

## Some Residually Solvable One-Relator Groups

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ABSTRACT. This communication records some observations made in the course of studying one-relator groups from the point of view of residual solvability. As a contribution to classification efforts we single out some relator types that render the corresponding one-relator groups residually solvable.

### 1. INTRODUCTION

It is well known that free groups are residually nilpotent and, consequently, residually solvable. The literature contains a sizable amount of information about structural, residual, virtual properties of one-relator groups. The purpose of this communication is to offer a collection of facts and examples gathered while attempting to characterize the residually solvable one-relator groups in terms (of the form) of the (single) defining relator. In what follows we prove sufficiency results for certain cases when the relator is a commutator, and then raise some questions.

The class of one-relator groups shows a varied pattern of behavior with respect to residual properties. We begin with reviewing some of the literature that motivated our interest in the topic. G. Baumslag in [3] showed that positive one-relator groups, which is to say that the relator has only positive exponents, are residually solvable. In the same paper he provided a specific example to demonstrate that not all one-relator groups are residually solvable. A free-by-cyclic group is necessarily residually solvable. As well are the free-by-solvable Baumslag–Solitar groups  $B_{m,n}$  (the groups with presentation  $\langle a, b; a^{-1}b^m a = b^n \rangle$  for pairs of non-zero integers  $m, n$ ), by a result of Peter Kropholler [15] who showed that in these groups the second derived subgroup is free. The Baumslag–Solitar groups

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$B_{1,n}$  ( $n \neq 0$  integer) are solvable but not polycyclic; and the non-Hopfian (therefore not residually finite) Baumslag–Solitar groups are not residually nilpotent.

It is worth mentioning here that a large class of residually solvable one-relator groups is indicated by [8]: G. Baumslag, Fine, Miller and Troeger established that many one-relator groups, in particular cyclically pinched one-relator groups, are either free-by-cyclic or virtually free-by-cyclic. Further, a recent result of M. Sapir and I. Spakulova in [17] tells that, with probability (measured in terms of the length of the relator) tending to 1, a one-relator group with at least 3 generators is residually finite, even virtually a residually (finite  $p$ )-group, and coherent, for all sufficiently large primes  $p$ . In his subsequent work [16] M. Sapir also focuses on residual properties of one-relator groups with at least 3 generators. Our attention is turned mainly toward two-generator one-relator groups.

Our two main results concern the situation where  $G$  is a one-relator group whose relator is a commutator. First we recall, in fair detail, some classes of one-relator groups whose behavior with regard to residual solvability has been established before. Then we show sufficiency, for residual solvability, of certain conditions imposed on the (single) defining relator of the one-relator group. Then, we provide examples illustrating the difficulty in determining residual solvability of one-relator groups with arbitrary commutator.

Clearly, attempts to find criteria for residual solvability could be facilitated by linkages to outcomes of recent and older studies on the (fully) residually freeness of one-relator groups, in particular. For example, surface groups are easily found residually solvable since they are known to be fully residually free [6]. Also, in [2] B. Baumslag shows residual freeness for one-relator groups of the type  $\langle a_1, \dots, a_k; a_1^{w_1} \dots a_k^{w_k} = 1 \rangle$ , where  $k > 3$ , and the  $w_i$ 's in the ambient free group on  $a_1, \dots, a_k$  satisfy certain conditions; thus residual solvability for such one-relator groups is immediate.

## 2. PRELIMINARIES

For convenience, we start with a list of some of the definitions, facts, and theorems we will rely on throughout.

**Theorem 2.1** (Von Dyck). *Suppose  $G = \langle X; R \rangle$  and  $D = \langle X; R \cup S \rangle$ , with presentation maps  $\gamma$  and  $\mu$  respectively. Then  $x\mu \mapsto x\gamma$  ( $x \in X$ ) defines a homomorphism of  $G$  onto  $D$ .*

**Theorem 2.2** (Freiheitssatz, [14]). *Let  $G$  be a one-relator group, i.e.,  $G = \langle x_1, \dots, x_q; r = 1 \rangle$ . Suppose that the relator  $r$  is cyclically reduced, i.e., the first and the last letters in  $r$  are not (formal) inverses of each other. If each of  $x_1, \dots, x_q$  actually appears in  $r$ , then any proper subset of  $\{x_1, \dots, x_q\}$  is a free basis for a free subgroup of  $G$ .*

W. Magnus' method of structure analysis [14] for groups with a single defining relation has the following immediate consequence:

