CONTENTS

EDITORIAL

This issue of the Bulletin *is special in some ways. It illustrates very well what I discussed in the editorial of Volume 63. Two more interviews with 'retirees', Professors David Lewis and Sean Dineen have been conducted by Gary McGuire and are contained here. Tony O'Farrell's survey article also pays tribute to the contributions and influence of two colleagues who recently retired from NUI Maynooth, David Walsh and Richard Watson. I personally find it most interesting to read about the development of (University) mathematics in Ireland during the years gone by. Contributions to the* Bulletin *of this kind are most welcome.*

These reminiscences are counterpointed by almost 20 pages of Abstracts of PhD Theses (and I am convinced even this does not reflect fully all the PhD theses which were completed in 2009). This development, too, is most gladly received: to get a succinct idea of what type of research the 'youngsters' are pursuing in the various locations in Ireland.

Bernd Kreussler's report on the 2009 Mathematical Olympiad completes the picture and tells us many of the details on this activity which is so enthusiastically supported and driven forward at many of the Irish Mathematics Departments—to the better of the future of Mathematics on the island.

—MM

NOTICES FROM THE SOCIETY

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Applying for I.M.S. Membership

1. The Irish Mathematical Society has reciprocity agreements with the American Mathematical Society, the Irish Mathematics Teachers Association, the New Zealand Mathematical Society and the Real Sociedad Matemática Española.

2. The current subscription fees (as from 1 January 2009) are given below:

Institutional member	160 euro
Ordinary member	25 euro
Student member	12.50 euro
I.M.T.A., NZMS or RSME reciprocity member	12.50 euro
AMS reciprocity member	15 US$

   The subscription fees listed above should be paid in euro by means of a cheque drawn on a bank in the Irish Republic, a Eurocheque, or an international money-order.

3. The subscription fee for ordinary membership can also be paid in a currency other than euro using a cheque drawn on a foreign bank according to the following schedule:

   If paid in United States currency then the subscription fee is US$ 30.00.
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   The amounts given in the table above have been set for the current year to allow for bank charges and possible changes in exchange rates.

4. Any member with a bank account in the Irish Republic may pay his or her subscription by a bank standing order using the form supplied by the Society.

5. Any ordinary member who has reached the age of 65 years and has been a fully paid up member for the previous five years may pay at the student membership rate of subscription.

6. Subscriptions normally fall due on 1 February each year.

7. Cheques should be made payable to the Irish Mathematical Society. If a Eurocheque is used then the card number should be written on the back of the cheque.

8. Any application for membership must be presented to the Committee of the I.M.S. before it can be accepted. This Committee meets twice each year.

9. Please send the completed application form with one year's subscription to:

   The Treasurer, I.M.S.
   Department of Mathematics
   St Patrick's College
   Drumcondra
   Dublin 9, Ireland

IRISH MATHEMATICAL SOCIETY

President's Report 2009

I would like to express the Society's gratitude to my predecessor, Dr. Russell Higgs, for the outstanding work that he did during his term of office. He represented the Society very ably and left it in a very healthy state.

The Society has been involved in several important events over the last year. Most prominently perhaps, in April at NUI Galway, we hosted our annual conference in conjunction with the annual meeting of the British Mathematical Colloquium. This event was a great success, and I would like to pay tribute to the three main local organisers, Professor Ted Hurley, Dr James Ward and Dr Claas Röver. Of course many others helped out in making this event happen and I think it is safe to say that it projected a very positive image of the Society to our colleagues in the UK and elsewhere.

The Fergus Gaines trophy is presented by the Society annually to the top performer in the Irish Mathematical Olympiad. This year the award will go to Jack McKenna of Newbridge College, Kildare. Society members, led by Dr Bernd Kreussler of Mary Immaculate College, also oversaw the participation of Ireland in the International Mathematical Olympiad which took place in Bremen, Germany during the month of July. The IMS would like to pay tribute to all its members who help to promote mathematics among our second level students.

In the world of research mathematics, the Society continues to offer sponsorship to members that are organising mathematical conferences in Ireland. This past year we supported the following events.

- International Workshop on Multi-Rate Processes and Hysteresis, UCC, 31 March–5 April 2008
- Complex Function Theory Meeting (in honour of Brian Twomey), UCC, 18 April 2008
- Operator Theory and Operator Algebras (in memory of Professor Gerard Murphy), UCC, 7–9 May 2008
- Third Conference on Mathematics Service Teaching, WIT, 26–27 May 2008
- 23rd British Topology Meeting, QUB, 25–27 August 2008
- Instructional Workshop on Subfactors and Planar Algebras, QUB, 26–28 August 2008.

Thanks to Dr Richard Timoney for agreeing to represent the Society on the newly formed RIA committee for the mathematical sciences. Dr Martin Mathieu has indicated that he will step down as editor of the Bulletin at the end of 2010 — we are currently seeking out a suitable candidate to replace him. I would like to thank Martin for his tremendous contribution to the Society and to the Irish mathematical community as editor over the last 9 years.

As of next year, the Society will return to the normal practice of holding its AGM at the September conference. Next year's September conference will take place in DIT. We have also agreed that the 2011 September meeting will take place in UL and the 2012 September meeting will take place in IT Tallaght.

James Cruickshank
President of the Irish Mathematical Society
11th September 2009

Minutes of the Meeting
of the Irish Mathematical Society
Annual General Meeting
11th September 2009

The Irish Mathematical Society held its Annual General Meeting from 15:30 to 16:40 on Friday 11 September at Trinity College Dublin. There were 18 members present at the meeting.

1. Minutes

The minutes of the last AGM were approved and signed.

2. Matters arising

R. Higgs confirmed that the Service Teaching Report had been sent to heads of third level institutions; however the committee has not received any responses.

3. Correspondence

Apart from routine correspondence, there were two items.

- A call for nominations for this year's Abel Prize. While the deadline for nominations was imminent, the Society may consider making nominations in future years.
- A request from Swets for an invoice for the Bulletin. However, the Society has no record of having provided Swets with the Bulletin, so they owe us nothing. The Treasurer has responded to Swets to this effect.

4. New Members

Dr Thomas Waters, NUIG; Mr Tom Barry, Canada Life; Derek Kitson, TCD. There was a discussion on how the Society's membership base could be expanded. Suggestions included mathematics graduates, lecturers in engineering, and members of the Irish Mathematics Teachers Association. It was suggested that we write to the IMTA to ask them if they would like to be added to our email circulation list.

5. President's Report

The President presented an interim report for 2009. He outlined some of the main activities of the Society over the year, including the conferences supported by the Society. He paid tribute to his predecessor R. Higgs for the healthy state in which he left the Society.

6. Treasurer's Report

The Treasurer presented her interim report for 2009. It showed a shortfall of €595.27. However, the Society is generally breaking even.

Five conferences received funding from the Society this year. It was noted that there are significantly more applications for funding in recent years, with the result that some late applications have had to be refused.

7. The Bulletin

The Editor (who sent his apologies) sent a report on the Bulletin.

Volume 63 of the IMS Bulletin, the summer 2009 issue was available online. It was noted once again that there are too few good research notes and very few surveys being submitted. The record of the IMS Meeting/BMC2009 in Galway in the Bulletin was minimal, as the Editor received an insufficient response from the organisers of the meeting to write about the meeting.

The President noted the Editor's intention to stand down at the end of 2010, and thanked him for his tireless work over the years.

8. Election to Committee

S. Breen was elected unopposed to serve as Treasurer for another two-year term, and S. O Rourke was elected unopposed to serve as Secretary for another two-year term. Moreover, S. Buckley, B. Guilfoyle, N. Kopteva and A. Wickstead agreed to serve another two-year term as Committee members.

The following were also elected unopposed to the committee:

Committee Member	Proposer	Seconder
R. Quinlan	M. Mackey	K. Hutchinson
C. Hills	M. Mackey	R. Higgs

The committee may later co-opt one more member to serve on the committee.

As editor, M. Mathieu will be invited to committee meetings. The total number of years each existing member will have been on the committee as of 31 December 2009 will be: J. Cruickshank (7), T. Carroll (6), N. O'Sullivan (6), R. Timoney (5), N. Kopteva (4), B. Guilfoyle (4), S. Breen (3), S. O Rourke (3), S. Buckley (2), C. Hills (2), A. Wickstead (2), S. Wills (2).

The following will then have one more year of office: J. Cruickshank, R. Timoney.

9. Future Meetings

The 2010 September Meeting will take place in Dublin Institute of Technology, the 2011 Meeting will take place in the University of Limerick, and the 2012 Meeting will take place in the Institute of Technology, Tallaght. The next three AGMs of the Society will take place at these meetings.

10. Any other business

- The Committee had considered whether to host a committee meeting of the European Mathematical Society in the future. However, even limited financial support for such a meeting would consume a large proportion of the IMS's annual budget. It was decided that the limited funds of the Society would be better used supporting mathematical conferences.
- Daphne Gilbert mentioned an anomaly whereby much of SFI funding awarded to non-EU students is absorbed by the higher fees they have to pay. It was suggested that this situation should be made known to candidates prior to their application. It was pointed out, however, that the inequity lies with the institution, rather than with the SFI itself. Several people felt that third-level institutions should be pressured to change these rules for SFI applicants.

  It was suggested that the Society write to the SFI asking them to make this anomaly clear in their guidelines.

Shane O Rourke
CIT.

These minutes still need to be approved by the next AGM.

ANNOUNCEMENTS OF MEETINGS AND CONFERENCES

This section contains the announcement of the annual meeting of the IMS and closely related conferences (satellites) as supplied by organisers. The Editor does not take any responsibility for the accuracy of the information provided.

23rd Annual Meeting of the IMS

Dublin Institute of Technology
September 2–3, 2010

The next IMS September meeting will be held at Dublin Institute of Technology on Thursday and Friday, the 2nd and 3rd of September 2010. It is organised by Chris Hills (chris.hills@dit.ie) and Dana Mackey (dana.mackey@dit.ie).

Further information regarding the programme, registration, accommodation and travel will be available at

www.maths.dit.ie/ims/meeting2010/

Abstracts of PhD Theses at Irish Universities 2009

Infinite Cycles in Boson Lattice Models

Gerard G. Boland
gerry.boland@ucd.ie

This is an abstract of the PhD thesis *Infinite Cycles in Boson Lattice Models* written by Gerard G. Boland under the supervision of Joseph V. Pulé at the School of Mathematical Sciences, College of Engineering, Mathematical and Physical Sciences, University College Dublin and submitted in September 2009.

We study the relationship between long cycles and Bose–Einstein condensation (BEC) in the case of several models. A convenient expression for the density of particles on cycles of length q is obtained, in terms of q unsymmetrised particles coupled with a boson field.

Using this formulation we reproduce known results on the Ideal Bose Gas, Mean-Field and Perturbed Mean-Field Models, where the condensate density exactly equals the long cycle density. Then we consider the Infinite-Range-Hopping Bose–Hubbard Model:

$$H_V^{\mathrm{BH}} = \frac{1}{2V}\sum_{x,y=1}^{V}(a_x^* - a_y^*)(a_x - a_y) + \lambda\sum_{x=1}^{V} n_x(n_x - 1)$$

in two cases, first for $\lambda = +\infty$, otherwise known as the hard-core boson model; and secondly for λ finite, representing a finite on-site repulsion interaction.

For the hard-core case, we find we may disregard the hopping contribution of the q unsymmetrised particles, allowing us to calculate an exact expression for the density of particles on long cycles. It is shown that only the cycle of length one contributes to the cycle density. We conclude that while the existence of a non-zero long cycle density coincides with the occurrence of Bose–Einstein condensation, the respective densities are not necessarily equal.

For the case of a finite on-site repulsion, we obtain an expression for the cycle density involving the partition function for a Bose–Hubbard Hamiltonian with a single-site correction again by neglecting the q unsymmetrised hop. Inspired by the Approximating Hamiltonian method we conjecture a simplified expression for the short cycle density as a ratio of single-site partition functions. In the absence of condensation we prove that this simplification is exact and use it to show that in this case the long-cycle density vanishes. In the presence of condensation we can justify this simplification when a gauge-symmetry breaking term is introduced in the Hamiltonian. Assuming our conjecture is correct, we compare numerically the long-cycle density with the condensate and again find that though they coexist, in general they are not equal.

On Abelian Ideals in a Borel Subalgebra of a Complex Simple Lie Algebra

Patrick Browne
p.browne8@nuigalway.ie

This is an abstract of the PhD thesis *On Abelian Ideals in a Borel Subalgebra of a Complex Simple Lie Algebra* written by Patrick Browne under the supervision of Dr. John Burns at the Department of Mathematics, National University of Ireland, Galway and submitted in December 2008.

Let $\mathfrak{g}$ be a complex simple Lie algebra and $\mathfrak{b}$ a fixed Borel subalgebra of $\mathfrak{g}$. We construct all maximal (with respect to containment) abelian ideals of $\mathfrak{b}$ by a variety of methods each having their own merits. We also derive formulas for their dimensions. Then we give a new proof of Kostant's theorem on the dimension of an abelian ideal. Finally we apply our results to give new examples of Einstein solvmanifolds.

We now give a brief historical account of interest in this area. In 1945, A. Malcev [1] determined the commutative subgroups of maximum dimension in the semisimple complex Lie groups. The maximal dimension of these commutative subgroups coincides with the maximal dimension of a commutative subalgebra of $\mathfrak{g}$. The next development was Kostant's [2] paper published in 1965, where he gave a connection between Malchev's result and the maximal eigenvalue of the Laplacian acting on the exterior powers ($\bigwedge^k \mathfrak{g}$) of the adjoint representation. In 1998 Kostant reported on the results of Peterson that the number of abelian ideals in the fixed Borel subalgebra of $\mathfrak{g}$ is $2^{\mathrm{rank}(\mathfrak{g})}$, and this paper was the genesis of much of the recent interest in this area. In [3], Panyushev and Röhrle while studying the relationship between spherical nilpotent orbits and abelian ideals of $\mathfrak{b}$, constructed all maximal abelian ideals, with the aid of a computer program [4], and observed a bijection between them and the set of long simple roots. Our method does not require the use of computer calculations. Suter in [5] found the maximal dimension of a maximal abelian ideal using the affine Weyl group, in terms of certain Lie theoretic invariants and gave a uniform explanation of the one to one correspondence between the long simple roots and the maximal abelian ideals. In [6] Papi and Cellini gave formulas for the dimension of all maximal abelian ideals in $\mathfrak{b}$, similar to that of Suter. The new formulas in this thesis are simpler and different in nature.

The methods used in the thesis rely heavily on the theory of graded Lie algebras. We also give an alternative proof of Kostant's theroem that does not require the representation theory used by Kostant. Finally our results are used to construct new examples of non compact Einstein solvmanifolds.

References

1. Mal'cev, A. I., *Commutative subalgebras of semisimple Lie algebras*, Izestiya akademii Nauk SSSR. Seriya Matematiceskaya, vol. 9 (1945), 291–300.
2. Kostant, B., *Eigenvalues of a Laplacian and commutative Lie subalgebras*, Topology, vol. 3 (1965), 147–159.
3. Panyushev, D. and Röhrle, G. , *Spherical orbits and abelian ideals*, Adv. Math., vol. 159 (2001), 229–246.
4. Röhrle, G., *On normal abelian subgroups in parabolic groups*, Annales de l'institut Fourier, vol. 5 (1998), 1455–1482.
5. Suter, R., *Abelian ideals in a Borel subalgebra of a complex simple Lie algebra*, Inventiones Mathematicae, vol. 156 (2004), 175–221.
6. Cellini, P. and Papi, P., *Abelian ideals of Borel subalgebras and affine Weyl groups*, Adv. Math., vol. 187 (2004), 320–361.

Explicit Small Classifying Spaces for a Range of Finitely Presented Infinite Groups

MAURA CLANCY
m.clancy@nuigalway.ie

This is an abstract of the PhD thesis *Explicit small classifying spaces for a range of finitely presented infinite groups* written by Maura Clancy under the supervision of Dr Graham Ellis at the School of Mathematics, Statistics and Applied Mathematics, National University of Ireland, Galway and submitted in February 2009.

While a classifying space B_G exists for any group G, in reality given a group presentation, finding a productive B_G is by no means trivial. This thesis unearths explicit small classifying spaces for a range of finitely presented infinite groups and uses these spaces to deduce homological information on the groups.

