

PROBLEMS

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We begin with three integrals.

Problem 76.1.

$$(a) \int_0^{\infty} \sin(x^2) dx \quad (b) \int_0^1 \frac{x-1}{\log x} dx \quad (c) \int_{-1}^1 \frac{\cos x}{e^{1/x} + 1} dx$$

I learnt the next problem from a popular-mathematics lecture given by Vicky Neale of the University of Oxford.

Problem 76.2. For each point z on the unit circle, let ℓ_z denote the closed line segment from z to z^2 . Consider the collection of those points in the closed unit disc that each lie at the intersection of two distinct line segments ℓ_z and ℓ_w . What shape is the complement in the unit disc of this collection of points?

We finish with a problem proposed by Wenchang Chu of Università del Salento, Italy.

Problem 76.3. Evaluate $\sum_{n=0}^{\infty} \frac{1}{2^n} \tan\left(\frac{x}{2^n}\right)$.

SOLUTIONS

Here are solutions to the problems from *Bulletin* Number 74.

The first problem was solved by the North Kildare Mathematics Problem Club, and it is their solution that we present. The solution was also known to Grahame Erskine of the Open University, who suggested the problem.

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Problem 74.1. Given a positive integer A , let B be the number obtained by reversing the digits in the base n expansion of A . The integer A is called a *reverse divisor* in base n if it is a divisor of B that is not equal to B .

For example, using decimal expansions, if we reverse the digits of the integer 15, then we obtain 51. Since 15 is not a divisor of 51, the integer 15 is not a reverse divisor in base 10.

For which of the positive integers n between 2 and 16, inclusive, is there a two-digit reverse divisor in base n ?

You may also wish to attempt the more difficult problem of classifying those positive integers n for which there is a two-digit reverse divisor in base n .

Solution 74.1. Given a positive integer n and two integers a and b with $1 \leq a, b < n$, let $[ab]_n = an + b$. The number $[ab]_n$ is defined to be a *two-digit reverse divisor* in base n if $[ba]_n$ is divisible by, but not equal to, $[ab]_n$.

By inspection, there are no two-digit reverse divisors in bases 1, 2 or 3. For $n > 3$, we claim that there is a two-digit reverse divisor in base n if and only if $n + 1$ is composite. To see this, suppose first that $n + 1$ is composite. Choose positive integers a and k with $n + 1 = (a + 1)(k + 1)$, and define $b = n - a - 1$. Then $bn + a = k(an + b)$, so we have a two-digit reverse divisor, unless $k = 1$. In the case $k = 1$, we define $a' = 1$ and $b' = n - 2$ to give $b'n + a' = a(a'n + b')$. This time we certainly have a two-digit reverse divisor, because $a = (n - 1)/2 > 1$.

Conversely, suppose that $n + 1$ is equal to a prime p , and suppose that $[ab]_n$ is a two-digit reverse divisor in base n . Let $k = (bn + a)/(an + b)$. One calculates that

$$p(b - ak) = (b - a)(k + 1).$$

Thus one of $b - a$ or $k + 1$ is divisible by p . But $1 \leq b - a < p$ and

$$1 < k + 1 < b + 2 \leq n + 1 = p,$$

so we have a contradiction. Hence, contrary to our earlier assumption, there does not exist a two-digit reverse divisor in base n . \square

The second problem was solved by Henry Ricardo (New York Math Circle, New York, USA), the North Kildare Mathematics Problem Club, and the proposer, Ángel Plaza (Universidad de Las Palmas de Gran Canaria, Spain). All solutions were similar. The solution

we present is that of Henry Ricardo. *Nesbitt's inequality* features in the solution, which says that for positive real numbers a , b , and c , we have

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2}.$$

This inequality can be proven by using two applications of the rearrangement inequality with the triples (a, b, c) and $(1/(a+b), 1/(b+c), 1/(c+a))$.