**Lemma 2.3.** *Let  $G = \langle b, x, \dots, c; r = 1 \rangle$  be a one-relator group. Suppose that  $b$  occurs in  $r$  with exponent sum zero and that upon re-expressing  $r$  in terms of the conjugates  $b^i x b^{-i} = x_i, \dots, b^i c b^{-i} = c_i$  ( $i \in \mathbb{Z}$ ) and renaming  $r$  as  $r_0$ ,  $\mu$  and  $\nu$  are respectively the minimum and maximum subscripts of  $x$  occurring in  $r_0$ . If  $\mu < \nu$  and if both  $x_\mu$  and  $x_\nu$  occur only once in  $r_0$  then  $N = gp_G(x, \dots, c)$  is free. If  $G$  is a two-generator group with generators  $b$  and  $x$ , then  $N$  is free of rank  $\nu - \mu + 1$ .*

**Definition 2.4.** A group  $G$  is *residually solvable* if for each  $w \in G$  ( $w \neq 1$ ), there exists a solvable group  $S = S(w)$  and an epimorphism  $\phi : G \rightarrow S$  such that  $w\phi \neq 1$ .

**Theorem 2.5** (Kahrobaei, [12], [13]). *Any generalized free product of two finitely generated torsion-free nilpotent groups, amalgamating a cyclic subgroup, is an extension of a residually solvable group by a solvable group. It is therefore residually solvable.*

**Theorem 2.6** (Kahrobaei, [12], [13]). *Any generalized free product of an arbitrary number of finitely generated nilpotent groups of bounded class, amalgamating a subgroup central in each of the factors, is an extension of a free group by a nilpotent group. It is therefore residually solvable.*

**Theorem 2.7** (Kahrobaei, [12], [13]). *The generalized free product of a finitely generated torsion-free abelian group and a nilpotent group is (residually solvable)-by-abelian-by-(finite abelian), consequently residually solvable.*

Note that the groups in all three of these theorems above satisfy the conditions of K. Gruenberg's portentous observation [10] that we record here as

**Lemma 2.8.** *Suppose  $P$  is any group,  $K \triangleleft P$  with  $P/K$  solvable and  $K$  residually solvable. Then  $P$  is residually solvable.*

## 3. THE SINGLE RELATOR IS A COMMUTATOR

We first recall a result from [4] for a particular class of non-positive one-relator groups. Let  $G$  be a group that can be presented in the form,

$$G = \langle t, a, \dots, c; uw^{-1} = 1 \rangle, \quad (1)$$

where  $u$  and  $w$  are positive words in the given generators and each generator occurs with exponent sum zero in  $uw^{-1}$ . Then  $G$  is residually solvable. In fact,  $G$  is free-by-cyclic.

Now consider the group

$$H = \langle t, a, \dots, c; [u, w] = 1 \rangle. \quad (2)$$

If  $u$  and  $w$  are positive,  $H$  can be recognized as one of the groups in the preceding class (1). Hence  $H$  is free-by-cyclic and therefore residually solvable. However, known examples show that residual solvability for  $H$  is not guaranteed once the requirement that both  $u$  or  $w$  be positive is relaxed:

**Example 3.1.** [4] If  $G = \langle a, b, \dots, c; [u, v] = 1 \rangle$ , where

$$u = a, \quad v = [a, b][w, w^b], \quad \text{and} \quad w = [a, b]^{-1}[a, b]^a,$$

then  $G$  is not residually solvable.

*Proof.* It follows from Magnus' solution of the word problem that  $w \neq 1$  in  $G$  [14]. Since  $[u, v] = 1$ , we find that

$$[a, b]^a [w, w^b]^a = [a, b][w, w^b],$$

so that

$$w = [a, b]^{-1}[a, b]^a = [w, w^b]([w, w^b]^a)^{-1}.$$

Thus  $w$  lies in every term of the derived series of  $G$ .  $\square$

In contrast, the next example is a residually solvable one-relator group.

**Example 3.2.** The group  $G = \langle a, b; [a, [a, b]] \rangle$  is free-by-cyclic.

*Proof.* We expand and re-express the relator,

$$r = [a, [a, b]] = a^{-1}[a, b]^{-1}a[a, b] = a^{-1}b^{-1}a^{-1}bab^{-1}ab. \quad (3)$$

Observe that in

$$r_0 = b_1^{-1}b_2b_1^{-1}b_0$$

$\mu = 2$ ,  $\nu = 0$ ,  $b_0$  and  $b_2$  both occur only once, so we can invoke Lemma 2.3. Therefore  $G$ , as a cyclic extension of the free group  $N = gp_G(b)$ , is residually solvable (cf. 2.8).  $\square$