In Chapter 2 we derive formulae for the second integral homology of any Artin group for which the $K(\pi, 1)$-conjecture is known to hold, and for the third integral homology of the braid group A_n and the affine braid group $\tilde{A}_n$. The derivation and proofs are based on the cellular chain complex $C_*(\tilde{X}_D)$, where $\tilde{X}_D$ is the universal cover of a classifying space B_G for the group $G \in \{A_n, \tilde{A}_n\}$. Chapter 3 defines *polytopal* groups, actions and classifying spaces. We prove that a group G is polytopal when G is the semi-direct product of two polytopal groups N and Q. We show that $\tilde{B}_N \times \tilde{B}_Q$ is the universal covering space of a polytopal classifying space $\tilde{B}_G$ for G, where $\tilde{B}_N$ (resp. $\tilde{B}_Q$) is the universal covering space of a polytopal classifying space B_N for N (resp. B_Q for Q). We further show that the cellular chain complex $C_*(\tilde{B}_G)$ can be obtained as the total complex of a double complex with $Dim(B_N)$ rows and $Dim(B_Q)$ columns, a fact alternatively proven by Thomas Brady in his paper "Free resolutions for semi-direct products". Chapter 4 centres on Bieberbach groups; we realise six of the ten 3-dimensional Bieberbach groups as semi-direct products $G = N \rtimes_\alpha Q$, where N is 2-dimensional Bieberbach and $Q = C_\infty$. This technique can be extended to determine, inductively, classifying spaces for higher dimensional Bieberbach groups. Chapter 5 introduces *twisted Artin groups* $\mathfrak{A}_{\vec{D}}$ and shows that in some cases there exists a polytopal classifying space whose t-dimensional cells are indexed by the finite type subsets, of size t, of the generating set S. We show that 3-generator twisted Artin groups of large type admit a two-dimensional classifying space. Using star graph techniques we show that such classifying spaces are non-positively curved for standard Artin groups of large type. In Chapter 6 we conjecture that certain groups are quasi-lattice-ordered and then use a GAP routine to experimentally investigate the word-reversing algorithm.

The Geometry of the Space of Oriented Geodesics of Hyperbolic 3-Space

Nikos Georgiou
nikos.georgiou@research.ittralee.ie

This is an abstract of the PhD thesis *The Geometry of the Space of Oriented Geodesics of Hyperbolic 3-Space* written by Nikos Georgiou under the supervision of Dr. Brendan Guilfoyle at the Institute of Techonolgy Tralee and submitted in June 2009.

In this thesis we construct a Kähler structure $(\mathbb{J}, \Omega, \mathbb{G})$ on the space $\mathbb{L}(\mathbb{H}^3)$ of oriented geodesics of hyperbolic 3-space $\mathbb{H}^3$ and investigate its properties. We prove that $(\mathbb{L}(\mathbb{H}^3), \mathbb{J})$ is biholomorphic to $\mathbb{P}^1 \times \mathbb{P}^1 - \overline{\Delta}$, where $\overline{\Delta}$ is the reflected diagonal, and that the Kähler metric $\mathbb{G}$ is of neutral signature, conformally flat and scalar flat.

We establish that the identity component of the isometry group of the metric $\mathbb{G}$ on $\mathbb{L}(\mathbb{H}^3)$ is isomorphic to the identity component of the hyperbolic isometry group. We show that the geodesics of $\mathbb{G}$ correspond to ruled minimal surfaces in $\mathbb{H}^3$, which are totally geodesic iff the geodesics are null.

We then study 2-dimensional submanifolds of the space $\mathbb{L}(\mathbb{H}^3)$ of oriented geodesics of hyperbolic 3-space, endowed with the canonical neutral Kähler structure. Such a surface is Lagrangian iff there exists a surface in $\mathbb{H}^3$ orthogonal to the geodesics of Σ.

We prove that the induced metric on a Lagrangian surface in $\mathbb{L}(\mathbb{H}^3)$ has zero Gauss curvature iff the orthogonal surfaces in $\mathbb{H}^3$ are Weingarten: the eigenvalues of the second fundamental form are functionally related. We then classify the totally null surfaces in $\mathbb{L}(\mathbb{H}^3)$ and recover the well-known holomorphic constructions of flat and CMC 1 surfaces in $\mathbb{H}^3$.

Topics in Computer Assisted Finite Group Theory

Robert Heffernan
bob.heffernan@gmail.com

This is an abstract of the PhD thesis *Topics in Computer Assisted Finite Group Theory* written by Robert Heffernan under the supervision of Prof. Des MacHale at the School of Mathematical Sciences, University College Cork and submitted in April 2009.

The phrase 'computer assisted' in the title of this thesis refers not to 'computational group theory' in the usual sense—that is, the developing of algorithms and data structures to compute information about groups—but to the *use* of the computer, particularly the GAP [4] algebra system, as a tool. The thesis is motivated by the opinion that the Small Groups Library [3] is one of the most interesting and useful resources in the arsenal of the finite group theorist. This library contains computer descriptions of all finite groups of order less than 2000 (excepting the groups of order 1024) as well as many groups of larger order. In the thesis we consider several different problems in finite group theory; in each case our first avenue of inquiry will have been to search the Small Groups Library.

In the first chapter we consider the sum of the degrees of the irreducible characters of a finite group G, which we denote by $T(G)$. This has been studied by Berkovich and Zhmud [2, Chapter 11] and by Berkovich and Mann [1]. When p^n divides the order of G, where p is a prime and $n \leq 6$, we produce bounds for $T(G)$. We then produce bounds for $T(G)$ when G has at most 14 conjugacy classes.

In the second chapter we consider $\Pr(G)$, the probability that two group elements commute, and the function $f(G) = T(G)/|G|$ which can also be seen as a indicator of the commutativity of G. These and other such indicators of commutativity have been studied by various authors (see, for instance, [6, 7, 5]). We prove several 'threshold' results that link the structure of the group G to the value of $\Pr(G)$ or $f(G)$. For example: if G is a finite non-abelian group such that $\Pr(G) > \frac{7}{16}$, then G' is cyclic. Similarly, if G is a finite non-abelian group such that $f(G) > \frac{5}{8}$, then G' is cyclic. Moreover, in both cases $|G|$ is even and these results are best possible. We also briefly investigate some other functions related to $\Pr(G)$ and $f(G)$.

In the third chapter our focus switches to automorphisms of finite groups. We classify the finite abelian groups A such that $|A| =$

$|\mathrm{Aut}(A)|$. We then classify the finite abelian groups A such that $|A|_2 = |\mathrm{Aut}(A)|_2$.

In the fourth chapter we consider holomorphs of finite groups. If G is a group then the holomorph of G, $\mathrm{Hol}(G)$, is the group $G \rtimes \mathrm{Aut}(G)$ where the action of $\mathrm{Aut}(G)$ on G is natural. We determine all p-groups P of order p^n, where $n \leq 11$, such that P occurs as the holomorph of some finite group. We then prove that if $H \cong \mathrm{Hol}(G)$, where G is a finite group and H has odd order, then H has order at least 3^{13}. This bound is best possible.

In the fifth chapter we define the notion of a *failsafe cover* of a finite group G. This is a group H such that G embeds in H in such a way that if we remove any non-identity element h from H we can still find an isomorphic copy of G in H that does not involve the element h. We give some examples of such failsafe covers and find some failsafe covers of minimal order for small-order groups.

In the final chapter we describe some partial progress made on two further problems.

References

[1] Y. Berkovich and A. Mann, *On sums of degrees of irreducible characters*, Journal of Algebra **199** (1998), 646–665.
[2] Y. Berkovich and E.M. Zhmud, *Characters of finite groups. Part 1*, Translations of Mathematical Monographs, vol. 172, American Mathematical Society, Providence, RI, 1998.
[3] Bettina Eick Hans Ulrich Besche and Eamonn O'Brien, *The small groups library.*
[4] The GAP Group, *GAP – Groups, Algorithms, and Programming, Version 4.4.10*, 2007.
[5] R.M. Guralnick and G.R. Robinson, *On the commuting probability in finite groups*, J. Algebra **300** (2006), no. 2, 509–528.
[6] W.H. Gustafson, *What is the probability that two group elements commute?*, Amer. Math. Monthly **80** (1973), 1031–1034.
[7] K.S. Joseph, *Research Problems: Several Conjectures on Commutativity in Algebraic Structures*, Amer. Math. Monthly **84** (1977), no. 7, 550–551.

Normalisers of Reflexive Operator Algebras

Martin McGarvey
mmcgarvey07@qub.ac.uk

This is an abstract of the PhD thesis *Normalisers of Reflexive Operator Algebras* written by Martin McGarvey under the supervision of Ivan Todorov at the Department of Pure Mathematics, Queen's University Belfast and submitted in September 2009.

If $\mathcal{A}$ is an operator algebra acting on a Hilbert space $\mathcal{H}$, a unitary operator U on $\mathcal{H}$ is called a unitary normaliser of $\mathcal{A}$ if

$$U\mathcal{A}U^* = \mathcal{A}.$$

Unitary normalisers of von Neumann algebras were first studied in the 1930s by Murray and von Neumann [3] where they used these operators to construct factors of type II and III.

Let $\mathcal{H}$ be a Hilbert space and $\mathcal{A}$ be a subalgebra of the algebra $\mathcal{B}(\mathcal{H})$ of all bounded linear operators on $\mathcal{H}$. A normaliser of $\mathcal{A}$ is an operator $T \in \mathcal{B}(\mathcal{H})$ such that $T^*\mathcal{A}T \subseteq \mathcal{A}$ and $T\mathcal{A}T^* \subseteq \mathcal{A}$. We let $N(\mathcal{A})$ denote the set of all normalisers of $\mathcal{A}$. It is obvious that for each operator algebra $\mathcal{A}$, we have $N(\mathcal{A}) = N(\mathcal{A}^*)$. Here $\mathcal{A}^* = \{A^* : A \in \mathcal{A}\}$ is the adjoint algebra of $\mathcal{A}$.

Question: Let $\mathcal{A}$ and $\mathcal{B}$ be reflexive operator algebras such that $N(\mathcal{A}) = N(\mathcal{B})$. Is it true that either $\mathcal{A} = \mathcal{B}$ or $\mathcal{A} = \mathcal{B}^*$?

If $\mathcal{A}$ and $\mathcal{B}$ are von Neumann algebras the answer to the above question is easily shown to be affirmative. The question becomes more interesting if we allow $\mathcal{A}$ and $\mathcal{B}$ to belong to various classes of non-selfadjoint operator algebras. The classes considered in [4], and in the present work, are subclasses of the family of CSL algebras. Two of the main results in [4] are that the question has an affirmative answer if $\mathcal{A}$ and $\mathcal{B}$ are either continuous nest algebras or totally atomic CSL algebras.

This thesis is dedicated almost entirely to the investigation of the above question. Given that it has an affirmative answer in the case $\mathcal{A}$ and $\mathcal{B}$ are continuous nest algebras, it seems natural to consider the question for larger classes of CSL algebras. In this thesis we are concerned with two such classes: the class of finite tensor products of continuous nest algebras and the class of nest subalgebras of von Neumann algebras. The latter were introduced by Gilfeather and Larson in [1] where they showed that any nest subalgebra of a von Neumann algebra containing a masa can be represented as a direct integral of nest algebras with respect to some direct integral decomposition of the underlying Hilbert space.

In this thesis we establish a result which relates the support of the tensor product of masa-bimodules to the Cartesian product of their supports. We introduce the concept of a normaliser orbit of an element of a reflexive operator algebra. We then show that the question has a positive answer whenever $\mathcal{A}$ is a finite length tensor product of continuous nest algebras and $\mathcal{B}$ is a CSL algebra.

We introduce doubly continuous nest subalgebras of von Neumann algebras and their invariant subspace lattices. We show that a doubly continuous multiplicity free CNvNSL is unitarily equivalent to the the tensor product of the Volterra nest with the projection lattice of a continuous masa. This allows us to prove that the question has a positive answer whenever $\mathcal{A}$ is a doubly continuous nest subalgebra of a von Neumann algebra and $\mathcal{B}$ is a CSL algebra.

We also investigate whether or not the question has a positive answer if only the sets of unitary normalisers of the CSL algebras are considered.

References

1. F. Gilfeather and D. R. Larson, *Nest subalgebras of von Neumann algebras*, Adv. Math. 46 (1982), 171-199.
2. M. McGarvey and I.G. Todorov, *Normalisers, nest algebras and tensor products*, Contemporary Mathematics, in press.
3. F.J. Murray and J. von Neumann, *On rings of operators*, Ann. of Math 37 (1936), 116-229.
4. I. G. Todorov, *Normalisers of CSL algebras*, Indiana Univ. Math. J. 53 (2004), no. 4, 1013-1036.

Dihedral Codes

Ian McLoughlin
mcloughlin.ian@gmail.com

This is an abstract of the PhD thesis *Dihedral Codes* written by Ian McLoughlin under the supervision of Professor Ted Hurley at the School of Mathematics, Statistics and Applied Mathematics, National University of Ireland, Galway and submitted in June 2009.

In this thesis we give new constructions of a number of extremal type II codes. Algebraic proofs are provided that the constructions do in fact yield the codes. The codes are constructed using group rings of which the underlying groups are dihedral. Type II codes are not cyclic, but the constructions here are similar to the constructions of cyclic codes from polynomials.

The first code we construct is the extended binary Golay code. It is constructed from a zero divisor in the group ring of the finite field with two elements and the dihedral group with twenty-four elements. We create a generator matrix of the code that is in standard form and is a reverse circulant generator matrix. The generator matrix generates the code as quasi cyclic of index two.

Algebraic proofs are given of the code's minimum distance, self-duality and doubly evenness. A list of twenty-three other zero divisors that we have found to generate the code is given. Trivial changes adapt the aforementioned algebraic proofs to any of these zero divisors. We also prove that the twenty-four zero divisors are the only ones of their form that will generate the code.

Next we construct the $(48, 24, 12)$ extremal type II code as a dihedral code. The new construction is similar to that of the extended Golay code. Again, proofs of the self-duality and doubly evenness are given. An algebraic proof of the minimum distance is achieved through the use of two different group ring matrices.

A number of different codes are then constructed, building on the first two constructions. We list some zero divisors that generate type II codes of lengths seventy-two and ninety-six. According to investigations by computer, these codes have minimum distances of twelve and sixteen respectively. No type II codes of each of these lengths are known that have greater respective minimum distances. Some techniques are detailed that vastly reduce the calculations involved in their analysis.

The constructions of the $(72, 36, 12)$ and $(96, 48, 16)$ codes are facilitated by the construction of extremal type II codes of all lengths a multiple of eight up to length forty. Type II codes only exist at lengths that are multiples of eight. Overall we showed that extremal type II codes can be constructed in dihedral group rings at every length a multiple of eight up to and including length forty-eight, and at some lengths beyond forty-eight. We also successfully investigate the possibility to construct some type I codes in the same way.

Structural and Universality Properties of Gelfand–Tsetlin Patterns and Their Generalizations

Anthony Paul Metcalfe
apmetcalfe@yahoo.ie

This is an abstract of the PhD thesis *Structural and universality properties of Gelfand–Tsetlin patterns and their generalizations* written by Anthony Paul Metcalfe under the supervision of Prof. Neil

O'Connell at the Mathematics Department of University College Cork and submitted in March 2009.

This thesis is primarily concerned with probability distributions on sets of interlaced particles. Such distributions arise naturally throughout the study of random matrix theory, most famously when considering the eigenvalues of the principal minors of random Hermitian matrices.

We construct a random walk on sets of interlaced particles on the discrete circle, using a dynamical rule inspired by the *Robinson–Schensted–Knuth*, or *RSK*, correspondence. Using this, we prove analogous results on the circle to some recently discovered results on sets of interlaced particles on the line.

We construct a measure on the set of interlaced particle configurations on the infinite cylinder. We prove a determinantal structure, and compute an associated space-time correlation kernel. We consider models which are equivalent to the above construction, namely tilings of the cylinder with rhombi, configurations of non-intersecting paths, and dimer configurations of the honeycomb lattice on the infinite cylinder.

Finally, we consider universality problems on sets of interlaced particles, that arise from free probability. We consider the asymptotic behavior of particles in the bulk of the configurations, as the size of the configurations increase. We consider configurations both on a line, and on a cylinder. Under certain assumptions, we show that the particles behave locally like a determinantal random point field, with correlation kernel given by the *sine* kernel.

Solving Polynomials with Integer Matrics

Raja Mukherji
rajamukherji@gmail.com

This is an abstract of the PhD thesis *Solving Polynomials with Integer Matrices* written by Raja Mukherji under the supervision of Professor Thomas J. Laffey at the School of Mathematical Sciences, University College Dublin and submitted in October 2009.

