

A Class of Weakly Nil-Clean Rings

PETER V. DANCHEV

ABSTRACT. We completely describe the structure of weakly nil-clean rings with the strong property. The main characterization theorem somewhat improves on results of Diesl in J. Algebra (2013) concerning strongly nil-clean rings and Breaz-Danchev-Zhou in J. Algebra Appl. (2016) concerning abelian weakly nil-clean rings.

1. INTRODUCTION AND BACKGROUND

Throughout the present paper all rings R considered shall be assumed to be associative and unital with identity element 1. As usual, $U(R)$ denotes the set of all invertible elements of R , $Id(R)$ the set of all idempotents of R and $Nil(R)$ the set of all nilpotents of R . Traditionally, $J(R)$ will always denote the Jacobson radical of R . All other notions and notations, not explicitly stated herein, are standard and may be found in [10].

The following concept appeared in [11].

Definition 1.1. A ring R is called *clean* if each $r \in R$ can be expressed as $r = u + e$, where $u \in U(R)$ and $e \in Id(R)$.

If, in addition, the existing idempotent is unique, then R is said to be *uniquely clean*. A clean ring R with $ue = eu$ is said to be *strongly clean*. If again the existing idempotent is unique, the ring is called *uniquely strongly clean* (see [4]).

It is well known that uniquely clean rings, being abelian clean rings, are strongly clean. The converse, however, does not hold in general. Nevertheless, uniquely clean rings are uniquely strongly clean, which containment cannot be reversed. However, [4, Example 4] demonstrates that uniquely clean rings are exactly the abelian uniquely strongly clean rings.

2010 *Mathematics Subject Classification.* 16D50; 16S34; 16U60.

Key words and phrases. Clean rings, Weakly nil-clean rings, Idempotents, Nilpotents, Jacobson radical.

Received on 18-10-2015; revised 30-4-2016.

In particular, in [9] the following concept was introduced:

Definition 1.2. A ring R is called *nil-clean* if each $r \in R$ can be written as $r = q + e$, where $q \in Nil(R)$ and $e \in Id(R)$.

If, in addition, the existing idempotent is unique, then R is said to be *uniquely nil-clean*. A nil-clean ring R with $qe = eq$ is said to be *strongly nil-clean*. If again the existing idempotent is unique, the ring is called *uniquely strongly nil-clean*.

It is well known that uniquely nil-clean rings, being abelian nil-clean rings, are strongly nil-clean. This implication is not reversible, however. Nevertheless, it follows from [3, Theorem 4.5] and [7] that uniquely nil-clean rings are precisely the abelian nil-clean rings (see [9, Theorem 5.9] as well). Also, commutative nil-clean rings are always uniquely nil-clean (compare with [8, Proposition 1.6]), and even it was proved in [9, Corollary 3.8] that strongly nil-clean rings and uniquely strongly nil-clean rings do coincide in general.

On the other hand, the latter concept of nil-cleanness was extended in [8] and [2], respectively, by defining the notion of *weak nil-cleanness* as follows:

Definition 1.3. A ring R is called *weakly nil-clean* if every $r \in R$ can be presented as either $r = q + e$ or $r = q - e$, where $q \in Nil(R)$ and $e \in Id(R)$.

If, in addition, the existing idempotent is unique, then R is said to be *uniquely weakly nil-clean*. A weakly nil-clean ring with $qe = eq$ is said to be *weakly nil-clean with the strong property*. If again the existing idempotent is unique, the ring is called *uniquely weakly nil-clean with the strong property*.

It was established in [2] and [8] that weakly nil-clean rings are themselves clean. Likewise, in [2] was established a complete characterization of abelian weakly nil-clean rings as those abelian rings R for which $J(R)$ is nil and $R/J(R)$ is isomorphic to a Boolean ring B , or to \mathbb{Z}_3 , or to $B \times \mathbb{Z}_3$. We notice also that uniquely weakly nil-clean rings were classified in [5] as abelian weakly nil-clean rings.

The objective of this article is to continue these two explorations by giving a complete description of the structure of the above-defined weakly nil-clean rings having the strong property. As an application, we will characterize the class of rings equipped with the strong nil-involution property, defined as follows:

Definition 1.4. We will say that a ring R has the *nil-involution property* if, for any $r \in R$, we have either $r = v+1+w$ or $r = v-1+w$, where $v \in Nil(R)$ and $w^2 = 1$.

If, in addition, $vw = wv$, then we call that R has the *strong nil-involution property*.

