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PROBLEMS

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Problems

I learned the first problem from Grahame Erskine of the Open University.

Problem 74.1. Given a positive integer A, let B be the number obtained by reversing the digits in the base n expansion of A. The integer A is called a *reverse divisor* in base n if it is a divisor of B that is not equal to B.

For example, using decimal expansions, if we reverse the digits of the integer 15, then we obtain 51. Since 15 is not a divisor of 51, the integer 15 is not a reverse divisor in base 10.

For which of the positive integers n between 2 and 16, inclusive, is there a two-digit reverse divisor in base n?

You may also wish to attempt the more difficult problem of classifying those positive integers n for which there is a two-digit reverse divisor in base n.

The second problem was proposed by Ángel Plaza of Universidad de Las Palmas de Gran Canaria, Spain.

Problem 74.2. Let $f:\mathbb{R}^+\longrightarrow\mathbb{R}^+$ be an increasing, convex function with $f(1)=1$, and let x, y, and z be positive real numbers. Prove that for any positive integer n,

$$\left(f\left(\frac{2x}{y+z}\right)\right)^n+\left(f\left(\frac{2y}{z+x}\right)\right)^n+\left(f\left(\frac{2z}{x+y}\right)\right)^n\geqslant 3.$$

The third problem was contributed by Finbarr Holland of University College Cork. To state this problem, we use the standard notation

$$f(n)\sim g(n)\quad\text{as}\quad n\to\infty,$$

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where f and g are positive functions, to mean that

$$\frac{f(n)}{g(n)} \to 1 \quad \text{as} \quad n \to \infty.$$

Problem 74.3. Prove that for $j = 0, 1, 2, \ldots,$

$$\sum_{k=0}^{n} k^j \binom{n}{k} \sim n^j 2^{n-j} \quad \text{as} \quad n \to \infty.$$

Solutions

Here are some solutions to the problems from *Bulletin* Number 72. The proposer sketched a method for solving Problem 72.1, but we have yet to receive a full solution. If we receive a full solution, then we'll publish it in a later issue.

The second problem was solved by the North Kildare Mathematics Problem Club and the proposer, Finbarr Holland. The solution that we give is equivalent to the submitted solutions.

Problem 72.2. Prove that the integral

$$\int_0^\infty \frac{x \sin x}{2 + 2\cos x - 2x \sin x + x^2}\, dx$$

exists as a Riemann integral, but not as a Lebesgue integral, and determine its value as a Riemann integral.

Solution 72.2. Let

$$f(x) = \frac{x \sin x}{2 + 2\cos x - 2x \sin x + x^2}.$$

The denominator is equal to $(1 + \cos x)^2 + (x - \sin x)^2$, so f is continuous on $[0, \infty)$, and hence it is Riemann integrable on any compact subdomain of $[0, \infty)$. Let

$$g(x) = \arctan\left(\frac{\cos x + 1}{\sin x - x}\right).$$

Then g is an antiderivative of f on $(0, \infty)$, so

$$\int_a^b f(x)\, dx = g(b) - g(a),$$

where $0 < a < b < \infty$. Since $g(a) \to -\pi/2$ as $a \to 0$, and $g(b) \to 0$ as $b \to \infty$, we deduce that, as a Riemann integral,

$$\int_0^\infty f(x)\,dx = \frac{\pi}{2}.$$

Next, to prove that f is not Lebesgue integrable on $(0, \infty)$, let

$$I_n = \int_\pi^{(2n+1)\pi} |f(x)|\,dx$$

for $n = 1, 2, \ldots$. Then

$$\begin{aligned}
I_n &= \sum_{k=1}^n \int_{(2k-1)\pi}^{(2k+1)\pi} |f(x)|\,dx \\
&= \sum_{k=1}^n \int_0^{2\pi} |f(x + (2k-1)\pi)|\,dx \\
&= \sum_{k=1}^n \int_0^{2\pi} \frac{(x + (2k-1)\pi)|\sin x|}{(1+\cos x)^2 + ((2k-1)\pi + (x - \sin x))^2}\,dx \\
&\geqslant \sum_{k=1}^n (2k-1)\pi \int_0^{2\pi} \frac{|\sin x|}{(1+\cos x)^2 + ((2k-1)\pi + (x - \sin x))^2}\,dx.
\end{aligned}$$

Let

$$a_k = (2k-1)\pi \int_0^{2\pi} \frac{|\sin x|}{(1+\cos x)^2 + ((2k-1)\pi + (x - \sin x))^2}\,dx.$$

Then

$$(2k-1)\pi a_k \to \int_0^{2\pi} |\sin x|\,dx = 4 \quad \text{as} \quad k \to \infty.$$

As the sum of the reciprocals of the odds numbers is infinite, we see that the sequence I_n is unbounded. Thus

$$\int_0^\infty |f(x)|\,dx = \infty,$$

so f is not Lebesgue integrable. □

The third problem was solved by the North Kildare Mathematics Problem Club and the proposer, Tom Moore of Bridgewater State University, USA. The two solutions are equivalent to the solution given below.

*Problem 72.3***.** For $n = 1, 2, \ldots,$ the triangular numbers T_n and square numbers S_n are given by the formulas

$$T_n = \frac{n(n+1)}{2} \quad \text{and} \quad S_n = n^2.$$

It is well known that every even perfect number is a triangular number. Prove that every even perfect number greater than 6 can be expressed as the sum of a triangular number and a square number.

Solution 72.3. The Euclid–Euler theorem says that every even perfect number can be expressed in the form $2^{p-1}(2^p - 1)$, where p and $2^p - 1$ are prime numbers, and every number of that form is an even perfect number. We use this representation of even perfect numbers to solve the problem.

After some basic algebraic manipulations, we can see that

$$T_n + S_{2n+1} = \tfrac{1}{2}(3n+1)(3n+2).$$

Let p be an odd prime number such that $2^p - 1$ is also a prime number. Notice that $2^p - 2$ is divisible by 3. Let $m = \frac{1}{3}(2^p - 2)$. Then with some further algebraic manipulations, we see that

$$T_m + S_{2m+1} = 2^{p-1}(2^p - 1).$$

This shows that every even perfect number other than the number 6 (which is given by $p = 2$) can be expressed as the sum of a triangular number and a square number. The number 6 cannot be expressed as the sum of a triangular number and a square number, as you can easily check. □

We invite readers to submit problems and solutions. Please email submissions to `imsproblems@gmail.com`.

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