For an $n \times n$ complex matrix A, and a polynomial $f(x)$ with complex coefficients, methods to find $n \times n$ complex matrices X satisfying $f(X) = A$ have been discussed for some time. A quite complete (and lengthy) discussion can be found in Evard and Uhlig's paper [2]. A brief summary of the relevant sections are given in Chapter 2.

In Chapter 3, we look at the case where A is an $n \times n$ rational or integer matrix and $f(x)$ has rational or integer coefficients. We first look at the rational case, and reduce the problem to the case where the characteristic polynomial of A is a power of an irreducible polynomial. We present a new method for enumerating and finding rational matrices X satisfying $f(X) = A$. We then discuss an existing method first described by Drazin [1]. We present a new method for finding rational solutions in the case when the characteristic polynomial of A is irreducible and conditions such that these solutions have only integer entries. We give a bound on the denominator(s) occuring in rational solutions and discuss how the bound can be used to find exact rational solutions by numerical methods.

In Chapter 4, we look at the case where A is an $n \times n$ integer matrix and $f(x)$ has integer coefficients and show how the results of Chapter 3 can be applied to the integer case.

In Chapter 5, we look at the case where $f(x) = x^m$, and discuss solutions X satisfying $X^m = A$, where X has entries in an algebraic extension field of $\mathbb{Q}$. We present new methods to compute exact solutions in the cases where $m = 2$ and $n \leqslant 6$ and where $m = 3$ and $n \leqslant 4$. We discuss the minimum degree of such an extension field and show that for $m = 2$ and $n \leqslant 5$, there must exist such an extension field of degree a power of 2 over $\mathbb{Q}$. We show that in the case $m = 2$ and $n = 6$, there need not exist a extension field of degree a power of 2 over $\mathbb{Q}$ and present an example of such a matrix A.

The topic of matrix similarity arises in much of this discussion. Working over a field, in particular $\mathbb{C}$ or $\mathbb{Q}$, is straightforward, using well known results such as the Jordan and Frobenius (Rational) canonical forms. However, matrix similarity over a ring, even over $\mathbb{Z}$, is more complicated. For example, the Latimer–MacDuffee theorem [3, III.16], [4] relates similarity over $\mathbb{Z}$ to ideal classes in the ring of algebraic integers in a field extension. In chapter 6, we look at the idealizer of such ideals, and present new results about their relation to the centralizer of an integer matrix, and their role in similarity over $\mathbb{Z}$.

References

1. Michael P. Drazin. Exact rational solution of the matrix equation $a = p(x)$ by linearization. *Linear Algebra and its Applications*, 426:502–515, 2007.

2. Jean-Claude Evard and Frank Uhlig. On the matrix equation $f(X) = A$. *Linear Algebra Applications*, 162-164:447–519, 1992.
3. M Newman. Integral matrices, 1972.
4. Olga Taussky. On a theorem of Latimer and MacDuffee. *Can. J. Math.*, 1:300–302, 1949.

Mathematics in the life of Éamon de Valera

Cáit Ní Shúilleabháin
caitnishuilleabhain@gmail.com

This is an abstract of the PhD thesis *Mathematics in the life of Éamon de Valera* written by Cáit Ní Shúilleabháin under the supervision of Professor Des MacHale and Mr. Gabriel Doherty at the School of Mathematical Sciences and Department of History at University College Cork and submitted in November 2009.

The objective of the thesis was to examine, in detail, the influence of mathematics in the life of Éamon de Valera. Throughout this abstract I will give a brief insight into some of the ways his interest in mathematics impacted on his private and public life. At the turn of the century, de Valera entered University College Blackrock with the intention of pursuing a degree in mathematics and mathematical physics. In his final year, he received an offer, which he accepted, of teaching in Blackrock's sister College in Rockwell, Co. Tipperary. It was reported that de Valera was a very talented teacher, and therefore, he was entrusted with teaching both Senior Grade students and undergraduate degree students, though he had to finish his own degree! The pressures of a full teaching load left him little time to study and as a result, he had to be content with a pass degree. This made further study difficult for him; however, he benefited from public lectures given by UCD lecturers Arthur Conway and Henry MacWeeney, and lectures given by the Astronomer Royal, E.T. Whittaker. After meeting Conway de Valera's interest in quaternions intensified. Conway reported that "he [de Valera] has in the past two years [1910–1912] gone deeply into the subject of Quaternions, and is at present prosecuting an important original research in them which promises to be of considerable interest." During February 1917 de Valera wrote a letter to Conway on the subject of celestial mechanics. The letter also contained a reference to the type of mathematics de Valera had been working on with Conway, namely conformal representation. It is difficult to ascertain what type of problems he was working on, but it may have been that he was working on the application of conformal representation to quaternions. Arising from his 3rd level teaching experience, and his friendship with Conway and Whittaker, de Valera's confidence grew and he applied for the chair of Mathematical Physics in UCC in 1912. In the straw vote de Valera secured 11 votes, two higher than the nearest competitor, E. H. Harper. However, in the actual vote they both scored 10 and the two names were sent along with the results of the final vote. Harper was elected to the chair, and de Valera managed to secure a post teaching mathematical physics in the recognised College of the NUI in Maynooth. Due to de Valera's participation in the 1916 rising, his formal study of mathematics ceased. He did, however, maintain an interest in the subject throughout his life and in some scenario's actively used mathematics to solve political problems. One instance of this can be found when he set about devising external association, a concept which would satisfy both Irish and British aspirations following the 1921 War of Independence. Although not all accepted his proposal, certain republicans, though not enamoured with it, were eventually won over by de Valera's mathematical explanations, which were based on set theory. The most important contribution de Valera made to mathematics both in Ireland and internationally was the foundation of the Dublin Institute for Advanced Studies in 1940. Upon founding the Institute, de Valera hoped that Ireland would "achieve a reputation comparable to the reputation which Dublin and Ireland had in the middle of the last century", which, it appears was the case according to former director, JL Synge, who maintained that "former scholars are to be found in chairs in many parts of the world, and the school may feel reasonably proud that Ireland has, with profit to herself, made an international contribution to physics". The foundation of DIAS, among other acts, culminated in de Valera's election as an honorary Fellow of the Royal Society in 1968. Though de Valera may not have had the opportunity to study mathematics formally after his entry into politics, he often practiced it for fun. His attitude towards mathematics can be summed up by Rényi, who once said, "If I feel unhappy, I do mathematics to become happy. If I am happy, I do mathematics to keep happy."

The Hochschild Cohomology Ring of a Quadratic Monomial Algebra

David O'Keeffe
david.okeeffe@nuigalway.ie

This is an abstract of the PhD thesis *The Hochschild Cohomology Ring of a Quadratic Monomial Algebra* written by David O'Keeffe under the supervision of Dr. Emil Sköldberg at the Department of Mathematics, National University of Ireland, Galway and submitted in January 2009.

Until recently, little was known about the multiplicative structure of the Hochschild cohomology ring for most associative algebras. During the last decade or so, more light has been shed on this topic, where several papers have been published regarding the structure of the Hochschild cohomology rings for various algebras.

In this thesis, we compute the Hochschild cohomology groups for a special class of associative algebras, the so-called class of Quadratic monomial algebras. These results are used to compute the Hochschild cohomology ring for the above class of algebras. We extend results obtained by Claude Cibils, where he computes the cohomology ring for the class of radical square zero algebras. The latter class of algebras form a subclass of the class of quadraric monomial algebras.

Chapter 1 consists of all the necessary background material that is referred to throughout this work. There are also some new proofs of already known results. However most of the original work is contained in chapters 2, 3 and 4.

Chapter 2 describes the Hochschild cohomology groups of a quadratic monomial algebra in terms of the cohomology of a graph. An alternative calculation of these groups was performed by Emil Sköldberg. Also in this chapter we describe the generating elements of the cohomology algebra.

Chapter 3 builds on the calculations of the previous chapter by computing the algebra structure on the cohomology algebra. We describe the product structure on the cohomology algebra as a composition of chain maps on a projective resolution.

Finally using the algebra structure calculated in Chapter 3 and the Hilbert series of a vector space, we show in Chapter 4, the cohomology algebra of a quadratic monomial algebra exhibits all possible behaviours as an algebra: It may be

- finite dimensional over k,
- finitely generated over k,
- infinitely generated over k,

where we write k to denote a field of arbitrary characteristic.

References

1. Claude Cibils, *Hochschild cohomology algebra of radical square zero algebras*, Algebras and modules, II CMS Conf. Proc., vol.24, Amer. Math. Soc., Providence, RI, 1998,pp. 93-101.
2. Emil Sköldberg, *Some Remarks on the Hochschild cohomology of graded algebras*, (preprint), available from *http://larmor.nuigalway.ie/emil/*.

Stochastic Delay Difference and Differential Equations: Applications to Financial Markets

CATHERINE SWORDS
catherine.swords2@mail.dci.ie

This is an abstract of the PhD thesis *Stochastic Delay Difference and Differential Equations: Applications to Financial Markets* written by Catherine Swords under the supervision of Dr John Appleby at the School of Mathematical Sciences, Dublin City University and submitted in September 2009.

This thesis deals with the asymptotic behaviour of stochastic difference and functional differential equations of Itô type. Numerical methods which both minimise error and preserve asymptotic features of the underlying continuous equation are studied. The equations have a form which makes them suitable to model financial markets in which agents use past prices. The second chapter deals with the behaviour of moving average models of price formation. We show that the asset returns are positively and exponentially correlated, while the presence of feedback traders causes either excess volatility or a market bubble or crash. These results are robust to the presence of nonlinearities in the traders demand functions. In Chapters 3 and 4, we show that these phenomena persist if trading takes place continuously by modelling the returns using linear and nonlinear stochastic functional differential equations (SFDEs). In the fifth chapter, we assume that some traders base their demand on the difference between current returns and the maximum return over several trading periods, leading to an analysis of stochastic difference equations with maximum functionals. Once again it is shown that prices either fluctuate or undergo a bubble or crash. In common with the earlier chapters, the size of the largest fluctuations and the growth rate of the bubble or crash is determined. The last three chapters are devoted to the discretisation of the SFDE presented in Chapter 4. Chapter 6 highlights problems that standard numerical methods face in reproducing long-run features of the dynamics of the general continuous-time model, while showing these standard methods work in some cases. Chapter 7 develops an alternative method for discretising the solution of the continuous time equation, and shows that it preserves the desired long-run behaviour. Chapter 8 demonstrates that this alternative method converges to the solution of the continuous equation, given sufficient computational effort.

Homology Stability for the Special Linear Group of a Field and Milnor–Witt K-theory

Liqun Tao
lqtao99@hotmail.com

This is an abstract of the PhD thesis *Homology Stability for the Special Linear Group of a Field and Milnor–Witt K-theory* written by Liqun Tao under the supervision of Dr. Kevin Hutchinson at the School of Mathematical Sciences, UCD and submitted in February 2009.

The main results of the thesis provide answers, for fields of characteristic zero, to some old questions of A. A. Suslin and C.-H. Sah about homology stability for the special linear group and to a more recent question of J. Barge and F. Morel as to whether Milnor–Witt K-theory arises as the first obstruction to homological stability.

Let F be an infinite field and let $f_{t,n}$ be the stabilization homomorphism $\mathrm{H}_n(\mathrm{SL}_t(F),\mathbb{Z}) \to \mathrm{H}_n(\mathrm{SL}_{t+1}(F),\mathbb{Z})$. C.-H. Sah [1] conjectured that $f_{t,n}$ is an isomorphism for all $t \geq n+1$. For $n \geq 2$ even, Barge and Morel [2] defined a natural surjective map from the cokernel of $f_{n-1,n}$ to the n-th Milnor–Witt K-theory group $K_n^{\mathrm{MW}}(F)$, and asked whether this map is an isomorphism for all infinite fields.

In this thesis, the following results are proved: Let F be a field of characteristic 0. Then $f_{t,n}$ is an isomorphism for $t \geq n+1$ and is surjective for $t = n$. When n is odd, $f_{n,n}$ is an isomorphism. When

n is even, the kernel of $f_{n,n}$ is naturally isomorphic to $I^{n+1}(F)$, the $(n+1)$st power of the fundamental ideal of the Witt Ring of the field F. When $n \geq 2$ is even, the question of Barge and Morel has a positive answer. When $n \geq 3$ is odd, then the cokernel of $f_{n-1,n}$ is naturally isomorphic to the subgroup $2K_n^{\mathrm{M}}(F)$ in the (additive) Milnor K-theory group $K_n^{\mathrm{M}}(F)$.

The thesis also proves a simple presentation of the additive group $K_1^{\mathrm{MW}}(F)$, which is closely related to Hermitian K-theory.

References

1. Chih-Han Sah, Homology of classical Lie groups made discrete. III. *J. Pure Appl. Algebra*, 56(3):269-312, 1989.
2. Jean Barge and Fabien Morel. Cohomologie des groupes linéaires, K-theorie de Milnor et groupes de Witt. *C. R. Acad. Sci. Paris. Sér I Math.*, 328(3):191-196, 1999.

On the Pathwise Large Deviations of Stochastic Differential and Functional Differential Equations with Applications to Finance

HUIZHONG WU
huizhong.wu4@mail.dcu.ie

This is an abstract of the PhD thesis *On the pathwise large deviations of stochastic differential and functional differential equations with applications to finance* written by Huizhong Wu under the supervision of Dr John Appleby at the School of Mathematical Sciences, Dublin City University and submitted in September 2009.

The thesis deals with the asymptotic behaviour of highly nonlinear stochastic differential equations, as well as linear and nonlinear functional differential equations. Both ordinary functional and neutral equations are analysed. In the first chapter, a class of nonlinear stochastic differential equations which satisfy the Law of the Iterated Logarithm is studied, and the results applied to a financial market model. Mainly scalar equations are considered in the first chapter. The second chapter deals with a more general class of finite-dimensional nonlinear SDEs and SFDEs, employing comparison and time change methods, as well as martingale inequalities, to determine the almost sure rate of growth of the running maximum of functionals of the solution. The third chapter examines the exact almost sure rate of growth of the large deviations for affine stochastic functional differential equations, and for equations with additive noise which are subject to relatively weak nonlinearities at infinity. The fourth chapter extends conventional conditions for existence and uniqueness of neutral functional differential equations to the stochastic case. The final chapter deals with large fluctuations of stochastic neutral functional differential equations.

Irish Math. Soc. Bulletin **64** *(2009), 31–42*

A little Help from my Friends

ANTHONY G. O'FARRELL

Abstract. This article is based on a talk given at a one-day meeting in NUI, Maynooth on the Fourth of April, 2008, held to honour David Walsh and Richard Watson.

1. Introduction

This is a tribute to my dear colleagues and friends David Walsh and Richard Watson, who were here before me in Maynooth, and who laboured with me in the day and the heat. They cheerfully shouldered with me a teaching load that would, apparently, kill the academics of today. The teaching load required, in order to ensure that our students were adequately trained, continued to be a problem until the presidency of Míchael Ledwith, in the early 1990's. It was not easy to pursue research while giving 275 lectures a year, but they gave it their best. At distinct times, both helped me in my investigations. David was sound on complex analysis and hard analysis, so we joined forces to tackle some problems that required technical estimates for integral kernels that solve the $\bar{\partial}$-problem. Richard had a sound background in algebra, and he worked with me on problems that could be addressed using algebras of smooth functions. Richard became my Ph.D. student, after a while, and then, after graduating, continued to work with me for a few years. My period of active collaboration with David was in the early eighties, and with Richard the nineties. Recently, both have taken some interest in my reversibility project.

Most of the sources referred to in what follows will be found in the references cited in our joint papers, which are listed in the bibliography below.

2. A Way to think of Complex Analysis

Holomorphic functions are the solutions to the $\bar\partial$ equation

$$\bar\partial f = 0,$$

where

$$\bar\partial f = \frac{\partial f}{\partial \bar z}\, d\bar z, \qquad \frac{\partial f}{\partial \bar z} = \frac{1}{2}\left\{\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right\}.$$

We shall refer to both $\bar\partial$ and $\frac{\partial}{\partial \bar z}$ as "the $\bar\partial$ operator" (pronounced *d-bar operator*), as convenient.

The $\bar\partial$ operator is skew:

$$\int_{\mathbb{C}} \phi \frac{\partial \psi}{\partial \bar z}\, dx\, dy = -\int_{\mathbb{C}} \psi \frac{\partial \phi}{\partial \bar z}\, dx\, dy,$$

whenever ϕ and ψ belong to the space $\mathcal{D}$ of C^∞ complex-valued functions on $\mathbb{C}$, having compact support.