The motivation for considering and exploring the sum of an invertible plus an idempotent, as well as other such variations of elements as above, is illustrated in details via concrete examples in [11], [9] and [2], respectively.

2. THE MAIN RESULT

We come now to our main result in which we give a comprehensive characterization of weakly nil-clean rings enabled with the strong property.

Theorem 2.1. *A ring R is weakly nil-clean with the strong property if, and only if, R is either a strongly nil-clean ring, or $R/J(R)$ is isomorphic to \mathbb{Z}_3 with $J(R)$ nil, or R is a direct product of two such rings.*

Proof. We show that for any such weakly nil-clean ring R will follow that $R \cong R_1 \times R_2$, where $R_1/J(R_1)$ is Boolean with $J(R_1)$ nil, and R_2 is either $\{0\}$ or $R_2/J(R_2) \cong \mathbb{Z}_3$ with $J(R_2)$ nil. In fact, in accordance with [2], we write that $R \cong R_1 \times R_2$, where R_1 is a nil-clean ring and R_2 is a weakly nil-clean ring for which 2 is invertible. Since R has the “strong” property, it follows at once that R_1 is strongly nil-clean. Thus, appealing to [7], $R_1/J(R_1)$ must be Boolean with nil $J(R_1)$.

As for the second direct factor, we claim that all units of R_2 are the sum or the difference of a nilpotent and 1, that is, they belong to $Nil(R_2) \pm 1$. In fact, if u is an arbitrary unit in R_2 , then being an element in R , one can write that $u = q + e$ or $u = q - e$, where q is a nilpotent of R_2 and e is an idempotent of R_2 which commute, i.e., $qe = eq$. Thus, $uq = (q \pm e)q = q^2 \pm eq = q^2 \pm qe = q(q \pm e) = qu$. Therefore, $u - q = e$ and $q - u = e$ are again units and simultaneously idempotents. This means that $e = 1$ in both cases, hence $u = q \pm 1$ as claimed. Moreover, in view of [2], 6 being a central nilpotent in R_2 lies in $J(R_2)$ which is nil. Since 2 inverts in R_2 , it follows immediately that 3 lies in $J(R_2)$, so that $R/J(R_2)$ is of characteristic 3. Furthermore, for any idempotent $e \in R_2$, the element $1 - 2e$ is an

involution because $(1 - 2e)^2 = 1$ and thus, by what we have already shown, either $1 - 2e = q + 1$ or $1 - 2e = q - 1$ for some $q \in Nil(R)$. Hence one of the following equalities $2e = -q$ or $2(1 - e) = q$ holds, so that $e = -\frac{q}{2}$ or $1 - e = \frac{q}{2}$. Since these are both idempotents and nilpotents, it follows at once that $e = 0$ or $e = 1$. Consequently, R_2 being indecomposable is abelian and, by [2], we deduce that $R_2/J(R_2)$ has to be isomorphic to \mathbb{Z}_3 , as stated. \square

As a consequence to the above fact, we have the following generalization of the corresponding fact from [9].

Corollary 2.2. *If R is a weakly nil-clean ring with the strong property, then for any idempotent e of R the corner ring eRe is a weakly nil-clean ring with the strong property. In particular, if $\mathbb{M}_n(R)$ is a weakly nil-clean ring having the strong property, then so is the ring R .*

Proof. First of all, we apply Theorem 2.1. Next, one observes that $Id(R) = \{0, 1\}$, provided $R/J(R)$ is isomorphic to \mathbb{Z}_3 . To that goal, if $r \in Id(R)$, then $r + J(R) \in Id(R/J(R)) = \{J(R), 1 + J(R)\}$. So, either $r \in J(R)$ or $r \in 1 + J(R) \leq U(R)$ which ensures that $r = 0$ or $r = 1$, respectively. This substantiates our assertion. Henceforth, the proof goes on utilizing the fact that, if R has all unipotent units, then eRe also has only unipotent units (see [7]).

For the second part, one sees that $R \cong E_{1n}\mathbb{M}_n(R)E_{1n}$ for the idempotent matrix E_{1n} with $(1, n)$ -entry 1 and the other entries 0, so we are finished. \square

Dropping off the “strong” condition, it is unknown at this stage whether or not if R is weakly nil-clean, then so does eRe for any idempotent e of R . Adapting some results from [2], we can conclude that the validity of the converse implication for the corner problem cannot be happen.

We are now ready to proceed by proving of the following.

