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PROBLEMS

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Problems

Let us begin with a classic.

Problem 75.1. What is the least positive integer n for which a square can be tessellated by n acute-angled triangles?

The second problem was proposed by Finbarr Holland of University College Cork. The inequality involving the exponential function that is considered in the problem is a generalisation of the useful inequalities

$$e^x \leqslant \frac{1}{1-x} \quad \text{and} \quad e^{2x} \leqslant \frac{1+x}{1-x} \quad (0 \leqslant x < 1),$$

which are strict inequalities unless $x = 0$.

Problem 75.2. Let

$$s_n(x) = \sum_{k=0}^{n} \frac{x^k}{k!}, \quad n = 0, 1, 2, \dots .$$

Suppose $0 < \alpha < 1$. Prove that when $n \geqslant 1$,

$$e^x \leqslant \frac{s_n(x) - \alpha x s_{n-1}(x)}{1 - \alpha x} \quad \text{for all } x \in [0, 1/\alpha)$$

if and only if $\alpha \geqslant 1/(n+1)$.

We finish with another inequality: the sort that might crop up in a mathematics olympiad.

Problem 75.3. Given positive real numbers a, b, and c, prove that

$$a + b + c \leqslant \sqrt[3]{abc}\left(\frac{a}{b} + \frac{b}{c} + \frac{c}{a}\right).$$

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Solutions

Here are solutions to the problems from *Bulletin* Number 73.

The first problem was solved by Angel Plaza (Universidad de Las Palmas de Gran Canaria, Spain), the North Kildare Mathematics Problem Club, and the proposer, Finbarr Holland. We present the solution of the North Kildare Mathematics Problem Club.

Problem 73.1. Let U_n denote the Chebyshev polynomial of the second kind of degree n, which is the unique polynomial that satisfies the equation $U_n(\cos\theta) = \sin((n+1)\theta)/\sin\theta$. The polynomial U_{2n} satisfies $U_{2n}(t) = p_n(4t^2)$, where

$$p_n(z) = \sum_{k=0}^{n} (-1)^k \binom{2n-k}{k} z^{n-k}.$$

Prove that p_n is irreducible over the integers when $2n+1$ is a prime number.

Solution 73.1. Define $q_n(t) = p_n(2t+2)$, so that $q_n(2t^2-1) = U_{2n}(t)$. Since $q_n(\cos 2\theta) = U_{2n}(\cos\theta)$, the n roots of q_n are the numbers $\cos(2k\pi/(2n+1))$ for $k = 1,\dots,n$. We prove that if p_n is reducible, then $2n+1$ is not prime.

Suppose that p_n is reducible over the integers. Then so is q_n, and one of the proper factors of q_n has $a = \cos(2\pi/(2n+1))$ as a root. It follows that the degree of the extension $\mathbb{Q}(a)$ over $\mathbb{Q}$ is less than n. Now let $b = i\sin(2\pi/(2n+1))$. Since $b^2 = a^2 - 1$, the degree of the extension $\mathbb{Q}(a,b)$ over $\mathbb{Q}$ is less than $2n$. Notice that $\mathbb{Q}(a,b)$ contains $a+b$, a primitive root of unity. Therefore the cyclotomic polynomial $x^{2n} + \cdots + x^2 + x + 1$ of degree $2n$ splits in $\mathbb{Q}(a,b)$. However, this polynomial is irreducible when $2n+1$ is prime, as is well-known, so $2n+1$ cannot be prime. $\square$

The second problem was solved by Henry Ricardo (New York Math Circle, New York, USA), the North Kildare Mathematics Problem Club, and the proposer (the Editor, who learned the problem from Tony Barnard of King's College London). The solution we present is an amalgamation of the submitted solutions. Henry Ricardo pointed out that the problem (and solution) appear elsewhere; for example, see Problem 1339 in Math. Mag. 64 (1991), no. 1.

Problem 73.2. Find all positive integers a, b, and c such that

$$\begin{aligned} bc &\equiv 1 \pmod{a} \\ ca &\equiv 1 \pmod{b} \\ ab &\equiv 1 \pmod{c}. \end{aligned}$$

Solution 73.2. Without loss of generality, suppose that $a \leqslant b \leqslant c$. Since $bc-1$, $ca-1$, and $ab-1$ are divisible by a, b, and c, respectively, we see that

$$(bc-1)(ca-1)(ab-1) = (abc)^2 - (abc)(a+b+c) + (ab+bc+ca) - 1$$

is divisible by abc. Hence $ab + bc + ca - 1$ is divisible by abc. But $0 < ab + bc + ca - 1 < 3bc$, so $a < 3$.

Next, we know that

$$(ca-1)(ab-1) = a^2(bc) - (ab+ca) + 1$$

is divisible by bc, so $(ab + ca) - 1$ is divisible by bc. But $0 < (ab + ca) - 1 < 2ac$, so $b < 2a$.

From the inequalities $a < 3$ and $b < 2a$ we see that either $a = 1$ and $b = 1$ or $a = 2$ and $b < 4$. In the former case we obtain the solution $(1, 1, m)$, where m is any positive integer. In the latter case, the congruence $bc \equiv 1 \pmod{a}$ tells us that b is odd, so $b = 3$. From the congruence $ab \equiv 1 \pmod{c}$ we deduce that $c = 5$, which gives the only other solution $(2, 3, 5)$. □

The third problem was solved by Adnan Al (Mumbai, India), Angel Plaza (Universidad de Las Palmas de Gran Canaria, Spain), Henry Ricardo (New York Math Circle, New York, USA), the North Kildare Mathematics Problem Club, and the proposer (the Editor, who learned the problem from Tony Barnard). It was also solved by Finbarr Holland, and it is his short solution that we present here. Several contributors noted that there is literature on this kind of problem; see, for example, S. Koumandos, *Remarks on a paper by Chao-Ping Chen and Feng Qi*, Proc. Amer. Math. Soc. 134 (2006), no. 5, 1365–1367.

Problem 73.3. Prove that

$$\frac{1}{10\sqrt{2}} < \frac{1}{2} \times \frac{3}{4} \times \frac{5}{6} \times \cdots \times \frac{99}{100} < \frac{1}{10}.$$

Solution 73.3. Let

$$v_n = \frac{1}{2} \times \frac{3}{4} \times \frac{5}{6} \times \cdots \times \frac{2n-1}{2n}.$$

Then a quick check shows that the sequence $\sqrt{n}v_n$ is strictly increasing and the sequence $\sqrt{2n+1}v_n$ is strictly decreasing. Since $v_1 = 1/2$, we obtain the more general collection of inequalities

$$\frac{1}{2\sqrt{n}} < v_n < \frac{1}{\sqrt{2n+1}}, \quad n = 2, 3, \ldots .$$ □

We invite readers to submit problems and solutions. Please email submissions to `imsproblems@gmail.com` in any format (we prefer Latex). Submissions for the summer Bulletin should arrive before the end of April, and submissions for the winter Bulletin should arrive by October. The solution to a problem is published two issues after the issue in which the problem first appeared. Please include solutions to any problems you submit, if you have them.

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