The adjoint operator acts on distributions:

$$\left\langle \phi, \left(\frac{\partial}{\partial \bar z}\right)^* f \right\rangle = \left\langle \frac{\partial}{\partial \bar z}\phi, f \right\rangle,$$

whenever $\phi \in \mathcal{D}$, and f belongs to $\mathcal{D}'$. In view of the skewness, we define

$$\frac{\partial f}{\partial \bar z} = -\left(\frac{\partial}{\partial \bar z}\right)^* f, \ \forall f \in \mathcal{D}',$$

so that the operator $\frac{\partial}{\partial \bar z}$ on $\mathcal{D}'$ is the weak-star continuous extension of $\frac{\partial}{\partial \bar z}$ on $\mathcal{D}$, when we regard $\mathcal{D}$ as a subset of $\mathcal{D}'$, under the identification of each $f \in L^1_{\mathrm{loc}}(dxdy)$ with the distribution *represented by* f, given by

$$\langle \phi, f \rangle = \int_{\mathbb{C}} \phi f dxdy, \ \forall \phi \in \mathcal{D}.$$

Complex Radon measures (Borel-regular complex-valued measures on $\mathbb{C}$, having finite total variation on each compact subset of $\mathbb{C}$) also represent distributions. The measure μ acts continuously on the space C^0_{cs} of continuous complex-valued functions on $\mathbb{C}$ having compact support, and equipped with the usual inductive limit topology, via

$$\langle \phi, \mu \rangle = \int_{\mathbb{C}} \phi f d\mu, \ \forall \phi \in C^0_{\mathrm{cs}}$$

and hence restricts to a continuous linear functional on $\mathcal{D}$. If we identify $f \in L^1_{\mathrm{loc}}$ with the measure $fdxdy$, then this generalises the previous remark. A measure is uniquely-determined by the corresponding distribution, because $\mathcal{D}$ is dense in C^0_{cs}, so we may identify the measure and distribution, without fear of confusion.

The $\bar\partial$ operator is linear and translation-invariant. It is also *elliptic*: this means that it is almost invertible; more precisely it has finite-dimensional kernel and cokernel, when restricted to a suitable space. When restricted to $\mathcal{D}'$, it has a very big kernel, the space of all entire functions. This statement is a case of Weyl's Lemma: a distributional solution $u \in \mathcal{D}'(U)$ (where $\mathcal{D}(U)$ is the space of C^∞ complex-valued functions on U) of $\bar\partial u = 0$ on an open set $U \subset \mathbb{C}$ is representable by a holomorphic function on U. This is a good thing, because it gives complex analysts a nontrivial field of study. But when restricted to $\mathcal{E}'$, the dual of the space of C^∞ functions on $\mathbb{C}$ with compact support, $\bar\partial$ is injective. The fundamental solution is the locally-integrable function $-1/\pi z$, i.e.

$$\frac{\partial}{\partial\bar z}\left(-\frac{1}{\pi z}\right) = \delta_0,$$

the point mass at 0. For $\phi \in \mathcal{D}$, we have

$$\frac{\partial}{\partial\bar z}\hat\phi = \phi,$$

where

$$\hat\phi = \left(-\frac{1}{\pi z}\right) * \phi,$$

the *Cauchy transform* of ϕ. We extend the transform to a map $\mathcal{E}' \to \mathcal{D}'$ by setting

$$\langle \phi, \hat f\rangle = -\langle \hat\phi, f\rangle,\ \forall \phi \in \mathcal{D},\ \forall f \in \mathcal{E}'.$$

We have

$$\frac{\partial}{\partial\bar z}\hat f = f,\ \forall f \in \mathcal{E}'.$$

In particular, for each $f \in \mathcal{E}'$, the distribution $\hat f$ is (represented by) a holomorphic function off $\text{spt} f$, the support of f. Because of all this, the Cauchy transform is intimately connected with analytic function theory, and one can use it to establish many interesting results.

3. Some Holomorphic Approximation Theorems

For $X \subset \mathbb{C}^n$, let $\mathcal{O}(X)$ denote the space of functions holomorphic near X. For compact Hausdorff X, let $C^0(X)$ denote the Banach space of all continuous, complex-valued functions on X, with the sup norm.

Theorem 3.1 (Hartogs–Rosenthal, 1931)**.** *Suppose $X \subset \mathbb{C}$ is compact and has area zero. Then $\mathcal{O}(X)$ is dense in $C^0(X)$.*

Proof. By the Separation Theorem for Banach spaces, it suffices to show that

$$L \in C^0(X)^* \cap \mathcal{O}(X)^\perp \Rightarrow L = 0.$$

By the Riesz Representation Theorem, the dual $C^0(X)^* = M(X)$, the space of (complex, Radon) measures supported on X.

Fix $\mu \in M(X)$ with $\mu \perp \mathcal{O}(X)$, i.e. $\int f\,d\mu = 0$ whenever $f \in \mathcal{O}(X)$.

Regarding μ as a distribution on $\mathbb{C}$, we find that $\hat{\mu}$ is the locally-integrable function given by

$$\hat{\mu}(\zeta) = \frac{1}{\pi}\int \frac{d\mu(z)}{z-\zeta},\ \forall \zeta \in \mathbb{C}.$$

Since the function $z \mapsto 1/(z-\zeta)$ belongs to $\mathcal{O}(X)$ for $\zeta \notin X$, we have $\hat{\mu} = 0$ $dxdy$-a.e., hence $\hat{\mu} = 0$ as a distribution, hence

$$\mu = \frac{\partial \hat{\mu}}{\partial \bar{z}} = 0.$$ □

Corollary 3.2 (A. Browder)**.** *Suppose $X \subset \mathbb{C}^n$ is compact, and each coordinate projection of X has area zero. Then $\mathcal{O}(X)$ is dense in $C^0(X)$.*

Proof. Denote $z = (z_1, \ldots, z_n)$, and $z_j = x_j + iy_j$. Fix $j \in \{1, \ldots, n\}$. Let $\pi_j : z \mapsto z_j$. Then $\pi_j(X)$ has area zero, so by Hartogs–Rosenthal $z \mapsto x_j$ and $z \mapsto y_j$ are uniform limits of functions (depending only on $\pi_j(z)$) that are holomorphic on a neighbourhood of $\pi_j^{-1}(\pi_j(X))$, and hence belong to $\mathcal{O}(X)$. Thus the uniform closure A on X of the algebra $\mathcal{O}(X)$ contains all the coordinate functions x_j and y_j, so by La Valleé Poussin's extension of Weierstrass' Polynomial Approximation Theorem (a special case of the Stone–Weierstrass Theorem), we conclude that $A = C^0(X)$. □

Consider the function spaces, for $0 < \alpha < 1$ and compact $X \subset \mathbb{C}^n$:

$$\mathrm{Lip}(\alpha, X) = \left\{ f \in C^0(X) : \sup_{z \neq w} \frac{|f(z) - f(w)|}{|z - w|^\alpha} < +\infty \right\},$$

and

$$\mathrm{lip}(\alpha, X) = \left\{ f \in \mathrm{Lip}(\alpha, X) : \sup_{0<|z-w|<\delta} \frac{|f(z) - f(w)|}{|z - w|^\alpha} \to 0 \text{ as } \delta \downarrow 0 \right\}.$$

With a suitable norm, $\mathrm{Lip}(\alpha, X)$ becomes a Banach algebra, and the subspace $\mathrm{lip}(\alpha, X)$ is a closed subalgebra, equal to the closure of $\mathcal{D}$ in $\mathrm{Lip}(\alpha, X)$. The elements of the dual $\mathrm{lip}(\alpha, X)^*$ may be represented in a manner somewhat similar to the Riesz representation, as follows.

Fix any $a_0 \in X$.

Given $L \in \mathrm{lip}(\alpha, X)^*$, there exist $\lambda \in \mathbb{C}$ and a measure μ on the product $X \times X$ having no mass on the diagonal, such that

$$Lf = \lambda f(a_0) + \int_{X \times X} \frac{f(z) - f(w)}{|z - w|^\alpha} d\mu(z, w),$$

whenever $f \in \mathrm{lip}(\alpha, X)$. If $L1 = 0$, then $\lambda = 0$. For such λ, this permits us to represent $\hat{L}$ by integration against an L^1_{loc} function:

$$\hat{L}(\zeta) = \frac{1}{\pi} \int \frac{(w - z)\, d\mu(z, w)}{(\zeta - z)(\zeta - w)|z - w|^\alpha}.$$

Using this, and essentially the same proof as given for the Hartogs–Rosenthal Theorem, one obtains [6, p. 387]:

Theorem 3.3. *If $X \subset \mathbb{C}$ is compact with area zero, then $\mathcal{O}(X)$ is dense in* $\mathrm{lip}(\alpha, X)$ *for* $0 < \alpha < 1$.

Corollary 3.4. *If $X \subset \mathbb{C}^n$ has all its coordinate projections of area zero, then $\mathcal{O}(X)$ is dense in* $\mathrm{lip}(\alpha, X)$.

4. Higher-dimensional Cauchy Transforms

The utility of the Cauchy transform in one dimension prompted people to seek a similar tool for problems of several complex variables. Here one *must* use forms. The kernel $-1/\pi z$ must be replaced by a $(2n-1)$-form of type $(n, n-1)$:

$$\Omega = \sum_{j=1}^{n} K_j(\zeta, z)\, d\bar{\zeta}_1 \wedge \cdots d\bar{\zeta}_{j-1} \wedge d\bar{\zeta}_{j+1} \wedge \cdots \wedge d\bar{\zeta}_n \wedge d\zeta_1 \wedge \cdots \wedge d\zeta_n,$$

such that

$$\phi(z) = \int \Omega(\zeta, z) \wedge \bar{\partial}\phi(\zeta) \tag{1}$$

holds for test functions ϕ. A form Ω that does this is called a *Cauchy–Leray–Fantappié form.* There are many such forms, and depending on the end in view, one prefers one or another. There are also more complex forms, involving boundary terms (analogous to Pompeiu's formula), useful for specific purposes.

In joint work with David Walsh, and the late Ken Preskenis, we obtained the following [8]:

Theorem 4.1. *Let $X \subset \mathbb{C}^n$ be compact and holomorphically-convex. Let $E \subset X$ be closed, and suppose that each point $a \in X \sim E$ has a neighbourhood $N \subset \mathbb{C}^n$ such that $X \cap N$ is a subset of a C^1 submanifold without complex tangents. Then*

$$\mathrm{clos}_{C^0(X)}\mathcal{O}(X) = C^0(X) \cap \mathrm{clos}_{C^0(E)}\mathcal{O}(X),$$

and

$$\mathrm{clos}_{\mathrm{Lip}(\alpha,X)}\mathcal{O}(X) = \mathrm{lip}(\alpha, X) \cap \mathrm{clos}_{\mathrm{Lip}(\alpha,E)}\mathcal{O}(X), \quad \text{for } 0 < \alpha < 1.$$

In other words, approximation problems on X reduce to approximation problems on the singular set $E \subset X$.

This generalised and extended to $\mathrm{Lip}(\alpha)$ earlier work of Range and Siu ($E = \emptyset$), Weinstock (X polynomially-convex), and ourselves [7] (see below).

The proof comes down to showing that if a distribution L that acts continuously on $\mathrm{lip}(\alpha, X)^*$ annihilates $\mathcal{O}(X)$, then L is supported on E. To do this, one constructs a kernel $\Omega(\zeta, z)$ such that Equation (1) holds for z on a neighbourhood U of X and $\phi \in \mathcal{D}(U)$, and a second kernel $\tilde{\Omega}(\zeta, z)$ such that $\tilde{\Omega}(\zeta, z) = \Omega(\zeta, z)$ for $z \in X$ and $\zeta \in U$, and $\tilde{\Omega}$ has coefficients $\tilde{K}_j(\zeta, z)$ that are holomorphic in $z \in U$ for each $\zeta \in U \sim X$, and have another technical property. This construction is based on work of Berndtsson, building on the special Bochner–Martinelli kernel. Then, representing L as before by a measure μ on $X \times X$ with no mass on the diagonal, we can represent

$$\begin{aligned}\langle \phi, L\rangle &= \int_{X\times X} \frac{1}{|z-w|^\alpha} \int_U \{\Omega(\zeta, z) - \Omega(\zeta, w)\} \wedge \bar{\partial}\phi(\zeta)\, d\mu(z, w)\\ &= \int_U \int_{X\times X} \frac{1}{|z-w|^\alpha} \{\Omega(\zeta, z) - \Omega(\zeta, w)\}\, d\mu(z, w) \wedge \bar{\partial}\phi(\zeta),\end{aligned}$$

by Fubini's Theorem. (There are substantial technical estimates involved in justifying this.)

It remains to show that

$$\int_{X\times X} \frac{1}{|z-w|^\alpha} \{\Omega(\zeta,z) - \Omega(\zeta,w)\}\, d\mu(z,w) = 0$$

for almost all $\zeta \in U$. The fact that $\Omega(\zeta,z) = \tilde{\Omega}(\zeta,z)$ for $z \in X$ and that the latter is holomorphic in z, and the technical properties (the most important of which is an "omitted sector property") allow us to approximate each coefficient in the integral by elements of $\mathcal{O}(X)$, and gives the desired result. For the details, see [8].

Corollary 4.2 (Range-Siu). *If $E = \emptyset$, then $\mathcal{O}(X)$ is dense in $C^0(X)$.*

Corollary 4.3. *Let $F \subset Y$, where Y is a compact subset of $\mathbb{C}^n$ and F is a closed subset of Y. Let f be a $\mathbb{C}^n$-valued function defined on a neighbourhood of Y, let $X = f(Y)$ and $E = f(F)$. Suppose that X is polynomially-convex, and the matrix $f_{\bar z}$ (with columns $\frac{\partial f}{\partial \bar z_j}$) is invertible on $Y \sim F$. Then*

$$\mathrm{clos}_{C^0(X)}\mathbb{C}[z,w] = C^0(X) \cap \mathrm{clos}_{C^0(E)}\mathbb{C}[z,w], \qquad (2)$$

and

$$\mathrm{clos}_{\mathrm{Lip}(\alpha,X)}\mathbb{C}[z,w] = \mathrm{lip}(\alpha,X) \cap \mathrm{clos}_{\mathrm{Lip}(\alpha,E)}\mathbb{C}[z,w]. \qquad (3)$$

Equation 2 is due to Weinstock.

Corollary 4.4. *Suppose ρ is a C^2 strictly plurisubharmonic function on a neighbourhood of* bdy$X$*, where X is a compact subset of $\mathbb{C}^n$, with interior D, and that* bdy$X = \{z : \rho(z) = 0\}$, $\{z : \rho(z) < 0\} \subset D$*, and $E =$* clos$D$*. Then Equations 2 and 3 hold.*

In this case, Equation 2 is due to Henkin and Leiterer.

5. Extending Smooth Functions

According to one view, Geometry is an aspect of Group Theory. But more accurately, Geometry *is* Ring Theory. To be absolutely precise, Geometry is Topological Ring Theory.

Let M be a C^k manifold, and $X \subset M$. Given $f : X \to \mathbb{R}$, when does there exist a C^k function $\tilde{f} : M \to \mathbb{R}$ such that the restriction $\tilde{f}|X = f$? This problem arises in many applications, and has been studied since the 1930's, with important work of Whitney and Glaeser. Richard Watson and I studied it in the early 1990's drawing on some ideas of mine that go back to the 1970's.

We deal now with real-valued functions, real vector spaces and algebras, and i is just an index, or multi-index, and not $\sqrt{-1}$ any more.

Let $C^k(M)$ now denote the algebra (under pointwise operations) of C^k real-valued functions on M. This is a Fréchet algebra (a complete metric algebra) with the natural topology. For $S \subset C^k(M)$, let

$$S^\perp = \{L \in C^k(M)^* : Lf = 0, \forall f \in S\},$$

where $C^k(M)^*$ denotes the space of continuous linear functionals $L : C^k(M) \to \mathbb{R}$. Note that $C^k(M)^*$ is a module over $C^k(M)$. For $X \subset M$, let

$$X_\perp = \{f \in C^k(M) : f(a) = 0, \forall a \in X\}.$$

Let $a_\perp = \{a\}_\perp$, when $a \in X$. Each $X_\perp$ is an ideal in $C^k(M)$. The ideal $(a_\perp)^{k+1}$ is generated by products of $k+1$ elements of $a_\perp$. Its annihilator $((a_\perp)^{k+1})^\perp$ consists of the so-called "k-th order point differential operators". In local coordinates $(x_1, \ldots, x_d)$, each $\partial \in ((a_\perp)^{k+1})^\perp$ takes the form

$$\partial f = \sum_{|i| \le k} \alpha_i \frac{\partial f}{\partial x_i}(a), \ \forall f \in C^k(M),$$

where $i = (i_1, \ldots, i_d) \in \mathbb{Z}_+^d$ denotes a multi=index, $|i| = \sum_j i_j$, and $\alpha_i \in \mathbb{R}$ are constants depending on ∂, but not on f.