Theorem 2.3. *Suppose that R is a ring equipped with the strong nil-involution property. Then $R/J(R) \cong \mathbb{Z}_3$ and $J(R)$ is nil. The converse is also true.*

Proof. We shall show that such a ring R is weakly nil-clean with the strong property, for which $3 \in J(R)$. To that purpose, given $r \in R$, we write $r = q + 1 + v$ or $r = q - 1 + v$ for some existing nilpotent q and involution v which commute. Thus $q + v$ is again a

unit, say u , and $r = u + 1$ or $r = u - 1$. Since 1 cannot be written as $u + 1$, it follows that $1 = u - 1$. So, $2 = u$ is a unit whence 2 inverts in R . Moreover, it is elementary to check that both $\frac{1+v}{2}$ and $\frac{1-v}{2}$ are idempotents. Since $2r \in R$, one sees that $2r = q + 1 + v$ or $2r = q - 1 + v = q - (1 - v)$ which implies that $r = \frac{q}{2} + \frac{1+v}{2}$ or that $r = \frac{q}{2} - \frac{1-v}{2}$. Since $\frac{q}{2}$ remains a nilpotent, it is now clear that R is weakly nil-clean having the strong property, as asserted. In conjunction with [2], it follows that 6 belongs to $J(R)$ and hence 3 lies in $J(R)$. We furthermore need apply the idea for proof in Theorem 2.1 to get the wanted claim.

Reciprocally, if $r \in R$, then $r + J(R)$ can be written as one of $J(R)$, $1 + J(R)$ or $-1 + J(R)$. Since $J(R) \subseteq Nil(R)$, one derives that either $r = q$ or $r = q + 1$ or $r = q - 1$, for some nilpotent q . Since $3 \in J(R)$ is a nilpotent, we infer that either $r = q + 1 + (-1) = q - 1 + 1$ or $r = (q + 3) - 1 + (-1)$ or $r = (q - 3) + 1 + 1$. But $q \pm 3$ remains a nilpotent and $(-1)^2 = 1^2 = 1$, so we are set. \square

Remark. It is worthwhile noticing that it follows from the proof of Theorem 2.1 above in a combination with [2] that a ring satisfies the (strong) nil-involution property if and only if it is a weakly nil-clean ring (with the strong property) for which 2 is invertible.

We are now in a position to obtain an element-wise characterization of weakly nil-clean elements with the strong property. To that aim, similarly to above, an element a of a ring R is called *clean* if $a = u + e$ where $u \in U(R)$ and $e \in Id(R)$. If $a = q + e$ with $q \in Nil(R)$ and $e \in Id(R)$, a is said to be *nil-clean*, while a is said to be *weakly nil-clean* provided $a = q + e$ or $a = q - e$. In addition, if q and e commutes, we will say that a is either strongly nil-clean or weakly nil-clean with the strong property.

It is in principle known and easy to prove that $a \in R$ is strongly nil-clean $\iff a^2 - a \in Nil(R)$. This can be substantially extended to the following:

Proposition 2.4. *An element $a \in R$ is weakly nil-clean having the strong property if, and only if, either $a^2 - a \in Nil(R)$ or $a^2 + a \in Nil(R)$.*

Proof. “ \implies ”. Writing $a = q + e$ or $a = q - e$ with $qe = eq$, we have $a^2 - a = q^2 - q + 2qe$ or $a^2 + a = q^2 + q - 2qe$. In both cases, these are nilpotents, as expected.

“ \Leftarrow ”. First, suppose that $a^2 + a \in Nil(R)$ whence $(a^2 + a)^n = 0$ for some $n \in \mathbb{N}$. Setting $e = (1 - (1 + a)^n)^n$, by the Newton binomial formula we deduce that $e = ka^n = 1 - m(1 + a)^n$ for some $k, m \in R$ depending only on a , and thus $1 - e = m(1 + a)^n$. It is immediate that $ae = ea$ because $ak = ka$ as well as $am = ma$ whence we observe that $e \in Id(R)$ since $e(1 - e) = ka^n \cdot m(1 + a)^n = km(a + a^2)^n = 0$. Furthermore, $a + e = a(1 - e) + e(1 + a)$ and hence $(a + e)^n = a^n(1 - e) + e(a + 1)^n = a^n m(1 + a)^n + ka^n(1 + a)^n = m(a + a^2)^n + k(a + a^2)^n = 0$. Finally, this means that $a + e \in Nil(R)$ and, therefore, $a = (a + e) - e \in Nil(R) - Id(R)$ with $(a + e)e = e(a + e)$. We can proceed in a similar way letting $a^2 - a \in Nil(R)$ (see cf. [12] too) to conclude that $a = (a - e) + e \in Nil(R) + Id(R)$ with $(a - e)e = e(a - e)$, so that in both cases a is a weakly nil-clean element with the strong property, as asserted. \square

Remark. This can also be directly deduced, because a is a weakly nil-clean element with the strong property \iff either a or $-a$ is a strongly nil-clean element. In fact, $a^2 - a \in Nil(R)$ or $(-a)^2 - (-a) = a^2 + a \in Nil(R)$.