Problem 74.2. Let $f : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be an increasing, convex function with $f(1) = 1$, and let x , y , and z be positive real numbers. Prove that for any positive integer n ,

$$\left(f\left(\frac{2x}{y+z}\right)\right)^n + \left(f\left(\frac{2y}{z+x}\right)\right)^n + \left(f\left(\frac{2z}{x+y}\right)\right)^n \geq 3.$$

Solution 74.2. By Nesbitt's inequality,

$$\frac{2x}{y+z} + \frac{2y}{z+x} + \frac{2z}{x+y} \geq 3.$$

Then, as the function $g(x) = f(x)^n$ is convex and increasing with $g(1) = 1$, we can use Jensen's inequality to write

$$\begin{aligned} g\left(\frac{2x}{y+z}\right) + g\left(\frac{2y}{z+x}\right) + g\left(\frac{2z}{x+y}\right) &\geq 3g\left(\frac{\frac{2x}{y+z} + \frac{2y}{z+x} + \frac{2z}{x+y}}{3}\right) \\ &\geq 3g(1) = 3, \end{aligned}$$

as required. \square

The third problem was solved by the North Kildare Mathematics Problem Club, Ángel Plaza, and the proposer Finbarr Holland (University College Cork). Finbarr has pointed out that in fact the result in the problem can quickly be derived using Bernstein polynomials. Bernstein polynomials are useful for approximating continuous functions; they provide one way of proving the Weierstrass approximation theorem. To solve the problem, simply calculate the n th Bernstein polynomial of the function $f(x) = x^j$ and use the value $x = 1/2$. Nonetheless, we present a complete, more elementary, solution here, which does not require Bernstein polynomials. The solution is essentially the same as that of the North Kildare Mathematics Problem Club and Ángel Plaza.

In this problem, we use the standard notation

$$f(n) \sim g(n) \quad \text{as } n \rightarrow \infty,$$

where f and g are positive functions, to mean that

$$\frac{f(n)}{g(n)} \rightarrow 1 \quad \text{as } n \rightarrow \infty.$$

Problem 74.3. Prove that for $j = 0, 1, 2, \dots$,

$$\sum_{k=0}^n k^j \binom{n}{k} \sim n^j 2^{n-j} \quad \text{as } n \rightarrow \infty.$$

Solution 74.3. Let $f_0(x) = (1+x)^n$, and for $j = 1, 2, \dots$ let

$$f_j(x) = x \frac{d f_{j-1}(x)}{d x}.$$

We shall prove by induction that for $j = 0, 1, 2, \dots$,

$$f_j(x) = n^j x^j (1+x)^{n-j} + p_j(x, n),$$

where p_j is a polynomial in x and n , and its degree as a polynomial in n is less than j . This equation holds when $j = 0$ (p_0 is the zero polynomial, which we assume has degree -1 in each variable). Suppose that it holds when $j = m - 1$. Then one can check that

$$f_m(x) = n^m x^m (1+x)^{n-m} + q(x, n),$$

where

$$q(x, n) = (m-1)n^{m-1}x^{m-1}(1+x)^{n-m} + x \frac{\partial}{\partial x} p_{m-1}(x, n).$$

The degree of q as a polynomial in n is less than m , so the inductive proof is complete. We deduce that, for $x \geq 1$,

$$f_j(x) \sim n^j x^j (1+x)^{n-j}, \quad j = 0, 1, 2, \dots$$

Now,

$$f_0(x) = (1+x)^n = \sum_{k=0}^n \binom{n}{k} x^k,$$

so

$$\sum_{k=0}^n k^j \binom{n}{k} = f_j(1) \sim n^j 2^{n-j}, \quad j = 0, 1, 2, \dots$$

□

Finally, we return to a problem of Niall Ryan (University of Limerick) from issue 72. Niall had a solution to this problem, but it was quite involved. The solution we include here, provided by Finbarr Holland, deals with the ‘odd case’ only.

Problem 72.1. For each integer $n \geq 0$, let

$$S_n = \sum_{m=0}^{\infty} \frac{K_{2m}}{2m+n+1} + \sum_{\substack{m=0 \\ 2m \neq n}}^{\infty} \frac{K_{2m}}{2m-n},$$

where

$$K_{2m} = \left[\frac{(2m)!}{2^{2m}(m!)^2} \right]^2.$$

Prove that

$$S_n = \begin{cases} 0, & n \text{ odd,} \\ 2K_n \left(\log 2 - \sum_{k=1}^n \frac{(-1)^{k+1}}{k} \right), & n \text{ even.} \end{cases}$$

Solution 72.1. Let

$$a_n = \frac{\Gamma(\frac{1}{2} + n)}{\Gamma(\frac{1}{2}) \Gamma(n+1)} = \frac{(2n)!}{2^{2n}(n!)^2} = \sqrt{K_{2n}}.$$