The *k-th order tangent space to M at a* is defined as

$$\mathrm{Tan}^k(M, a) = C^k(M)^* \cap ((a_\perp)^{k+1})^\perp,$$

and the *k-th order tangent space to X at a* is defined as

$$\mathrm{Tan}^k(M, X, a) = \mathrm{Tan}^k(M, a) \cap (X_\perp)^\perp,$$

the set of k-th order point differential operators ∂ at a such that ∂f depends only on the values of f on X. The two disjoint unions

$$T^k(M) = \dot{\bigcup}_{a \in M} \mathrm{Tan}^k(M, a),$$
$$T^k(M, X) = \dot{\bigcup}_{a \in M} \mathrm{Tan}^k(M, X, a)$$

are called the *k-th order tangent bundle of M*, and the *k-th order tangent sheaf of X*, respectively. The stalks $\mathrm{Tan}^k(M, X, a)$ have the structure of finite-dimensional modules over a finite-dimensional real algebra, and provide numerical C^k invariants for the pair (M, X), since the tangent construction behaves functorially. A C^k function $F : M \to M'$ between C^k manifolds induces an algebra homomorphism

$$F^{\#} : \begin{cases} C^k(M') & \to & C^k(M) \\ g & \mapsto & g \circ F \end{cases}$$

and a C^k-module homomorphism

$$F_{\#} = (F^{\#})^* : C^k(M)^* \to C^k(M')^*.$$

If F maps X into X', then $F_{\#}$ maps the stalk $\mathrm{Tan}^k(M, X, a)$ to the stalk $\mathrm{Tan}^k(M', X', f(a))$, and so induces a map $F_* : T^k(M, X) \to T^k(M', X')$. We established the following [9]:

Theorem 5.1. *Let X be a closed subset of a C^k manifold M, $f : X \to \mathbb{R}$ be continuous, and*

$$\pi : \begin{cases} M \times \mathbb{R} & \to & M \\ (x, y) & \mapsto & x \end{cases}$$

be the projection. Then f has a C^k extension to M if and only if the map

$$\pi_* : T^k(M \times \mathbb{R}, f) \to T^k(M, X)$$

is bijective.

This result, and the k-th order tangent concept, are not particularly difficult, but are completely fundamental for the extension problem. They reduce the extension problem to the problem of deciding whether or not two integral dimensions (of $\mathrm{Tan}^k(M \times \mathbb{R}, f, (a, f(a))$ and $\mathrm{Tan}^k(M, X, a))$ agree at each point $a \in X$. We were gratified by the favourable reception of our paper, which included a congratulatory letter from Malgrange. It remained a problem to come up with a constructive procedure for deciding the question. Whitney himself dealt with this in dimension one, for all k. We provided a way to do it in 1993, in case $k = 1$, for all dimensions. In recent years, C. Fefferman and co-workers have gone as far as can be done in providing a constructive procedure for general k. See the website [5] where this monumental corpus may be downloaded. Their work employs, *inter alia* the k-th order sheaf we introduced and our result. Fefferman was unaware of our work, having taken the concept and result from the 2003 Inventiones paper of Bierstone, Milman and Pawluckii [4]. I supplied a copy of our paper to Pawlucki in December 1997, at his request. These authors included our paper among their references, but did not attribute the concept to us. They referred to our paper only in order to make a gratuitously dismissive remark about it. I am at a loss to understand this behaviour. They called the tangent sheaf the *Zariski paratangent bundle* (although it is not in general a bundle), and subsequently a number of authors have referred to it as *the Zariski paratangent bundle of Bierstone, Milman and Pawlucki*. The bundle $T^k(M)$ was originally introduced by Pohl and Feldman, but the sheaf $T^k(M, X)$ appeared first in our paper.

I must also mention the work of Declan O'Keeffe [10], who proved the analogous result for $C^{k+\alpha}$ extensions ($k \in \mathbb{N}$, $0 < \alpha < 1$), and who also used T^k to study algebraic curve singularities in $\mathbb{C}^2$.

6. Approximating C^∞ Functions

In conclusion, here is a brief summary of the joint work with the late Graham Allan, Grayson Kakiko and Richard, on Segal's problem. The problem called for a characterization of the closed subalgebras of the algebra $C^\infty(M, \mathbb{R})$, for a smooth manifold M. The problem is local, so that we may take $M = \mathbb{R}^d$. There is not much loss in generality in considering subalgebras that are topologically-finitely-generated, i.e. those of the form

$$A(\Psi) = \mathrm{clos}_{C^\infty(\mathbb{R}^d)} \mathbb{R}[\Psi] = \mathrm{clos}_{C^\infty(\mathbb{R}^d)} \{g \circ \Psi : g \in C^\infty(\mathbb{R}^r, \mathbb{R}),$$

where $\Psi = (\psi_1, \ldots, \psi_r) \in C^\infty(\mathbb{R}^d)$. In 1950, Nachbin conjectured a solution, analogous to Whitney's Spectral Theorem for closed ideals. Let $\mathbb{R}[[x_1, \ldots, x_n]]$ denote the algebra of formal power series in n indeterminates, let

$$T'_a : C^\infty(\mathbb{R}^d, \mathbb{R}^r) \to \mathbb{R}[[x_1, \ldots, x_d]]^r$$

denote the truncated Taylor series map, for each $a \in \mathbb{R}^d$, and let $T_a f = f(a) + T'_a f$ be the full Taylor series. (Note that x_j would have to be replaced by $(x_j - a_j)$ in these series in applications of Taylor's Theorem.)

In case $\Psi \in C^\infty(\mathbb{R}^d, \mathbb{R}^r)$ is injective, Nachbin's conjecture comes down to

$$A(\Psi) = \bigcap_{a \in \mathrm{crit}\,\Psi} T_a^{-1} \mathbb{R}[[T'_a \Psi]], \tag{4}$$

which may be stated in loose terms as: $f \in A(\Psi)$ if and only if f has the "right kind" of Taylor series at each point.

Theorem 6.1 (Tougeron 1971)**.** *Suppose that for each compact $K \subset \mathbb{R}^d$ there exists $\alpha > 0$ and $\beta > 0$ such that*

$$|\Psi(x) - \Psi(y)| \geq \alpha |x - y|^\beta, \ \forall x, y \in K.$$

Then Equation (4) holds.

Corollary 6.2. *If Ψ is injective and real-analytic, then Equation (4) holds.*

Our main result was this [1]:

Theorem 6.3. *If Ψ is injective and $d = 1$, then Equation (4) holds.*

The proof involves some hard analysis, to overcome the problems around the accumulation points of the critical set.

We say that Ψ is *flat at a* if $T'_a \Psi = 0$. We also proved the following useful result [2].

Theorem 6.4. *If Ψ is injective and $f \in C^\infty(\mathbb{R}^d)$ is flat at each critical point of Ψ, then $f \in A(\Psi)$.*

Corollary 6.5. *If Ψ is injective and flat on* $\mathrm{crit}\Psi$*, then Equation (4) holds.*

Corollary 6.6. *If Ψ is injective and* $\mathrm{crit}\Psi$ *is discrete, then Equation (4) holds.*

In further work [3], we studied $A(\Psi)$ as a Fréchet algebra, for injective Ψ. We established that it is always regular, and that membership of $A(\Psi)$ is a local property.

7. Notes

It is convenient to take this opportunity to correct the definition of proxy distance given on p. 49 of our paper [2] in the proceedings of the meeting at Blaubeuren. This should read:

Definition. Let $E \subset \mathbb{R}^d$ be closed and $\kappa_m \geq 1$ $(m = 0, 1, 2, \ldots)$. A function $d_E : \mathbb{R}^d \to [0, +\infty)$ that is C^∞ on $\mathbb{R}^d i \sim E$ is called a $\{\kappa_m\}$ *proxy distance for E* if

$$\frac{1}{\kappa_0} d_E(x) \leq \mathrm{dist}(x, E) \leq \kappa_0 d_E(x), \ \forall x \in \mathbb{R}^d$$

and

$$|D^m d_E(x)| \leq \kappa_m \cdot \mathrm{dist}(x, E)^{1-m}, \ \forall x \in \mathbb{R}^d, \forall m \geq 1.$$

No other change to the paper is needed, and it all remains true. (The reason for the change is that d_E *cannot* be C^∞ on the whole of $\mathbb{R}^d$ when $\emptyset \neq E \neq \mathbb{R}^d$; in the subsequent application d_E is always composed with functions ϕ that vanish near 0, so $\phi \circ d_E$ is C^∞ on $\mathbb{R}^d$.)

I would also like to point out to readers of [8] that the interesting case of Example 3.2 is when $X \neq \operatorname{int} \operatorname{clos} X$. It is even interesting when X has no interior.

References

[1] G.R. Allan, G. Kakiko, A.G. O'Farrell and R.O. Watson, Finitely-generated algebras of smooth functions, in one dimension, *J. Funct. Anal.* **158** (1998), 458–74.

[2] G.R. Allan, G. Kakiko, A.G. O'Farrell and R.O. Watson, Algebras of smooth functions. In: E. Albrecht and M. Mathieu (eds.), Banach Algebras '97 (Blaubeuren), de Gruyter, Berlin, 1998, pp. 43–53.

[3] G.R. Allan, G. Kakiko, A.G. O'Farrell and R.O. Watson, Localness of $A(\Psi)$ algebras, *Math. Proc. Roy. Ir. Acad.* **101A** (2001), 61–70.

[4] E. Bierstone, P.D. Milman and W. Pawlucki, Differentiable functions defined in closed sets. A problem of Whitney, *Invent. math.* **151** (2003), 329–52.

[5] C. Fefferman. Selected Papers. `http://www.math.princeton.edu/facultypapers/Fefferman/`

[6] A.G. O'Farrell, Annihilators of rational modules, *J. Funct. Anal.* **19** (1975), 373–89

[7] A.G. O'Farrell, K.J. Preskenis and D. Walsh, Polynomial approximation on graphs, *Math. Ann.* **266** (1983), 73–81.

[8] A.G. O'Farrell, K.J. Preskenis and D. Walsh, Holomorphic approximation in Lipschitz norms, *Contemp. Math.* **32** (1984), 187–94.

[9] A.G. O'Farrell and R.O. Watson, The tangent stars of a set, and extensions of smooth functions, *J. Reine Angew. Math.* **430** (1992), 109–37.

[10] D. O'Keeffe, The tangent stars of a set and extensions of functions in Lipschitz classes, *Math. Proc. Roy. Ir. Acad.* **97A** (1997), 5–13.

Anthony G. O'Farrell,
Department of Mathematics,
National University of Ireland,
Maynooth, Co. Kildare
Ireland
anthonyg.ofarrell@gmail.com

Received on 4 January 2010.

Irish Math. Soc. Bulletin **64** *(2009), 43–54*

Ireland's Participation in the 50th International Mathematical Olympiad

BERND KREUSSLER

The 50th International Mathematical Olympiad (IMO) took place in Bremen (Germany) from 12th July until 22nd July 2009. With 565 participants (59 of whom were girls) from 104 countries, this was the IMO with the largest participation so far. It was the first time that a three-digit number of countries participated.

The Irish delegation consisted of six students (see Table 1), the Team Leader, Bernd Kreussler (MIC Limerick), the Deputy Leader, Gordon Lessells (UL) and the Official Observer, Donal Hurley (UCC).

1. Team Selection and Preparation

The IMO is the most prestigious mathematical contest for second level students in the world. Participation in this event is already considered to be a great honour. In order to be able to gain any marks at the IMO exams, it is not sufficient, even for the brightest of students, to rely solely on the Leaving Certificate Mathematics Syllabus (or the equivalent in other countries). Leading countries such as China and Russia are able to identify the most talented students at a very early age and then organise very intense training programmes for them.

Name	School	Year
Jack McKenna	Newbridge College, Newbridge, Co. Kildare	6th
David McCarthy	Midleton CBS, Midleton, Co. Cork	6th
Colman Humphrey	St. Andrew's College, Booterstown, Co. Dublin	6th
Cillian Power	Christian Brothers College, Cork	5th
Colin Egan	Clonkeen College, Blackrock, Co. Dublin	5th
Vicki McAvinue	St. Angela's School, Ursuline Convent, Waterford	4th

Table 1. The Irish contestants at the 50th IMO

In Ireland, we are currently able to identify students with exceptional mathematical talent in three different ways: top performance in the Junior Certificate Examination in Mathematics, or excellent results in the PRISM competition or recommendation by a person who has identified this talent (maths teachers, parents, school principals). At five different locations all over Ireland (UCC, UCD, NUIG, UL and NUIM), mathematical enrichment programmes are offered to the students who came to our attention through one of these sources. These classes run each year from December/January until April and are offered by volunteer academic mathematicians from these universities or nearby third-level institutions.

The selection contest for the Irish IMO team is the Irish Mathematical Olympiad (IrMO), which was held for the 22$^{\text{nd}}$ time on Saturday, 9$^{\text{th}}$ May, 2009. The IrMO contest consists of two 3-hour papers on one day with five problems on each paper. The participants of the IrMO normally attended the enrichment classes in one of the five centres; however, particpation is open to all interested secondary school students. This year, a total of 69 students took part in the IrMO, which was held at the five centres listed above. The top performer is awarded the Fergus Gaines cup; this year this was Jack McKenna. The best six students (listed in order in Table 1) were invited to represent Ireland at the IMO in Bremen.

During the week before the students travelled to Bremen, they gathered for an intensive training camp at the University of Limerick. In preparation for this camp, they were sent a number of problems which they were expected to have solved by the time they had arrived in Limerick. The camp was organised as usual in a very efficient way by Gordon Lessells. The sessions with the students were directed this year by Jim Cruickshank, Donal Hurley, Kevin Hutchinson, Bernd Kreussler, Tom Laffey, Jim Leahy, Gordon Lessells and Rachel Quinlan. They focussed on problem solving techniques in the core topics that are covered by IMO problems: algebra, combinatorics, elementary geometry and number theory.

In addition to the six members of the Irish IMO-team, six of the best students, who were known to the organisers and who will be eligible to participate in future IMOs, were also invited to attend the camp for three days. This measure aims at building a base for future success.

2. Jury Meetings — The Problem Selection

Donal Hurley and I travelled on the 12th of June to Bremen because the Jury was scheduled to convene on Saturday, 13th June. The location where the students will reside and where the contest is to be held is always well known in advance. The Jury of the IMO however, which is the prime decision making body for all IMO matters, resides at a location which is kept as secret as possible. This is one of the measures which helps ensure that the contest problems will not become publicly known before the exams take place. This year, the hideout of the Jury was in Bremerhaven, the seaport of the free city-state of Bremen. This town was founded in 1827 and is now one of the most important German trade ports. It is located on the North Sea, about 60 kilometres north of Bremen, at the estuary of the river Weser.

The Jury is composed of the Team Leaders of the participating countries and a Chairperson. The Observers at the Jury's site participate, without voting rights, in all Jury meetings. The Chairman for the 50th IMO was Prof. Hans-Dietrich Gronau, who has been the leader of the German IMO-team for many years. He chaired the Jury meetings in a very efficient way, so that the Jury completed all its tasks well within the very tight preassigned timeframe.

In recent years, the seating plan for the Jury meetings was determined by the alphabetical order of a three letter country code (IRL for Ireland). The organisers decided to change the seating plan for the Jury meetings this year in such a way that the countries were seated in the order of their initial participation in the IMO. The first two, according to this order, were Romania and Hungary; Ireland, participating since 1988, was on position 45. Because of the changes in the political landscape during the past 20 years, some nations were seated behind Ireland, even though students from the same geographic location had participated much earlier than students from Ireland. For example, the former Soviet Union belonged to the group of seven countries which participated in the first IMO in Braşov (Romania) in 1959, but the representative of the Russian Federation was seated at position 59, next to the representative from Trinidad and Tobago.

The host country's problem selection committee selected a shortlist of 30 problems from the problems submitted to them in advance. There was general agreement among the Jury members that the shortlist was very well prepared: for the vast majority of the problems two solutions were given, one problem even came with five distinct solutions. Most of Saturday was available for the leaders and observers to get familiar with the problems. Later, at 4 p.m., the solutions were handed out and the evening and night were spent studying them.

