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PROBLEMS

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Problems

We have a selection of inequalities to prove this issue, each of a significantly different type. The first was contributed by Ángel Plaza of Universidad de Las Palmas de Gran Canaria, Spain.

Problem 78.1. Given positive real numbers a, b, c, u and v, prove that

$$\frac{a}{bu+cv}+\frac{b}{cu+av}+\frac{c}{au+bv}\geqslant\frac{3}{u+v}.$$

The next problem arose in a course taught at the Open University.

Problem 78.2. Let $a_1,\dots,a_n$ be distinct complex numbers. Prove that

$$\left|1+\prod_{\substack{i=1\\ i\neq j}}^{n}(a_j-a_i)\right|\geqslant 1$$

for at least one of the integers $j=1,\dots,n$.

We finish with an inequality proposed by Finbarr Holland of University College Cork.

Problem 78.3. Suppose that the continuous function $f:[0,1]\to\mathbb{R}$ is twice differentiable on $(0,1)$ and the second derivative f'' is square integrable on $[0,1]$. Suppose also that $f(0)+f(1)=0$. Prove that

$$120\left|\int_0^1 f(t)\,\mathrm{d}t\right|^2\leqslant\int_0^1|f''(t)|^2\,\mathrm{d}t,$$

and show that 120 is the best-possible constant in this inequality.

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Solutions

Here are solutions to the problems from *Bulletin* Number 76.

The three integrals in the first problem were taken from *Inside interesting integrals* by Paul Nahin. All three integrals were solved by Finbarr Holland, Henry Ricardo of the New York Math Circle, USA, and the North Kildare Mathematics Problem Club. Solutions can also be found in Nahin's book.

Holland and Ricardo point out that integral (a) is one of the well-known Fresnel integrals. The value of this integral was first discovered by Euler, although his solution wasn't rigorous by today's standards. The integral can be found using contour integration; however, we supply a more elementary solution provided by Ricardo, who cites *On the evaluation of certain improper integrals* by Robert M. Young, Math. Gaz. 75 (March, 1991). This solution was also given in Nahin's text.

The method of solution to (b) was supplied by Ricardo and Nahin. The solution to (c) was offered by all those who submitted solutions, and by Nahin himself.

Problem 76.1.

$$\text{(a)} \int_0^\infty \sin(x^2)\,\mathrm{d}x \qquad \text{(b)} \int_0^1 \frac{x-1}{\log x}\,\mathrm{d}x \qquad \text{(c)} \int_{-1}^1 \frac{\cos x}{e^{1/x}+1}\,\mathrm{d}x$$

Solution 76.1. (a) Define a function $G:\mathbb{R}\to\mathbb{R}$ by

$$G(t) = \left(\int_0^t e^{ix^2}\,\mathrm{d}x\right)^2 + i\int_0^1 \frac{e^{it^2(x^2+1)}}{x^2+1}\,\mathrm{d}x.$$

One can check that $G(0) = i\pi/4$ and $G'(t) = 0$ for all $t\in\mathbb{R}$. Hence $G(t) = i\pi/4$ for all $t\in\mathbb{R}$. By taking a limit we see that

$$\left(\int_0^\infty e^{ix^2}\,\mathrm{d}x\right)^2 = i\frac{\pi}{4},$$

and on expanding the left-hand side and equating real and imaginary parts we deduce that

$$\int_0^\infty \sin(x^2)\,\mathrm{d}x = \sqrt{\frac{\pi}{8}}.$$

(b) Observe that, for $x > 0$,

$$\int_0^1 x^y \, \mathrm{d}y = \int_0^1 e^{y \log x} \, \mathrm{d}y = \frac{x-1}{\log x}.$$

Hence

$$\begin{aligned}\int_0^1 \frac{x-1}{\log x} \, \mathrm{d}x &= \int_0^1 \int_0^1 x^y \, \mathrm{d}y \, \mathrm{d}x \\ &= \int_0^1 \int_0^1 x^y \, \mathrm{d}x \, \mathrm{d}y \\ &= \int_0^1 \frac{1}{y+1} \, \mathrm{d}y \\ &= \log 2.\end{aligned}$$

(c) As cosine is an even function, we have

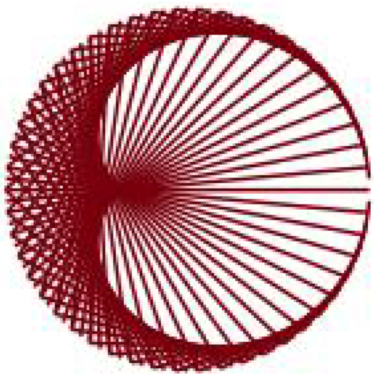
$$\begin{aligned}\int_{-1}^1 \frac{\cos x}{e^{1/x}+1} \, \mathrm{d}x &= \int_0^1 \left(\frac{\cos x}{e^{1/x}+1} + \frac{\cos x}{e^{-1/x}+1} \right) \mathrm{d}x \\ &= \int_0^1 \cos x \, \mathrm{d}x \\ &= \sin 1.\end{aligned}$$

□

The second problem was solved by the North Kildare Mathematics Problem Club.