We aim to prove that for $n = 0, 1, 2, \dots$,

$$S_{2n+1} = \sum_{m=0}^{\infty} a_m^2 \left(\frac{1}{2m+2n+2} + \frac{1}{2m-2n-1} \right) = 0.$$

Clearly, it is enough to prove that $T_n = 0$ for $n = 0, 1, 2, \dots$, where

$$T_n = \sum_{j=0}^n (-1)^j \binom{n}{j} S_{2j+1} = 0.$$

To establish this, notice that

$$\sum_{j=0}^n (-1)^j \binom{n}{j} \frac{1}{2z+2j+2} = \frac{n!}{2 \prod_{j=0}^n (z+j+1)} = \frac{n! \Gamma(z+1)}{2 \Gamma(z+n+2)},$$

and

$$\begin{aligned} \sum_{j=0}^n (-1)^j \binom{n}{j} \frac{1}{2z-2j-1} &= \frac{(-1)^n n! 2^n}{\prod_{j=0}^n (2z-2j-1)} \\ &= \frac{(-1)^n n! \Gamma(z - \frac{2n+1}{2})}{2 \Gamma(z + \frac{1}{2})}. \end{aligned}$$

Next we appeal to a well-known result about hypergeometric series, which can be found, for example, in Section 10.31 of *An introduction to the theory of functions of a complex variable* by E.T. Copson

(OUP, 1970). Recall that the hypergeometric series $F(a, b; c; z)$ is defined for $|z| < 1$ by the equation

$$\sum_{m=0}^{\infty} \frac{\Gamma(a+m)\Gamma(b+m)}{\Gamma(c+m)} \frac{z^m}{m!} = \frac{\Gamma(a)\Gamma(b)}{\Gamma(c)} F(a, b; c; z),$$

and it is defined for $|z| \geq 1$ by analytic continuation. In fact, by Abel's continuity theorem, the series above can also be used to define $F(a, b; c; 1)$. The result we need, from page 251 of Copson's text, is that

$$F(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}$$

if $\operatorname{Re}[c-a-b] > 0$. Using this observation, we see that

$$\begin{aligned} T_n &= \frac{n!}{2} \sum_{m=0}^{\infty} a_m^2 \left(\frac{\Gamma(m+1)}{\Gamma(n+2+m)} + (-1)^n \frac{\Gamma(-\frac{2n+1}{2}+m)}{\Gamma(\frac{1}{2}+m)} \right) \\ &= \frac{n!}{2\pi} \sum_{m=0}^{\infty} \left(\frac{\Gamma(\frac{1}{2}+m)\Gamma(\frac{1}{2}+m)}{m!\Gamma(n+2+m)} + (-1)^n \frac{\Gamma(\frac{1}{2}+m)\Gamma(-\frac{2n+1}{2}+m)}{m!\Gamma(1+m)} \right) \\ &= \frac{n!}{2\pi} \left(\frac{\Gamma(\frac{1}{2})^2 F(\frac{1}{2}, \frac{1}{2}; n+2; 1)}{\Gamma(n+2)} + (-1)^n \frac{\Gamma(\frac{1}{2})\Gamma(-\frac{2n+1}{2}) F(\frac{1}{2}, -\frac{2n+1}{2}; 1; 1)}{\Gamma(1)} \right) \\ &= \frac{n!}{2\pi} \left(\frac{\pi\Gamma(n+1)}{\Gamma(n+\frac{3}{2})^2} + (-1)^n \frac{\Gamma(-\frac{2n+1}{2})\Gamma(n+1)}{\Gamma(n+\frac{3}{2})} \right) \\ &= \frac{n!}{2\pi} \left(\frac{\pi\Gamma(n+1)}{\Gamma(n+\frac{3}{2})^2} + (-1)^n \frac{\pi\Gamma(n+1)}{\sin(-(n+\frac{1}{2})\pi)\Gamma(n+\frac{3}{2})^2} \right) \\ &= 0. \end{aligned}$$

The well-known functional equation $\Gamma(z)\Gamma(1-z) = \pi/\sin(\pi z)$ is used in the second-last line. \square

We invite readers to submit problems and solutions. Please email submissions to imsproblems@gmail.com in any format (we prefer Latex). Submissions for the summer Bulletin should arrive before the end of April, and submissions for the winter Bulletin should arrive by October. The solution to a problem is published two issues after the issue in which the problem first appeared. Please include solutions to any problems you submit, if you have them.

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