The main task of the Jury meetings on Sunday, 12$^{\text{th}}$ July, was the selection of the six contest problems from the shortlist. Under the skilful direction of the Chairman, this task was completed before 4 p.m. on Sunday. During the evening meeting, the final formulations of the six problems were discussed so that the representatives of the language groups of the five official languages (English, French, German, Russian and Spanish) were able to produce their versions over night. After approving these five versions at a Jury meeting on Monday morning, the rest of the day was available for the translation of the contest problems into 50 languages, in addition to the five mentioned before. Most of these translated versions were produced with the aid of LaTeX or MS-Word, but three of them were handwritten: the Armenian, the Mongolian and the Singhalese translations.

3. The Problems

First Day.

Problem 1. Let n be a positive integer and let $a_1, \dots, a_k$, $k \geq 2$ be distinct integers in the set $\{1, \dots, n\}$ such that n divides $a_i(a_{i+1} - 1)$ for $i = 1, \dots, k-1$. Prove that n does not divide $a_k(a_1 - 1)$.

(Australia)

Problem 2. Let ABC be a triangle with circumcentre O. The points P and Q are interior points of the sides CA and AB, respectively. Let K, L and M be the midpoints of the segments BP, CQ and PQ, respectively, and let Γ be the circle passing through K, L and M. Suppose that the line PQ is tangent to the circle Γ. Prove that $OP = OQ$. *(Russian Federation)*

Problem 3. Suppose that $s_1, s_2, s_3, \dots$ is a strictly increasing sequence of positive integers such that the subsequences

$$s_{s_1}, s_{s_2}, s_{s_3}, \dots \quad \text{and} \quad s_{s_1+1}, s_{s_2+1}, s_{s_3+1}, \dots$$

are both arithmetic progressions. Prove that the sequence $s_1, s_2, s_3, \dots$ is itself an arithmetic progression. *(United States of America)*

Second Day.

Problem 4. Let ABC be a triangle with $AB = AC$. The angle bisectors of $\angle CAB$ and $\angle ABC$ meet the sides BC and CA at D and E, respectively. Let K be the incentre of triangle ADC. Suppose that $\angle BEK = 45°$. Find all possible values of $\angle CAB$. *(Belgium)*

Problem 5. Determine all functions f from the set of positive integers to the set of positive integers such that, for all positive integers a and b, there exists a non-degenerate triangle with sides of lengths

$$a,\ f(b)\ \text{ and }\ f(b + f(a) - 1).$$

(A triangle is *non-degenerate* if its vertices are not collinear.) *(France)*

Problem 6. Let $a_1, a_2, \ldots, a_n$ be distinct positive integers and let M be a set of $n-1$ positive integers not containing $s = a_1 + a_2 + \cdots + a_n$. A grasshopper is to jump along the real axis, starting at the point 0 and making n jumps to the right with lengths $a_1, a_2, \ldots, a_n$ in some order. Prove that the order can be chosen in such a way that the grasshopper never lands on any point in M.

(Russian Federation)

4. The Contest

The six Irish contestants, accompanied by the Deputy Leader, Gordon Lessells, arrived in Bremen in the late evening of Monday, 13$^{\text{th}}$ July. The Opening Ceremony took place on Tuesday morning at the event centre “Pier 2” in Bremen. The start and the end of the ceremony were marked by a Breakdance performance. In addition to the usual speeches and a short mathematical entertainment entitled “Calculating without a calculator”, we saw a parade of all 104 participating teams in the order in which their countries initially participated in the IMO. Some contestants were missing because they hadn’t yet arrived, but all made it in time for the first exam on Wednesday.

On Tuesday afternoon the students enjoyed free time so that they could relax prior to the exams on the following two days. After a reception at the 600 years old city hall of Bremen, the Jury, however, were obliged to go back to work. On the agenda for the meetings in the afternoon and evening were the discussion and finalisation of the detailed marking schemes for all six contest problems. The problem captains had done a marvellous job: the only major demand of the Jury was a more explicit marking scheme for Problem 3. Spontaneous applause was awarded to the marking scheme for Problem 2 which included a flow chart type graph giving a very detailed overview of possible partial marks for a few of the anticipated solutions.

The two exams took place on the 14th and 15th of July, starting at 9 o'clock each morning. The venue for the exams was a single large hall, so that all students had equal conditions. The seats were arranged in six blocks, each of which did not accommodate more than one student from the same country.

On each day, $4\frac{1}{2}$ hours were available to solve three problems. During the first 30 minutes, the students were allowed to ask questions if they had difficulties in understanding the formulation of a contest problem. These questions were scanned in Bremen and sent electronically to Bremerhaven, where the printout was given to the relevant Team Leader. Each question was discussed and answered in front of the Jury, so that equal standards were applied to all contestants. The approved answer was then sent back electronically to Bremen. Most of the 90 questions on day one were seeking clarification what is an interior point of a triangle (Problem 2). On the second day, there were only 42 questions asked by the contestants.

5. Marking and Coordination

After this last questions-and-answers session on Thursday morning, the Team Leaders and Observers left the beautiful sea-side location in Bremerhaven to join the Deputy Leaders and the contestants on the campus of Jacobs University Bremen. This is a private university which was founded in 1999 and currently has about 1200 students from over 90 countries. The campus of this university is located on the former grounds of the Roland Barracks, which was in use by the German Army until 1998.

Because the scripts of the first day had arrived in Bremerhaven on the Wednesday evening, we already did a first reading before we moved to Bremen. After our arrival, we immediately began to read them in detail, with the help of Gordon. The most remarkable fact about these scripts was that David had produced the record number of 37 pages, neither numbered nor in correct order, for his attempted solution of Problem 1. All three of us, naturally, studied every single page and we tried to put these pages in some kind of order hoping to find the material in it which would be the basis for the points he deserved.

After their return from the second exam, we briefly met the students. They didn't seem optimistic about their performance. During the following two days the Leader, Deputy and Observer were fully occupied with our preparation for the coordination.

At the same time, the students started to relax after their two days of exams. On Friday and Saturday they enjoyed excursions to several destinations, including a Shipyard, a Transrapid Track and the newly opened Klimahaus Bremerhaven. They had a choice and so could spend these days according to their taste. In the evenings, a Football competition was organised. Two of the Irish students participated in a team with students from Georgia, Zimbabwe and South Africa and lost narrowly in the Final.

The marking of the scripts at the IMO is undertaken by two independent groups. One group consists of the Team Leader, the Deputy Leader and the Official Observer. The second group consists of the coordinators, who were appointed by the local organisers. The two groups met according to a tight schedule which was distributed before our departure from Bremerhaven. For each problem there were five coordination tables with two coordinators at each. This means that each pair of coordinators had to study the solutions for one of the problems of about 120 contestants from 20 countries. All coordinators were former IMO participants or involved in the training of contestants for many years, or both.

The coordination is a very important part of the IMO, because it is the well established method to mark fairly the scripts of the students from so many different nations. It is important for the representatives of the teams to be well prepared for each of the half-hour meetings with the coordinators. For example, if a student came up with an incomplete solution which is not exactly covered by the agreed marking scheme, a fully worked out solution which starts with the work done by the contestant could be the argument required to obtain partial points. Sometimes, when the relevant part of the student's work was difficult to find in the forest of unfinished ideas, the many hours we had spend reading and discussing the scripts did pay off. We found that the coordinators were very knowledgable, were always well prepared and ready to listen to our presentation.

Name	P1	P2	P3	P4	P5	P6	total	ranking
Jack McKenna	2	2	0	2	0	0	6	416
David McCarthy	3	1	0	0	0	0	4	444
Colman Humphrey	0	1	0	1	0	0	2	482
Cillian Power	1	1	0	0	0	0	2	482
Colin Egan	1	1	0	2	0	0	4	444
Vicki McAvinue	0	0	0	2	0	0	2	482

TABLE 2. The results of the Irish contestants

In general they were generous in agreeing with our proposals. Table 2 shows the results of the Irish contestants.

The Jury tries to choose the problems in such a way that Problems 1 and 4 are easier than Problems 2 and 5. Problems 3 and 6 are usually designed to be the hardest problems. Table 3 shows that Problems 2 and 4 need to be swapped in order to achieve this aim, but otherwise the results fit very well into this pattern.

	P1	P2	P3	P4	P5	P6
0	83	101	357	188	270	540
1	56	106	127	79	42	2
2	28	43	16	37	50	1
3	20	51	5	23	33	10
4	17	16	2	17	6	6
5	16	15	5	69	4	2
6	21	19	2	52	7	1
7	324	214	51	100	153	3
average	4.804	3.710	1.019	2.915	2.474	0.168

TABLE 3. For each problem, how many contestants have got how many points

Problem 6, the grasshopper problem, turned out to be one of the hardest problems ever: only three students achieved the maximum of 7 marks for this problem, whereas 540 of the 565 contestants could not gain any mark for it. Nevertheless, if the medal cut-offs are taken as an indicator, this IMO does not seem to have been harder than previous ones. Gold medals were awarded to 49 students who scored at least 32 points, 98 students with points in the range 24–31 got silver medals and 135 students who scored at least 14 points but not more than 23 were awarded a bronze medal.

Although the IMO is a competition for individuals only, it is interesting to compare the total scores of the participating countries. This year's top teams were from China (221 points), Japan (212 points) and Russia (203 points). Ireland, with 20 points in total, finished in 89th palce. Two students, Dongyi Wei from China and Makoto Soejima from Japan, achieved the perfect score of 42 points. The detailed results and statistics can be found on the official IMO website `http://www.imo-official.org`.

6. Celebration of the 50TH Anniversary

The International Mathematical Olympiad is the oldest and largest among all international academic competitions for secondary school students. Exactly 50 years ago, in 1959, the first IMO, with 7 participating countries, was held in Romania, where at least since 1897, national mathematical competitions were organised. At the end of the 1970s, about 20 countries were regularly participating in the IMO. Since then, the number of participating countries monotonically increased to over 100 in Bremen 2009. Because there was no IMO held in 1980, the fiftieth anniversary of the first IMO coincided with the 50th IMO.

The German organisers organised a marvellous anniversary celebration, which took place on Sunday, 19th July, at the Bremen Musical Hall. As the highlight of this event, they invited six of the world's leading mathematicians, all of them also successful former IMO participants: Belá Bollobás, Timothy Gowers, Lásló Lovász, Stanislav Smirnov, Terence Tao and Jean-Christophe Yoccoz. Three of them (Gowers, Tao and Yoccoz) have been awarded a Fields Medal, which is considered to be the equivalent of the Nobel Prize for mathematicians.

Each of these six guests gave a short talk centrered around the relationship between IMO problems and mathematical research problems. During his speech at the closing ceremony, József Pelikán, the chairman of the IMO Advisory Board, gave a succinct characterisation of the two types of problems: IMO problems are like wild animals you meet in a zoo whereas dealing with mathematical research problems is more like meeting such animals in the jungle. Considering an IMO problem, you can be sure that there exists a nice solution, whereas this is not always the case for a mathematical research problem.

During two extensive breaks at the anniversary celebration, the students (and, indeed, others) surrounded the six speakers like celebrities. They queued up for autographs and took group photos with them, just as others would do in the presence of famous actors or sports heroes. On the following day, most of the six celebrities also participated in the excursion to Wangerooge, which is the most eastern of the East Frisian Islands.

The closing ceremony took place on Tuesday, 20$^{\text{th}}$ July, in the concert hall "Die Glocke" in Bremen. Two Fields Medalists were among those who handed over medals to the succesful contestants. At the end, the IMO banner was passed on to the delegation from Kazakhstan, where the 51$^{\text{st}}$ IMO will be held from 2$^{\text{nd}}$ July until 14$^{\text{th}}$ July 2010.

7. Outlook

The next countries to host the IMO will be

2010	Kazakhstan	`http://www.imo2010org.kz/`
2011	Netherlands	
2012	Argentina	

While no formal decision has been made about the location of the IMO 2013, it was announced that Coratia has expressed its interest in organising the IMO that year. Two countries have expressed their interest in holding the IMO in 2014 and 2015, details of which are to be given at the next IMO in Kazakhstan.

8. Conclusions

Comparing the Irish performance with that of other nations, it seems not to be sufficient to involve the majority of the students, with exceptional mathematical talent, in mathematical enrichment activities which do not begin until after the Junior Certificate. The youngest participant of this year's IMO, for example, was eleven years old; he was from Peru and got a bronze medal. The PRISM competition (`http://www.maths.nuigalway.ie/PRISM/`) is a promising and valuable initiative to attract younger students. It seems to be important that younger students, who were successful in this competition or whose interest in mathematics was sparked by it, should have the possibility of getting involved in guided problem solving training.

In addition, it was sufficient in Bremen to get a bronze medal by solving two problems completely. In recent years, two of the IMO problems were always from geometry. Therefore, a strong background in elementary geometry will certainly help improve the scores of our participants. A starting point for improved performance of future Irish contestants at the IMO might be to focus on a solid foundation in basic geometry, in particular for students in their Junior Cycle. Geometry is also particularly suitable in introducing interested students to mathematical problem solving. Those who really engage in this activity will experience the satisfaction of success whenever they solve a problem having spent some hours tackling it. Such satisfaction is a most important motivation in continuing to think independently about mathematical problems. Students who do not develop this internal interest in problem solving will rarely be able to win a medal at an IMO.

9. Acknowledgements

Ireland could not participate in the International Mathematical Olympiad without the continued financial support of the Department of Education and Science, which is gratefully acknowledged. Thanks to its Minister, Mr Batt O'Keeffe TD, and the members of his department, especially Eamonn Murtagh and Doreen McMorris, for their continuing help and support.

Further, we would like to thank the CEO of the State Examinations Commission, Mr Padraig McNamara, and his staff, for providing information to school principals on the top performers in Junior Certificate Mathematics 2008. This information greatly assisted us in identifying a cohort of mathematically talented students.

Also, thanks to the Royal Irish Academy, its officers, the Committee for Mathematical Sciences, and especially Gilly Clarke, for continuing support in obtaining funding.

The essential part of the preparation of the contestants is the work with the students done in the enrichment programmes at the five universities. This work is carried out for free by volunteers in their spare time. Thanks go to this year's trainers at the five Irish centres:

At UCC: Tom Carroll, Finbarr Holland, Donal Hurley, Edward Lee, Anca Mustata.

At UCD: Omran Ahmadi, Peter Clifford, Melissa Erdmann, Mark Flanagan, Marius Ghergu, Mary Hanley, Kevin Hutchinson, Raymond Kinane, Tom Laffey, Gary McGuire, Colin Wilmott.

At NUIG: Jim Cruickshank, Niall Madden, Rachel Quinlan, Jerome Sheahan and Emil Sköldberg.

At UL: Mark Burke, Eugene Gath, Bernd Kreussler, Jim Leahy, Gordon Lessells.

At NUIM: Stefan Bechtluft-Sachs, Caroline Brophy, Stephen Buckley, Peter Clifford, Katarina Domijan, Kurt Falk, David Malone, John Murray, Anthony G. O'Farrell, Lars Pforte, Adam Ralph, David Redmond, Rade Stanojevich and Richard Watson.

Thanks also to the above named universities for permitting the use of their facilities in the delivery of the enrichment programme, and especially to the University of Limerick for their continued support and hosting of the pre-olympiad training camp.

Finally, thanks to the German hosts for their excellent organisation of this year's IMO and the worthy celebration of the 50th anniversary of the International Mathematical Olympiad.

Bernd Kreussler,
Department of Mathematics
and Computer Studies,
Mary Immaculate College,
Limerick, Ireland
bernd.kreussler@mic.ul.ie

Received on 27 November 2009.

An Interview with Professor David W. Lewis

GARY MCGUIRE

1. Introduction

On 20 July 2009 I conducted an informal interview with David, who officially retired in 2009.

2. Background and Education

GMG: As a child were you interested in mathematics?

DL: To the extent that a child can be, yes I was. I was also interested in things like football and sports and so on, and I wasn't interested in mathematics in an academic sense until late in secondary school. In primary school I liked mathematics, and I was always good at mathematics. In secondary school I was even more interested in physics, and astronomy. If I had any ideas of studying a subject it was physics or astronomy. It was only in the last year of my secondary school that I realized I was better at mathematics than anything else. That's when I decided to study it at university.

GMG: What part of the world was this?

DL: I was born and grew up in Douglas, Isle of Man. I went to school there, both primary and secondary. My secondary school was called Douglas High School.

GMG: Did you have any good teachers in school?