Recollect that a ring R is called *weakly Boolean* if each element of R is idempotent or minus idempotent. Since it is self-evident that the element a or $-a$ is nil-clean \iff a is weakly nil-clean \iff $-a$ is weakly nil-clean, adapting the idea from [12, Proposition 3.9] along with [6], one can infer an other confirmation of Theorem 2.1 in a more convenient form. Namely, R is a weakly nil-clean ring with the strong property \iff $J(R)$ is nil and $R/J(R)$ is weakly Boolean \iff $J(R)$ is nil and either $R/J(R) \cong B$, or $R/J(R) \cong \mathbb{Z}_3$, or $R/J(R) \cong B \times \mathbb{Z}_3$, where B is a Boolean ring.

Generally, if $R/J(R)$ is a reduced weakly nil-clean ring having the strong property and $J(R)$ is nil, then R is a weakly nil-clean ring having the strong property. In fact, by what we have proved above in Proposition 2.4, the relation $(a + J(R))^2 + (a + J(R)) = a^2 + a + J(R) \in Nil(R/J(R)) = J(R)$ tells us that $a^2 + a \in J(R) \subseteq Nil(R)$, as required. Same for $a^2 - a \in Nil(R)$, and so again with the aid of Proposition 2.4 we are finished.

On the other side, in [1] was proven that there is a nil-clean element which is not clean. However, every strongly nil-clean element has to be clean. In fact, even much more is true:

Proposition 2.5. *Each weakly nil-clean element having the strong property is clean.*

Proof. Writing $b = n + e$ or $b = n - e$ with $ne = en$ for some nilpotent n and idempotent e , we have either $b = (n + 2e - 1) + (1 - e)$ or $b = (n - 1) + (1 - e)$. In both cases, b is a clean element because $(2e - 1)^2 = 1$ and $2e - 1$ commutes with n , so that $n + 2e - 1$ is a unit as well as so is $n - 1$, whereas $1 - e$ is an idempotent. \square

3. CONCLUDING DISCUSSION

We close the work with the following challenging problem.

Conjecture 1. A ring R is weakly nil-clean if, and only if, R is either a nil-clean ring or $R/J(R)$ is isomorphic to \mathbb{Z}_3 with $J(R)$ nil or R is a direct product of these two rings.

We notice that this question will be resolved in the affirmative provided that the following holds:

Conjecture 2. A ring R satisfies the nil-involution property if, and only if, $R/J(R) \cong \mathbb{Z}_3$ and $J(R)$ is nil.

Indeed, to show that the “and only if” part of Conjecture 1 is true, we decompose R as the direct product of a nil-clean ring and a ring with the nil-involution property. In fact, since by [2] we know that $6^n = 0$ for some $n \in \mathbb{N}$ and since $(2^n, 3^n) = 1$, i.e., there exist non-zero integers u, v such that $2^nu + 3^nv = 1$, it plainly follows that $R = 2^nR \oplus 3^nR$ because $2^nR \cap 3^nR = \{0\}$. In fact, to show that this intersection is really zero, given $x = 2^na = 3^nb$ for some $a, b \in R$, we then have $2^na u = 3^nb u$. However, $a(1 - 3^nv) = 3^nb u$ whence $3^n(av + bu) = a$. Multiplying both sides by 2^n , we derive that $0 = 2^na = x$, as required. So, with the Chinese Remainder Theorem at hand, or directly by the above-given direct decomposition of R into the sum of two ideals, we deduce that $R \cong L \times P$, where $L \cong R/2^nR \cong 3^nR$ and $P \cong R/3^nR \cong 2^nR$. Utilizing [2], it follows that both L and P are weakly nil-clean as epimorphic images of R . But it is obvious that $2 \in J(L)$, so appealing once again to [2], we conclude that L is nil-clean, as claimed.