*Problem 76.2***.** For each point z on the unit circle, let ℓ_z denote the closed line segment from z to z^2. Consider the collection of those points in the closed unit disc that each lie at the intersection of two distinct line segments ℓ_z and ℓ_w. What shape is the complement in the unit disc of this collection of points?

Solution 76.2. A sketch of a selection from the family of lines is given below.



This suggests that the set is the perimeter and inside of a cycloid.

In fact, it is clear that the set is bounded by the envelope of the family of lines joining $e^{i\theta}$ to $e^{2i\theta}$, and these are given by the equations

$$y = \beta(\theta) + m(\theta)(x - \alpha(\theta)),$$

where

$$\alpha(\theta) = \cos\theta, \quad \beta(\theta) = \sin\theta \quad \text{and} \quad m(\theta) = \frac{\sin 2\theta - \sin\theta}{\cos 2\theta - \cos\theta}$$

for $0 \leqslant \theta < 2\pi$. The envelope is parametrized in terms of θ by

$$x = \frac{(\alpha m)' - \beta'}{m'} \quad \text{and} \quad y = \beta + m(x - \alpha),$$

where f' denotes the derivative of f with respect to θ. We calculate

$$m' = \frac{-3}{4\cos^3\theta - 3\cos\theta - 1},$$
$$(\alpha m)' = \frac{4\cos^4\theta - 5\cos^2\theta - 3\cos\theta + 1}{4\cos^3\theta - 3\cos\theta - 1},$$

so the envelope takes the form

$$x = \frac{2}{3}\cos\theta(\cos\theta + 1) - \frac{1}{3},$$
$$y = \frac{2}{3}\sin\theta(\cos\theta + 1).$$

If we translate this to the right by $\frac{1}{3}$ and then scale by a factor $\frac{3}{2}$, it becomes the standard cycloid

$$x = \cos\theta(\cos\theta + 1),$$
$$y = \sin\theta(\cos\theta + 1),$$

which has its cusp at 0 and is symmetrical about its chord $[0, 2]$. Therefore our envelope has its cusp at $-\frac{1}{3}$ and is symmetrical about its chord $[-\frac{1}{3}, 1]$. □

The third problem was solved by Henry Ricardo, Ángel Plaza, the North Kildare Mathematics Problem Club, and the proposer, Wenchang Chu of Universitá del Salento, Italy. Ricardo points out that the result (starting with $n = 1$) is Problem 3.1.19 of *Problems in Mathematical Analysis I* by W. J. Kaczor and M. T. Nowak (American Mathematical Society, 2000), and it is proved as a consequence of a more general series result (Problem 3.1.17). It is this solution that we supply (which is also that of the proposer).

Problem 76.3. Evaluate $\displaystyle\sum_{n=0}^{\infty} \frac{1}{2^n} \tan\left(\frac{x}{2^n}\right)$.

Solution 76.3. If $x = 0$, then the sum is 0. If x is a nonzero integral multiple of $\pi/2$, then clearly the sum is undefined, as one of the terms is undefined. Suppose then that x is not an integral multiple of $\pi/2$.

From the identity $\tan x = \cot x - 2\cot 2x$, we obtain

$$\tan\left(\frac{x}{2^n}\right) = \cot\left(\frac{x}{2^n}\right) - 2\cot\left(\frac{x}{2^{n-1}}\right).$$

Hence

$$\begin{aligned}\sum_{n=0}^{N} \frac{1}{2^n} \tan\left(\frac{x}{2^n}\right) &= \tan x + \sum_{n=1}^{N} \frac{1}{2^n} \tan\left(\frac{x}{2^n}\right) \\ &= \tan x + \sum_{n=1}^{N} \frac{1}{2^n} \left(\cot\left(\frac{x}{2^n}\right) - 2\cot\left(\frac{x}{2^{n-1}}\right)\right) \\ &= \tan x - \cot x + \frac{1}{2^N} \cot\left(\frac{x}{2^N}\right).\end{aligned}$$

Now

$$\frac{1}{2^N} \cot\left(\frac{x}{2^N}\right) = \frac{\cos(x/2^N)}{x} \cdot \frac{x/2^N}{\sin(x/2^N)} \ \rightarrow\ \frac{1}{x}$$

as $N \rightarrow \infty$. Therefore

$$\sum_{n=0}^{\infty} \frac{1}{2^n} \tan\left(\frac{x}{2^n}\right) = \tan x - \cot x + \frac{1}{x} = \frac{1}{x} - 2\cot 2x. \qquad \square$$

We invite readers to submit problems and solutions. Please email submissions to `imsproblems@gmail.com` in any format (we prefer Latex). Submissions for the summer Bulletin should arrive before the end of April, and submissions for the winter Bulletin should arrive by October. The solution to a problem is published two issues after the issue in which the problem first appeared. Please include solutions to any problems you submit, if you have them.

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