DL: Yes, one teacher I had for the last two years, named Henry Corlett, he was very very good. He had a degree from Cambridge University, and he was excellent. Before that I had a good teacher named Ted Kelly, whose nickname was The Mekon. You probably don't remember a comic called The Eagle, which had a character called Dan Dare. There was another character called the Mekon, the enemy of Dan Dare, who had a big bald head. Ted Kelly had a big bald head, so we called him the Mekon. But he was a very good teacher. Henry Corlett was a tremendous teacher, I met him a few years ago in the Isle of Man. He is now retired of course, but still going strong.

GMG: How did you decide what to study at university?

DL: In sixth form, we were divided into two streams, arts and science. In my day, this was the 1960s, science was very much the "in" subject. Maybe science is not so popular nowadays, but certainly in those days science was a subject people would go for. Whereas, for things like medicine, well I knew almost nobody who did medicine. Medicine was only for doctors' sons and daughters. Also very few people did commerce and law.

The teachers seemed to advise us that way. I had a friend who wanted to do engineering, but I remember a teacher advising him not to. But he wanted to do it, so he ignored them and went to Liverpool to study engineering, and he was very happy. It seemed that teachers looked down on the professional subjects at that time.

GMG: You went to university in Liverpool. Did many students from the Isle of Man go there?

DL: They would have done. In those days certainly Liverpool was the closest point, and was the easiest point to get to. There was a boat twice a day between Douglas and Liverpool, which had been running for at least the previous hundred years. Many people from the Isle of Man went to Liverpool to work and live. At my time, most people going to university would go to Liverpool university. That probably changed over the next twenty or thirty years, as travel became easier. Most people went to a university in the UK. I can recall a couple of people going to TCD, but I can't recall anyone going to UCD.

GMG: During your undergraduate days, did you study mostly mathematics?

DL: Yes, but when I started I was interested in studying physics, and also applied maths. I did physics for two years. I was quite interested in applied maths, however I found that I didn't like it very much, but I liked the pure maths very much. It had a kind of cleanness and purity to it that the applied maths didn't seem to possess. I found that applied maths made assumptions that were hardly justifiable, which didn't appeal to me.

GMG: Did you have any good lecturers at Liverpool?

DL: Yes, one man called Michael (M.C.R.) Butler, was very good, he was an Australian algebraist. He was married to an applied mathematician, Sheila Brenner. I was very impressed by Butler. Another good person was Geoff Horrocks, he never used any notes and gave these superb lectures, never making a mistake. He got the professorship in Newcastle afterwards.

3. Early Mathematical Career

GMG: You began your PhD in Liverpool, I believe.

DL: Yes, after I finished my undergraduate degree at Liverpool I began a PhD under Terry (C.T.C.) Wall. The topic was surgery obstruction groups, however it was quite algebraic and had a lot to do with Hermitian forms and quadratic forms. I was trying to do some topological aspects to start with, but I found that I was more suited to the algebraic aspects, so I became an algebraist instead of a topologist. Most of Wall's students found him hard to understand, including me. He wasn't the world's greatest expositor.

In those days at Liverpool there was a tremendous emphasis on topology. Somebody seemed to believe that topology was the "in" subject of the 1960s. They appointed a lot of topologists to the staff, both algebraic and differential. We were all encouraged to do topology.

GMG: How did you come to take a position at UCD?

DL: My PhD funding ended in 1968 and I needed a job, so I looked in the newspapers and I saw an ad for a job in UCD. I applied, got invited for an interview, and then I was offered a position, which I accepted. I was interviewed by Dick Timoney, Phil Gormley, Maurice Kennedy, Tony Christofides, and Tommy Nevin who was dean of science. I remember that I got the impression that Dick Timoney was head, but in fact it was Phil Gormley who was head.

I had visited Ireland before, coming from the Isle of Man you are more or less half-way between Ireland and England. When I was in Liverpool some people used to accuse me of being Irish, and when I came to Dublin people used to accuse me of being English!

Two other people were appointed in mathematics in UCD that same year, Tom Laffey and David Tipple. This was quite an increase in numbers in mathematics, because before that there was only Phil Gormley, Dick Timoney, Maurice Kennedy, Stephen O'Brien, Fergus Gaines, and Paddy McNeice who used to teach the engineers.

I was appointed as an assistant lecturer. There was even a grade below that, which was called assistant. We had no contract, and we were appointed from year to year. There was no guarantee you would be appointed the following year. You just had to trust that you would. Dick Timoney told us not to worry, that we would be appointed. You had to become statutory lecturer, which was an NUI position, before you were safe. To be dismissed as a statutory lecturer required a court order signed by three high court judges!

GMG: Phil Gormley was head at that time.

DL: Yes, it seemed like Dick Timoney was head because he did much of the work, but Gormley was head. Phil Gormley was a man of few words, but he chose his words very carefully. His instructions for marking exams were very interesting, he gave a simple four word instruction: "No Marks For Rubbish." Nowadays we have careful marking schemes, so his instructions would not be considered best practice these days.

Gormley was interested in classical analysis, and he used to say "let's do some real mathematics." I don't know why he appointed people in algebra, because he wouldn't have considered algebra as real mathematics. He came from the Derry area, and went to St. Columb's College there. His two brothers Tom Gormley and Paddy Gormley were nationalist members of parliament at Stormont.

Dick Timoney was very efficient at running the department. He had been given a personal chair in the 1960s, so he was also professor of mathematics.

The first year I was at UCD Phil Gormley asked me to attend a committee on the Leaving Certificate mathematics syllabus. I didn't know anything about the Leaving Cert at the time, not having grown up in Ireland. It turned out that they had already developed a syllabus, and Gormley was fed up with them because he didn't agree with what they were doing, so he asked me to go instead of him. At least we got a free dinner in Wynn's hotel. A lot of transformation geometry was on the syllabus then, which has mostly been removed now.

GMG: Maurice Kennedy was also a professor?

DL: He was an associate professor, he was appointed in the 1960s to that grade, as was Stephen O'Brien. The famous story in UCD is that the next round of promotions to associate professor did not happen until the 1980s! So anyone who missed out on promotion to associate professor in the 1960s had to wait about 18 years for another chance.

GMG: Where was your first office?

DL: My first office was in Earlsfort Terrace. I had to travel to Belfield to lecture to first science, on Saturday mornings. In about 1970 the arts building (now called the Newman Building) was opened and we were given offices there, on the second floor of block F. We remained there until about 2002 when we were moved to the Science buildings.

GMG: So you ended up at the third point of a triangle.

DL: That's right, I have lived in three places, Douglas, Liverpool, and Dublin, which form a nice triangle.

4. Later Mathematical Career

GMG: How was research going in the 1970s?

DL: I had not yet finished my PhD when I got the UCD job, so I was working to finish that. I published a couple of papers before I actually submitted my thesis. Tom Laffey and Fergus Gaines started an algebra seminar in UCD, which was interesting and helpful. Tom started to work with Fergus on matrix theory questions. The older people in the department didn't do research, at that time.

GMG: What were your early papers on?

DL: I wrote about Hermitian forms, defining a signature on some manifolds. It was algebra with applications to manifolds, and topology. I wrote a couple of papers on that kind of stuff. I became much more algebraic after that, working on the algebraic theory of quadratic forms, central simple algebras and other kinds of algebras.

GMG: Who were the big influences on your career?

DL: One big influence on me was David K. Harrison, who visited Dublin in the early 1970s. He gave a series of lectures on quadratic forms and the work of Pfister and so on. I didn't know this work before then, but I got interested in that area after his lectures. It was close to my previous work, but more algebraic.

Then Lam's book came out after that, and that was also a big influence. Lam refers to lectures at the University of Kentucky by Harrison, so Harrison seems to have influenced the area although he didn't publish much in the area. He ended up in Eugene, Oregon, I think.

Locally, Tom Laffey was a good influence on me. He was always available to talk to about algebra, or indeed, about any area of mathematics. Farther afield, other people I started to collaborate with in the 1980s were an influence on me. Working with Jean-Pierre Tignol in Louvain-la-Neuve was very good for me. He is a great guy to work with. Also Jan van Geel in Ghent has been a great collaborator for me. This connection led to an Erasmus programme.

GMG: Could you tell me more about this Erasmus exchange.

DL: We set up an Erasmus programme between Ghent and UCD. Mainly it was myself and van Geel, but also Tom Laffey and Fergus Gaines were involved. Students from either university could visit the other university, for a year, and take some courses. Mostly it seemed that Belgian students came to UCD, rather than the other way around. I'm not sure why, but perhaps it was the language. The exchange still goes on, in fact. The first student to visit here was in 1989, and there was one here in 2009. Two of the students became PhD students of mine. So Belgium has been a big influence on me.

GMG: You were also involved in a European network.

DL: I got involved with people in several European countries, France, Belgium, Germany, and so on. We applied for and got money to set up an EU research training network, it would fund postdocs and PhD students to visit another university. Staff could also visit other places. It was very good and beneficial. The idea of the EU was that the leaders of tomorrow, whether they be scientific or industrial or political, if they got together as students and got to know each other they would be more friendly in the future and it would make the EU stronger. They put money into it, in the 1990s, which they did not do in the 1970s or 1980s.

5. Mathematics In General

GMG: How do you think mathematics has changed over your career?

DL: In general, I think people go more for the applications of mathematics now than they did. In my day in the 1960s people were more into pure maths. Obviously research funding is much more important now than it was then, and people go more for applied or applicable mathematics. There are more mathematicians around, more mathematics is being done, of all kinds. But there is a swing towards applied mathematics.

GMG: Is that good or bad?

DL: There is certainly room for both. You need to have a reservoir of pure mathematics in order to have some mathematics that can be applied at a later stage. If you don't have that reservoir, you've got nothing to apply. So they both play an important role.

Computer science has also had an effect on what people study. I remember back in the 1970s when a man named McConalogue was head of computer science here. Computer science was not given good facilities in those days, and the entire staff were located in portacabins in a car park. Now they have their own building. How things have changed!

GMG: How do you do research, do you work whenever you get an idea?

DL: I don't work in the middle of the night, if I get an idea then. I would wait until the morning to work on it. Sometimes I have woken up in the morning with a good idea. Ideas can come at strange times. I remember one idea came to me when I was laying a carpet.

GMG: Did you finish laying the carpet?

DL: Oh yes, I finished laying the carpet before working on that idea. My wife would not have accepted the excuse that I had just had a brilliant idea so I was not going to lay the carpet anymore!

GMG: How should Ireland develop in mathematics?

DL: I think Ireland should certainly continue to do research. It's a shame that not much money is going into basic research at present. Given the current financial status, it doesn't seem that the mathematical models were much use, does it. I think human behaviour is far too complicated to predict. Anyway, Ireland is a small country so we can't be expert in everything, there aren't enough mathematicians to do that. We have to specialize a bit.

Nowadays there is a lot more collaboration with people abroad. The old slogan for the navy used to be "join the navy and see the world" but I think we could say "do mathematics and see the world." I have been able to travel to many countries and continents. Mathematics is sort of a universal language, so you can talk to people all over the world. You can meet someone you've never heard of, in a place you've never heard of, and you can find that they have read some paper you wrote. It's kind of satisfying, in a way.

GMG: How has UCD changed over your career?

DL: When I first started at UCD I had the naive belief that the administration was there just to ensure that the academic work, teaching and research, went as smoothly as possible. I realized that some administrators believe that the administration exists for its own sake, and not for the academic work. There was a dispute in the library in the early 1970s and there was a strike with pickets, and the university was closed for a week. The registrar at the time was Tom Murphy, who went on to be president. I still remember him saying on the television that "it's only the academic work that has been affected by this strike. Normal administration is going on as usual." That made me realize that the administration goes on and has nothing to do with the academic work.

There were many fewer administrators in the early days, than there are now. I'm not sure that things work any better nowadays. Of course there has to be a certain amount of administration, but I used to think that a university was about teaching and research.

GMG: Have you any thoughts on the Mathematical Olympiad?

DL: I think it's good and it helps teach students to solve problems. The problems are very hard, and usually require some cunning and clever trick, and I'm not sure this is a good thing. It is not necessarily representative of mathematics. You can ask if it is the right way to learn mathematics. Students who come to UCD having attended olympiad training think that everything can be done by some clever trick, but sometimes a student has to learn a theory, build up knowledge, and apply this knowledge. I have nothing against the olympiad, however I have not got involved in it.

GMG: Do you think mathematicians need to be better at PR and salesmanship?

DL: Yes, I think this is important. I think that we generally are not very good at PR, we don't seem to be persuading any more students to do mathematics than we ever did. I'm not sure why that is, perhaps it is a hard subject and people go for easier options. Perhaps all the medical soaps depict being a doctor as glamorous, and this makes people do medicine, so perhaps we need a mathematical soap on television. We aren't good at PR and it's important that we get some good PR. Mathematicians tend to be modest and not good at selling the subject, so being a mathematician and being good at PR don't seem to go together. It's not in our nature to blow our own trumpets. We should have people doing it for us if we can't do it ourselves.

GMG: David, thank you very much.

Gary McGuire,
School of Mathematical Sciences,
University College Dublin,
Dublin 4, Ireland
gary.mcguire@ucd.ie

Received on 2 September 2009.

An Interview with Professor Sean Dineen

GARY MCGUIRE

1. Introduction

On 10 November 2009 I conducted an informal interview with Sean, who officially retired in 2009.

2. Background and Education

GMG: As a child were you interested in mathematics?

SD: I wasn't top of my class, but I had some good teachers. We had to do a lot of mental arithmetic in school. Now, it wasn't mental arithmetic like you have it. We had pounds, shillings and pence, we had hundredweights, tonnes, stones and pounds of weight, and so on. We had to work out these things in our head, and the teacher used to give sixpence to whoever got the right answer first. And I can tell you, the competitive element really pushed us on because we had no pocket money in those days.

I was always good at maths in school, and I never had a problem, but I wasn't exceptional. In my class in primary school was Donal Hurley, who went on to get his PhD from Cornell in mathematics. So two PhDs in mathematics came from that small class in Clonakilty, Co. Cork.

Sixth and seventh class were together in the school. Sixth class used to do algebra, working with polynomials in x. Seventh class, for those who would not go on to secondary school, did geometry and proved theorems. There was one student in that class who got full marks in all subjects in the primary cert. It was quite competitive there. Clonakility was good in sport too and that was also competitive.

My parents were teachers, and they started the boys secondary school in Clonakilty, in 1938. There was no boys secondary school before that. My father died in 1953 and my mother continued to run the school. At the time Clonakilty was part of the diocese of Ross, which is now part of Cork and Ross, and that apparently had something to do with the fact that there were very few boys secondary schools in the area.

GMG: You went to University College Cork for your undergraduate studies.

SD: By the time I got to my Leaving Certificate year, I was certain that I wanted to study mathematics in university. I entered UCC in 1961. There were about 14 in the class doing honours mathematics. In those days you had to qualify for honours at the end of first year in order to proceed at honours level. I remember I was afraid that I would qualify for doing honours in physics or chemistry, and that I would not qualify in maths. But I did qualify in maths. I really liked chemistry, and I qualified in chemistry also.

GMG: Was attendance taken?

SD: Yes. We all had an assigned seat, which was done alphabetically. The porter came in to every lecture and took a note of which seats were empty. There was a strict rule that if you didn't attend a certain percentage of lectures, you weren't allowed sit the exam.

GMG: Were there good students in your class?

SD: Oh yes, I think about half the class of 14 went on to do a PhD. Tony Hollingsworth, who was scientific director of the European Meteorological Centre in Reading, and Brendan MacWilliams, another meteorologist, who used to write a column in the Irish Times, were in my class. Niall Horgan, who did very well in applied mathematics, was also there. It was very competitive.

The dean would announce the exam results orally, in the quadrangle, and then post them on a noticeboard. The next day they would be published in the Cork Examiner, in order of merit. There were lots of McCarthy's, of course, so if you were McCarthy you were never sure of your results when they were read out. You had to wait until the official posting.

There was a great atmosphere in the class. During the year, we used to cover the ceramic tables in the restaurant with mathematics, and we could wipe it off afterwards. We drank coffee and smoked, everyone smoked in those days. In third year the exams were in September, so we had the summer to study. We had no email of course, and phone calls were expensive. So we used to write a lot of letters to each other about solving the problems that were set.

We set up the student Mathematical Society in our second year. We visited the other student societies to see how things were operated. In my first year we had Finbarr Holland for tutorials, he was a masters student. And Mick Mortell, later president of UCC, was a tutor in mathematical physics. Paddy Barry came back when I was in third year. For statistics we had Tagdh Carey, who also went on to be president of UCC. You either did maths and statistics, or maths and maths physics. I took the statistics option.