As for P , we may assume that $P \neq 0$. Thus $3 \in J(P)$ and, in addition, $2 \in U(P)$. Applying [2] and [9], we infer that P is indecomposable and not nil-clean. Moreover, a new application of [2] implies that $J(P)$ is nil. Letting now $a \in P$, there exist $b \in Nil(P)$ and $e \in Id(P)$ such that $a = b + e$ or $a = b - e$. In the first case,

$a = ((b + 3e) - 1) + (1 - 2e)$ with $(1 - 2e)^2 = 1$. Moreover, as $b^m = 0$ for some integer $m > 0$ and $3 \in J(P)$, it readily follows that $(b + 3e)^m \in J(P)$, whence $b + 3e$ is a nilpotent because $J(P)$ is nil. Next, if $a = b - e$, then $a = ((b - 3e) + 1) + (-1 + 2e)$ with $(-1 + 2e)^2 = 1$. As above, since $b^m = 0$ and $3 \in J(P)$, it easily follows that $(b - 3e)^m \in J(P)$, so that $b - 3e$ is a nilpotent as $J(P)$ is nil. This finally enables us that P satisfies the nil-involution property, as claimed. We furthermore apply Conjecture 2 to get the desired claim.

To demonstrate now that the “if” part of Conjecture 1 is valid, exploiting [2] and Conjecture 2, it is enough to prove that any ring equipped with the nil-involution property is weakly nil-clean. In fact, one easily sees that $3 \in J(R)$ and hence $2 \in U(R)$. Let now $a \in R$. Then $-2a = v + w$, where $v \in Nil(R) \pm 1$ and $w^2 = 1$. If $v = b + 1$ with $b \in Nil(R)$, then $a = (-\frac{b}{2}) - \frac{1+w}{2}$ with $-\frac{b}{2} \in Nil(R)$ and $\frac{1+w}{2}$ an idempotent. If now $v = b - 1$ with $b \in Nil(R)$, then $a = (-\frac{b}{2}) + \frac{(1-w)}{2}$ with $-\frac{b}{2} \in Nil(R)$ and $\frac{1-w}{2}$ an idempotent. So, R is weakly nil-clean, as needed.

This completes the proof.

Note also that it is not too hard to verify that the sufficiency in Conjecture 2 is always fulfilled, so that it suffices to establish only the necessity.

REFERENCES

- [1] D. Andrica and G. Calugareanu, *A nil-clean 2×2 matrix over integers which is not clean*, J. Algebra Appl. **13** (2014).
- [2] S. Breaz, P. Danchev and Y. Zhou, *Rings in which every element is either a sum or a difference of a nilpotent and an idempotent*, J. Algebra Appl. **15** (2016).
- [3] H. Chen, *On uniquely clean rings*, Comm. Algebra **39** (2011), 189–198.
- [4] J.L. Chen, Z. Wang and Y. Zhou, *Rings in which elements are uniquely the sum of an idempotent and a unit that commute*, J. Pure Appl. Algebra **213** (2009), 215–223.
- [5] A. Cîmpean and P. Danchev, *Weakly nil-clean index and uniquely weakly nil-clean rings*, preprint (2016).
- [6] P.V. Danchev, *Weakly semi-boolean unital rings*, preprint (2015).
- [7] P.V. Danchev and T.Y. Lam, *Rings with unipotent units*, Publ. Math. Debrecen **88** (2016), 449–466.

- [8] P.V. Danchev and W.Wm. McGovern, *Commutative weakly nil clean unital rings*, J. Algebra **425** (2015), 410–422.
- [9] A.J. Diesl, *Nil clean rings*, J. Algebra **383** (2013), 197–211.
- [10] T.Y. Lam, *A First Course in Noncommutative Rings*, Second Edition, Graduate Texts in Math., Vol. **131**, Springer-Verlag, Berlin-Heidelberg-New York, 2001.
- [11] W.K. Nicholson, *Lifting idempotents and exchange rings*, Trans. Amer. Math. Soc. **229** (1977), 269–278.
- [12] J. Šter, *Rings in which nilpotents form a subring*, arXiv: 1510.07523v1 [math.RA] 26 Oct., 2015.

Peter Danchev is Professor of Mathematics at Plovdiv and his interests are Abelian Groups and Group Algebras, Associative Rings (both commutative and non-commutative), and Matrix Theory.

DEPARTMENT OF MATHEMATICS, PLOVDIV STATE UNIVERSITY, PLOVDIV
4000, BULGARIA

E-mail address: pvdanchev@yahoo.com