#### 4. CONNECTION BETWEEN GENERALIZED FREE PRODUCTS AND ONE-RELATOR GROUPS

Over the years since W. Magnus developed his treatment of one-relator groups the increased interest in them yielded many new results. Karrass and Solitar in 1971 showed that a subgroup of a one-relator group is either solvable or contains a free subgroup of rank two. G. Baumslag and Shalen showed that every one-relator group with at least four generators can be decomposed into a generalized free product of two groups where the amalgamated subgroup is proper in one factor and of infinite index in the other. Fine, Howie and Rosenberger [7], and Culler and Morgan [9] showed that any one-relator group with torsion and at least three generators can be decomposed, in a non-trivial way, as an amalgamated free product. These results made it seem reasonable to expect that a closer look at the residual solvability of generalized free products of two groups could provide further tools for detection of residual solvability of one-relator groups. The following result confirms that assumption.

**Theorem 4.1.** *The group  $G = \langle a, b; [a, w] \rangle$ , where  $w = [a, b]^n$  ( $n \in \mathbb{N}$ ), is residually solvable.*

*Proof.* Put  $N = gp_G(b)$ , the normal closure of  $b$  in  $G$ . Using the Magnus break-down, we consider:

$$N_0 = \langle b_0, b_1, b_2; (b_1 b_0)^n = (b_2^{-1} b_1)^n \rangle. \quad (4)$$

Now let

$$x_0 = b_1^{-1} b_0, \quad x_1 = b_2^{-1} b_1, \quad y = b_1.$$

Tietze transformations confirm that

$$N_0 = \langle x_0, x_1, y; (x_0)^n = (x_1)^n \rangle = \langle x_0, x_1; (x_0)^n = (x_1)^n \rangle * \langle y \rangle. \quad (5)$$

Next let  $K = \langle x_0, x_1; (x_0)^n = (x_1)^n \rangle$ . Clearly

$$K = \{ \langle x_0 \rangle * \langle x_1 \rangle; \langle x_0^n \rangle = \langle x_1^n \rangle \} \quad (6)$$

Since each factor of  $K$  is abelian, by Theorem 2.5  $K$  is residually solvable. The free factor of  $N_0$ ,  $\langle y \rangle$  is also residually solvable. Therefore  $N_0$  is residually solvable, and it follows for every  $i \in \mathbb{N}$  that  $N_i$  is residually solvable. If we put  $N_{i,j} = gp(N_i, N_{i+1}, \dots, N_j)$ , the preceding approach gives

$$N_{i,j} = \langle x_i \rangle *_{\langle x_i^n \rangle = \langle x_{i+1}^n \rangle} \langle x_{i+1} \rangle * \dots * \langle x_j \rangle *_{\langle x_j^n \rangle = \langle x_{j+1}^n \rangle} \langle x_{j+1} \rangle * \langle y \rangle.$$

Therefore, by Theorem 2.6,  $N_{i,j}$  is residually solvable. A task that remains for completing the proof is to show that the ascending union  $N = \cup_{r<0;s>0} N_{r,s}$  is residually solvable, which will be taken care of by the following Proposition 4.2. Granted that, the residual solvability of  $G$  follows with the use of Corollary 2.8.  $\square$

**Proposition 4.2.**  $N = \cup_{r<0;s>0} N_{r,s}$  is residually solvable.

*Proof.* We will retain notation from the proof of Theorem 4.1 and start with the assumption that  $N_{i,j}$  is residually solvable for all  $i, j \in \mathbb{N}$  ( $i \leq j$ ). For the derived series of  $N = \cup_{r<0;s>0} N_{r,s}$ , we have

$$\delta_i N = \delta_i(\cup_{r<0;s>0} N_{r,s}).$$

Every element  $g \in \delta_i N$  is a finite product of commutators of elements from a (finite) subset of the  $N_{r,s}$  groups. So  $g \in \delta_i N_{r,s}$  for suitably small value of  $r < 0$  and suitably large value of  $s > 0$ . Thus  $\delta_i N = \cup_{r<0;s>0} \delta_i N_{r,s}$ . Now, if  $j, k$  are a pair of fixed integers the infinite union above can be rewritten as

$$\delta_i N = \cup_{r<0;s>0} \delta_i N_{j+r,k+s},$$

so that,

$$\delta_i N \cap N_{j,k} = (\cup_{r<0;s>0} \delta_i N_{j+r,k+s}) \cap N_{j,k}.$$

Equivalently,

$$\delta_i N \cap N_{j,k} = \cup_{r<0;s>0} (\delta_i N_{j+r,k+s} \cap N_{j,k}).$$

