 \leq \int_0^1 |f''(t)|^2 dt,$$

and show that 120 is the best-possible constant in this inequality.

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SOLUTIONS

Here are solutions to the problems from *Bulletin* Number 76.

The three integrals in the first problem were taken from *Inside interesting integrals* by Paul Nahin. All three integrals were solved by Finbarr Holland, Henry Ricardo of the New York Math Circle, USA, and the North Kildare Mathematics Problem Club. Solutions can also be found in Nahin's book.

Holland and Ricardo point out that integral (a) is one of the well-known Fresnel integrals. The value of this integral was first discovered by Euler, although his solution wasn't rigorous by today's standards. The integral can be found using contour integration; however, we supply a more elementary solution provided by Ricardo, who cites *On the evaluation of certain improper integrals* by Robert M. Young, *Math. Gaz.* 75 (March, 1991). This solution was also given in Nahin's text.

The method of solution to (b) was supplied by Ricardo and Nahin. The solution to (c) was offered by all those who submitted solutions, and by Nahin himself.

Problem 76.1.

$$(a) \int_0^{\infty} \sin(x^2) \, dx \quad (b) \int_0^1 \frac{x-1}{\log x} \, dx \quad (c) \int_{-1}^1 \frac{\cos x}{e^{1/x} + 1} \, dx$$

Solution 76.1. (a) Define a function $G : \mathbb{R} \rightarrow \mathbb{R}$ by

$$G(t) = \left(\int_0^t e^{ix^2} \, dx \right)^2 + i \int_0^1 \frac{e^{it^2(x^2+1)}}{x^2+1} \, dx.$$

One can check that $G(0) = i\pi/4$ and $G'(t) = 0$ for all $t \in \mathbb{R}$. Hence $G(t) = i\pi/4$ for all $t \in \mathbb{R}$. By taking a limit we see that

$$\left(\int_0^{\infty} e^{ix^2} \, dx \right)^2 = i\frac{\pi}{4},$$

and on expanding the left-hand side and equating real and imaginary parts we deduce that

$$\int_0^{\infty} \sin(x^2) \, dx = \sqrt{\frac{\pi}{8}}.$$

(b) Observe that, for $x > 0$,

$$\int_0^1 x^y \, dy = \int_0^1 e^{y \log x} \, dy = \frac{x - 1}{\log x}.$$

Hence

$$\begin{aligned} \int_0^1 \frac{x - 1}{\log x} \, dx &= \int_0^1 \int_0^1 x^y \, dy \, dx \\ &= \int_0^1 \int_0^1 x^y \, dx \, dy \\ &= \int_0^1 \frac{1}{y + 1} \, dy \\ &= \log 2. \end{aligned}$$

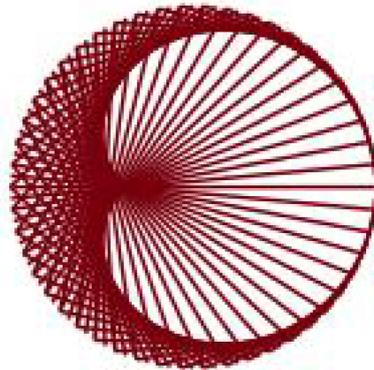
(c) As cosine is an even function, we have

$$\begin{aligned} \int_{-1}^1 \frac{\cos x}{e^{1/x} + 1} \, dx &= \int_0^1 \left(\frac{\cos x}{e^{1/x} + 1} + \frac{\cos x}{e^{-1/x} + 1} \right) \, dx \\ &= \int_0^1 \cos x \, dx \\ &= \sin 1. \quad \square \end{aligned}$$

The second problem was solved by the North Kildare Mathematics Problem Club.

Problem 76.2. For each point z on the unit circle, let ℓ_z denote the closed line segment from z to z^2 . Consider the collection of those points in the closed unit disc that each lie at the intersection of two distinct line segments ℓ_z and ℓ_w . What shape is the complement in the unit disc of this collection of points?

Solution 76.2. A sketch of a selection from the family of lines is given below.



This suggests that the set is the perimeter and inside of a cycloid.