Further, each term in the union can be written as

$$\delta_i N_{j+r,k+s} \cap N_{j,k} = (\delta_i N_{j,k+s} \cap N_{j,k}) \cap (\delta_i N_{j+r,k} \cap N_{j,k}).$$

And because  $s < 0$  and  $r > 0$ , an argument fashioned after that in [3, p. 175, Lemma 4.3] yields after suitable conjugations that

$$\delta_i N_{j,k+s} \cap N_{j,k} = \delta_i N_{j,k} \text{ and}$$

$$\delta_i N_{j+r,k} \cap N_{j,k} = \delta_i N_{j,k}.$$

So each term in the union can be re-expressed as

$$\delta_i N_{j+r,k+s} \cap N_{j,k} = \delta_i N_{j,k}.$$

Notice that this expression is independent of  $r$  and  $s$ . Thus, we get

$$\delta_i N \cap N_{j,k} = \delta_i N_{j,k} \text{ ([3, pg. 175, line 16]).}$$

We claim that

$$\delta_i N \cap N_{j,k} = \delta_i N_{j,k}$$

implies that  $N$  is residually solvable. To see this, let  $g$  be a non-trivial element of

$$N = \cup_{r < 0; s > 0} N_{r,s}.$$

Then there is an integer  $j = j(g) \in \mathbb{N}$  such that  $g \in N_{-j,j}$ . By our (inductive) hypothesis at the outset  $N_{-j,j}$  is residually solvable. Consequently, there exists an integer  $i \in \mathbb{N}$  such that  $g \notin \delta_i N_{-j,j}$ . Then, since  $\delta_i N \cup N_{-j,j} = \delta_i N_{-j,j}$ , we see that  $g \notin \delta_i N \cap N_{-j,j}$ . But  $g \in N_{-j,j}$ . So it must be the case that  $g \notin \delta_i N$ . Thus we have found a normal subgroup  $\delta_i N \triangleleft N$  with the property that  $g \notin \delta_i N$  and  $N/\delta_i N$  is solvable. Hence  $N$  is residually solvable.  $\square$

## 5. THE RELATOR IS A BASIC COMMUTATOR

The tools of the Magnus theory were of good use for proving residual solvability through gaining information about the structure of the two-generator one-relator groups where the relator is a particular type of basic commutator.

We begin with recalling P. Hall's [11] definition of the basic commutators (in terms of the free group  $F$  on  $\{x_1, \dots, x_q\}$ ) and their linear ordering (in terms of their *weights*).

**Definition 5.1.** Basic Commutators.

- (1) The basic commutators of weight one with their linear order are  $x_1 < x_2 < \dots < x_q$ ; for their weights we write  $wt(x_i) = 1$ .
- (2) Having defined the basic commutators of weight less than  $n$ , the basic commutators with weight  $n$  are of the form  $c_n = [c_i, c_j]$  where  $c_i$  and  $c_j$  are all the basic commutators satisfying  $wt(c_i) + wt(c_j) = n$ ,  $c_i > c_j$ , and such that if  $c_i = [c_s, c_t]$ , then  $c_j \geq c_t$ .

In the following we will use the notation  $s_1 = x$  and  $s_{k+1} = [s_k, y]$  for positive integers  $k$ .

**Theorem 5.2.** *The group  $G = \langle x, y; r = [s_k, y] \rangle$  is free-by-cyclic, therefore residually solvable.*

*Proof.* Following the Magnus theory we put  $x_i = y^{-i} x y^i$  for this two generator case. Using induction on the weight of the commutator and the relationship

$$[s_k, y] = s_k^{-1} (s_k)^y \quad (k > 0),$$

we see that the minimum index and maximum index in  $r_0$  are 0 and  $k$ , respectively, and both of  $x_0$  and  $x_k$  occur only once in  $r_0$ . By Lemma 2.3 it follows, similarly to previous cases, that  $G$  is free-by-cyclic.  $\square$

## 6. OPEN PROBLEMS

- (1) Is it algorithmically decidable whether a one-relator group is residually solvable?
- (2) Are one-relator groups generically residually solvable? In other words, are they in most cases residually solvable? I. Kapovich conjectured [5] that many one-relator groups are finitely generated free-by-cyclic.
- (3) Do there exist residually finite one-relator groups that are not residually solvable? (This is recasting a question in [1] in this context.)
- (4) As defining relators, certain basic commutators were shown in this paper to render the respective one-relator groups residually solvable. Would all basic commutators have that property? If not, can the techniques used here be extended to one-relator groups with further types of basic commutator for their defining relator?
- (5) Find further examples of non-positive one relator groups that fail to be residually solvable.
- (6) Find examples of residually solvable one-relator groups that are not free-by-cyclic.

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