In fact, it is clear that the set is bounded by the envelope of the family of lines joining $e^{i\theta}$ to $e^{2i\theta}$, and these are given by the equations

$$y = \beta(\theta) + m(\theta)(x - \alpha(\theta)),$$

where

$$\alpha(\theta) = \cos \theta, \quad \beta(\theta) = \sin \theta \quad \text{and} \quad m(\theta) = \frac{\sin 2\theta - \sin \theta}{\cos 2\theta - \cos \theta}$$

for $0 \leq \theta < 2\pi$. The envelope is parametrized in terms of θ by

$$x = \frac{(\alpha m)' - \beta'}{m'} \quad \text{and} \quad y = \beta + m(x - \alpha),$$

where f' denotes the derivative of f with respect to θ . We calculate

$$m' = \frac{-3}{4 \cos^3 \theta - 3 \cos \theta - 1},$$

$$(\alpha m)' = \frac{4 \cos^4 \theta - 5 \cos^2 \theta - 3 \cos \theta + 1}{4 \cos^3 \theta - 3 \cos \theta - 1},$$

so the envelope takes the form

$$x = \frac{2}{3} \cos \theta (\cos \theta + 1) - \frac{1}{3},$$

$$y = \frac{2}{3} \sin \theta (\cos \theta + 1).$$

If we translate this to the right by $\frac{1}{3}$ and then scale by a factor $\frac{3}{2}$, it becomes the standard cycloid

$$x = \cos \theta (\cos \theta + 1),$$

$$y = \sin \theta (\cos \theta + 1),$$

which has its cusp at 0 and is symmetrical about its chord $[0, 2]$. Therefore our envelope has its cusp at $-\frac{1}{3}$ and is symmetrical about its chord $[-\frac{1}{3}, 1]$. \square

The third problem was solved by Henry Ricardo, Ángel Plaza, the North Kildare Mathematics Problem Club, and the proposer, Wenchang Chu of Università del Salento, Italy. Ricardo points out that the result (starting with $n = 1$) is Problem 3.1.19 of *Problems in Mathematical Analysis I* by W. J. Kaczor and M. T. Nowak (American Mathematical Society, 2000), and it is proved as a consequence of a more general series result (Problem 3.1.17). It is this solution that we supply (which is also that of the proposer).

Problem 76.3. Evaluate $\sum_{n=0}^{\infty} \frac{1}{2^n} \tan\left(\frac{x}{2^n}\right)$.

Solution 76.3. If $x = 0$, then the sum is 0. If x is a nonzero integral multiple of $\pi/2$, then clearly the sum is undefined, as one of the terms is undefined. Suppose then that x is not an integral multiple of $\pi/2$.

From the identity $\tan x = \cot x - 2 \cot 2x$, we obtain

$$\tan\left(\frac{x}{2^n}\right) = \cot\left(\frac{x}{2^n}\right) - 2 \cot\left(\frac{x}{2^{n-1}}\right).$$

Hence

$$\begin{aligned} \sum_{n=0}^N \frac{1}{2^n} \tan\left(\frac{x}{2^n}\right) &= \tan x + \sum_{n=1}^N \frac{1}{2^n} \tan\left(\frac{x}{2^n}\right) \\ &= \tan x + \sum_{n=1}^N \frac{1}{2^n} \left(\cot\left(\frac{x}{2^n}\right) - 2 \cot\left(\frac{x}{2^{n-1}}\right) \right) \\ &= \tan x - \cot x + \frac{1}{2^N} \cot\left(\frac{x}{2^N}\right). \end{aligned}$$

Now

$$\frac{1}{2^N} \cot\left(\frac{x}{2^N}\right) = \frac{\cos(x/2^N)}{x} \cdot \frac{x/2^N}{\sin(x/2^N)} \rightarrow \frac{1}{x}$$

as $N \rightarrow \infty$. Therefore

$$\sum_{n=0}^{\infty} \frac{1}{2^n} \tan\left(\frac{x}{2^n}\right) = \tan x - \cot x + \frac{1}{x} = \frac{1}{x} - 2 \cot 2x. \quad \square$$

We invite readers to submit problems and solutions. Please email submissions to imsproblems@gmail.com in any format (we prefer Latex). Submissions for the summer Bulletin should arrive before the end of April, and submissions for the winter Bulletin should arrive by October. The solution to a problem is published two issues after the issue in which the problem first appeared. Please include solutions to any problems you submit, if you have them.

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