During my first year the professor of mathematics, Paddy Kennedy, was away, so we had Siobhan O'Shea teaching us. Kennedy came back in Easter of my first year, and he lectured us then. He was a brilliant lecturer. He insisted on each student wearing the gown, you were thrown out if you didn't have your gown.

GMG: Students had to wear gowns?

SD: Oh yes, each student had their own gown. When you registered at the start of first year you had to buy a gown. There were no applications through the CAO in those days. You just queued up in the arts line, or the science line, or the pre-med line, or whatever, and you registered. And you had to buy a gown, and show your receipt. Then you went to the bursar and paid your fees. And then the dean of the faculty shook hands with you. The fees were 65 pounds for the year.

GMG: Did you do a masters degree in UCC?

SD: Yes I did an MSc. During that year I lectured to commerce, 250 students, and to third year honours. I was very involved with student politics that year and I organized all the student dances. I had to deal with a new MSc course, where you took two majors, instead of one major and two minors. I took real analysis, I was the only one. I think Maurice Kennedy from UCD was the one who designed the course. He took all the best books, like Kelly, Taylor, Halmos, Loomis, and called them the real analysis course. They were great books, and still are, but I had no preparation for that level and I struggled with those books all year. The department in UCC specialized in complex analysis.

One curious thing is that Boole was never mentioned in UCC. I remember that it was in my final year as a PhD student in America when I found out that Boole had been a professor in Cork. No-one mentioned it. And we didn't even know about Hamilton, we did the Cayley–Hamilton theorem but no-one mentioned that Hamilton was from Dublin.

GMG: What books had a big impression on you?

SD: I remember Cantor's little book on transfinite numbers as a great book. It was in the library and Paddy Kennedy mentioned that if you want to know more about infinite sets, go and look at this book. So I went to the library and read it. I'm not sure if students nowadays look up books that are outside the prescribed texts.

There was a book on Taylor series by Dienes, and also Bromwich's book on infinite series, that I remember studying for a long time on my own. I thought they were fantastic, and were useful later to me in my research. Bromwich actually lectured in Galway for a year, and later I got to know his grandson who was an engineer in Dublin during the 70's.

I also subscribed to the Oxford journal in my final year and tried to read the articles. But nobody told us to do that, we just explored by ourselves.

3. Early Mathematical Career

GMG: You then went to the USA for your PhD?

SD: Yes, I was the first student in pure mathematics from UCC to go to the USA for my PhD. There were a few in applied mathematics before me, like Paddy Quinlan, Mick Mortell. I didn't want to go to England because the financial support was uncertain. I applied to 4 universities in the USA, Maryland, Syracuse, Princeton, and Stanford. Stanford said I was too late. Princeton said I could enroll but that I would not receive financial support. It came down to Maryland or Syracuse, both of whom offered me a teaching assistantship, and I chose Maryland. I remember there was no-one to tell me which to pick. I had nothing to go on. So I chose Maryland because it was larger, it had 40,000 students, and Syracuse had 7,000 students. I reckoned you could get on better with a larger choice. Also it was nearer the equator and nearer the sea!

GMG: Were there a lot of students at Maryland?

SD: There were about 250 graduate students in mathematics! About 100 were doing the PhD, and the rest were doing a masters. This was the 1960s and there was a big expansion in Washington DC due to the space programme. There were lots of people from NASA, the Goddard space centre and other government institutes, who were sitting in on maths courses. There must have been another 250 people taking advanced mathematics courses! I remember in functional analysis, the capacity was 40 but there were 80 people who wanted to take it, so there was another class of 40 going on simultaneously.

There were lots of jobs in various colleges in the area teaching maths, which were well paid. So people with masters degrees got good jobs afterwards.

There was a lot of grant money too. I remember one professor asked me at the end of first year if I wanted a summer job, paid from his grant. I said no, I am going back to Ireland for the summer. He said that's no problem, just sign here.

I remember the preliminary PhD exams, because they were on my birthday. You had to take four exams in one day, in algebra, analysis, topology, and probability. We were trained in Ireland to do well in comprehensive exams like that, and I had no problems. Americans found them more difficult. I had more trouble with the language exam, because we had to do two languages. I already knew some German, but I had trouble with French. I eventually passed it.

Also in Maryland I was a fellow student of a grandson of George Salmon.

GMG: You established a Brazilian connection while in Maryland.

SD: My first day in America I met a Brazilian guy named Thomas Aloysius Walsh Dwyer. That might sound like an Irish name to you, but he was Brazilian. He became a great friend of mine, and we had the same initial advisor. I had the travelling studentship, so that gave me a bit of freedom. Tom told me about Leopoldo Nachbin, who was at IMPA (Instituto Matematica Pura e Aplicada) in Rio de Janeiro, and had written a book on approximation theory, and another one on the Haar Integral. I thought his books were great, so simple and clear. So I looked at his research, and I liked it too. I wondered if I could transfer to Rochester, where he had a joint position, the other half of his position being at IMPA. He said that I should do all my course requirements in Maryland, and he arranged that John Horvath would be my official supervisor in Maryland, and that I would go to Rio for my research. So Nachbin was my unofficial supervisor. He had other students in Brazil, like Soo-Bong Chae who was from Korea, Richard Aron and Phil Boland.

Nachbin had spent time in Paris, and he knew Henri Cartan and many other famous mathematicians. One time on sabbatical in Paris, Cartan arranged that Nachbin rent an apartment in the same block as Cartan. Nachbin's maid was from Rio and Cartan's maid was from Portugal, so of course the maids talked in Portuguese. Nachbin overheard them one time. Nachbin's maid asked Cartan's maid: what does your boss do? Cartan's maid answered, I don't know but he can't be very good at it, because he's always inside in his room tearing up paper!

Lots of well known mathematicians visited Rio, so there was a great mathematical atmoshere there. I took a course from I.N. Herstein, the algebraist. Laurent Schwartz and Francois Treves were there also. There was an excellent social atmosphere.

GMG: Did they pay you well in Brazil?

SD: Yes, they were interested in making IMPA a world class institute, so they paid quite good salaries to all staff, including the PhD students. I was never as well off, in fact, as I was down there. Because I was receiving the studentship from the NUI, I wrote to let them know that I was getting extra money for teaching, and I asked them if that was allowed. I believe it was discussed at a meeting of the senate of the NUI, because technically it was not allowed, but in the end they let me go ahead.

We had a great time there, we lived in Ipanema, near the beach. Every Saturday the PhD students went to the beach and we had a party afterwards.

GMG: What did you do after your PhD?

SD: I spent one year in Johns Hopkins, as an instructor. There were no positions in Ireland. Then I applied for a scholarship at the Dublin Institute for Advanced Study, which I got. So I was there for two years, 1970-1972. In the middle of my first year a job came up in UCD, which I applied for. The problem was that DIAS wanted a decision for my second year sooner than UCD could confirm the position, so I said to UCD that I had to take the DIAS post for another year. UCD agreed to take my application as if it were for the following year, so I spent another year (my second) at DIAS and then I got the UCD position which I took. I have been there ever since.

GMG: How was research in the 1970s?

SD: During the early 1970s, Maurice Kennedy (associate professor at UCD, and afterwards registrar) decided that he wanted research done, but that teaching was the top priority. Dick Timoney ran the department. Maurice felt that we should do research at night, in our spare time. He also wanted everyone to be included. So Maurice got us all to buy a book which was on operator theory with an algebraic slant, Bonsall and Duncan on numerical range. He organized for everyone to come to a seminar at 8pm in F209 (in the UCD Arts Block) on a weeknight. Afterwards we retired to McCluskey's pub in Donnybrook. That's how it was done. That started the year I was in DIAS, as far as I remember.

That year another job came up in UCD, and Phil Boland applied and got it. He was another student of Nachbin. Then Richard Aron got a job in TCD, so there were three of us here in the one area. Then Nachbin started sending us his PhD students from Rochester. We ran seminars on various things. People like Lelong and Stein had students, in Europe, and they started to come here. So in 1973-74 we had an international year in infinite-dimensional complex analysis, where we ran our seminar and had many visitors.

We had no funding whatsoever at that time. We had a travel grant from the university, which was very helpful to us.

4. Later Mathematical Career

GMG: Have you stayed with infinite-dimensional complex analysis for your whole career?

SD: Yes, but there have been many different strands to it. In the 1970s it was very topological, locally convex spaces, pseudo-convexity, holomorphic convexity, analytic continuation and things like that. At the end of the 1970s Phil Boland moved into statistics, Richard Aron went permanently to Kent State, so that stream had sort of finished. But if you want to stay active as a research mathematician, you have to reinvent yourself regularly. In 1978 I went back to Rio for six months and met Wilhelm Kaup, who is one of three brothers who all worked in complex analysis, believe it or not. I was there to write my first book. Kaup gave some lectures on bounded symmetric domains, Lie groups, Lie algebras, complex differential geometry, vector fields, all in infinite dimensional spaces. This was quite new to me and I found it very interesting. The next month I visited Kent State and I gave a talk on this topic, to help myself learn it. There was no-one else in UCD working on this topic. Richard Timoney had come back to TCD after completing his PhD with Lee Rubel in Urbana-Champaign, and we decided that the two of us would study this area. During the 1980s then, Richard and I concentrated on bounded symmetric domains.

In the 1990s I got interested in spectral theory, and I started working with Robin Harte. But all the areas relate to each other, so some extent, and you start to pull them together a bit.

GMG: How was the funding situation through your career?

SD: There was none in the 1970s, 1980s or 1990s even. In UCD in the 1980s, when I was head of department, we wrote a book of problems and we sold it to all the undergraduates. All homeworks were assigned from this book. The money from this paid for visitors, and visits, and conferences here, and it really kept the research going.

There never was any funding really for mathematics in Ireland until the late 1990s. As a profession we don't actually need much funding, that's one of our problems. Our students and postdocs need funding, but we don't need much. And we are considered to be one of the sciences, so the higher-ups think that if you are not bringing in funding, then you must be no good! We are not treated as an arts subject in that sense. We are like an arts subject in many ways, we don't need equipment. You can't verify a mathematical result by experiment, so in that sense we are not a science. Also, we build on previous work, we don't discard theories based on experiment. In mathematics, what the Greeks did is still there, and we build on it. By its nature, mathematics has to be a unit and has to integrate itself together. It's very important that different areas be related to each other. The basic theorems are still there, the concept of a prime number is still the same as it ever was. We don't invent a new kind of prime number that is better than the old one, although we may expand the concept. In some sciences, you have theories which get replaced by different theories after a while. That doesn't happen in mathematics.

GMG: Has mathematics, and doing mathematics, changed over the last forty years or so?

SD: Back in the 1960s, there were many people who were intrinsically interested in the subject. I spoke earlier of 80 people doing functional analysis at Maryland, and things like that. People were studying mathematics because they were interested in it. What I think is happening now is that, with all the grants and so on, it's turning into a bit of a business. What's still good is that there are original mathematicians leading the field. Mathematics will always be an individual effort. The subject is becoming a business, and people have to get grants and they have to publish. For thirty years, I didn't have to publish, and what kept me publishing was that I was part of a team, and I wanted to do good for my team.

There is a certain amount of sharpening pencils these days. There is a lot of technical mathematics being produced now, because people have to publish. On the other hand, there is fantastic mathematics being produced as well, like Fermat's Last Theorem, and the Poincaré Conjecture. And there are people like Terry Tao, Gilles Pisier, Tim Gowers, who are doing great things. So you have a very high standard of mathematics appearing, no doubt about that.

Compared to 30 and 40 years ago, written articles don't give as much of a lead in. In the old days, people used to give an introduction for people not in the precise area. I think that has changed. Another change is that in the 1970s there were a lot of conference proceedings, and some very good articles were put in conference proceedings. But nowadays such proceedings don't count for citations and they are being sidelined and people don't want to publish there. People respond to the environment they find themselves in. So if the citation index becomes important, people will respond to that. I think that a time will come when we will realize that all this is rubbish. The stuff that is of value will be kept going.

GMG: How important is teaching to you?

SD: Teaching is very important. I remember my advisor Nachbin saying to me once, you could prove Fermat's Last Theorem in your bathroom, but if you didn't tell anyone, nobody would know. You have to communicate it. You communicate with teaching, and if you are challenged with a question, you should be able to answer it.

I think the teaching of first year students is very important. We should be quite tough, but we should also spend a lot of time with first year students, trying to have as many small classes as we can. We have students who become enthused by maths, when they are pushed. One economics and finance student recently was very enthused by maths in first year, and now he is doing the higher diploma in maths. There is skill in teaching first year students. In first year, you can challenge them, and set the standard for later years, and this applies to students who are not studying a maths degree but are doing a degree with a large maths component, like engineering or economics and finance. You change them from thinking immediately that they can or can't do this problem, to thinking that it will take 15 minutes, and perhaps longer, before they know whether they can or can't do it.

5. General Questions

GMG: What is your feeling about Irish undergraduates doing a PhD in Ireland versus doing it abroad?

SD: I'm not dogmatic about this, I think the mixture we have now with about half our PhD students from Ireland and half from abroad is a good mix. I do believe that spending some time in another university is a good thing. Going abroad for a PhD, in a new atmosphere, is of course a good experience. Staying in Ireland has the advantage of keeping you in touch with what is going on in Ireland.

If people go abroad for years, they can get into new areas and come back to Ireland to find that there is nobody in Ireland they can talk to. It can be difficult to come home to work in an area that nobody locally is interested in.

GMG: Do you think the Erasmus programme is good ?

SD: The Erasmus exchange programme has been good for UCD, and for Ireland. It's something I was very involved in. We had 20 or 25 of the brightest students in Europe here, going through UCD. They were very good for our students, they lifted the level. I used to get something like 30 quid for each student. When it was all put together, we had enough for an orientation week at the beginning and this created a great bond between the students. Then UCD said that we had to have the same number of students going in and out, which was bad for us because we have few students doing mathematics and only a small number of those were willing to go abroad. Quotas like that just don't work. But I think it was good for Irish mathematics, and for UCD, to have Erasmus students going through.

GMG: How have universities changed over your career?

SD: Mathematically we have got a lot more professional. Almost everyone is publishing papers regularly, and we have regular seminars. We have PhD students and postdocs. In UCD we have been combined with Mathematical Physics and Statistics into one school. In the 1990s we were practically the last subject to change from having pass and honours degrees, to only honours degrees. We had to change. We came up with the Mathematical Studies degree in arts, and honours mathematics basically moved to science. Although, until recently you could still do it through arts, which I thought was very important and I regret that being taken out. Mathematical studies became an important degree for teacher training, and the best students in that group can go on and do the diploma and masters.

In the past an academic who had an idea for a project, say something that would be good for students, could get others on board and discuss it at a faculty meeting. Nowadays, a lot of initiatives are coming from the top, like getting more overseas students. These are based on financial considerations, or political considerations, or spin, rather than on their intrinsic merit. At faculties, if you had an idea you had to propose it and defend it in front of your colleagues, and if there was something wrong with it, that would come out at the meeting. You went to the faculty for academic reasons. Then, for the financial implementation you went to the bursar.

I think getting rid of faculties was a mistake. This was where academic decisions were made. Every year some changes were made, they were on the table and debated and voted on. One other advantage was that you met people from other departments and faculties. Nowadays it is difficult to meet colleagues from other schools. This is sad, because then the beaurocrats control the full flow of information. They are interested in perception and spin. I think it will come around again, they will see that it isn't working. It's important that young lecturers coming in know that there is an alternative way.

Previously it was very straightforward, you were a researcher and a teacher. As head of department, you were supposed to lead as a professional mathematician, as a researcher and teacher. The amount of time spent on administration was quite small. This has now changed, and being head of school requires most of your time on administration.

GMG: What are your thoughts on the Mathematical Olympiad?

SD: I think it's great, although I haven't been involved in it. There isn't time to do everything. I think Fergus Gaines and Tom Laffey did great work. As I said earlier, anything that is competitive, as long as it doesn't go overboard, is good. A little bit of competition is good. For example, when I was head, I said to the commerce faculty that we would allow three students to do honours mathematics. Admission would be by interview. So I then had twenty commerce students lined up outside my office trying to be one of the three. If I had said twenty could do it, I bet there would not have been the same level of interest!

GMG: Have you any thoughts about the numbers of school leavers going into mathematics?

SD: There are a number of factors. The points system has really distorted everything. There is a whole dynamic about the Leaving Cert in this country. For instance, the media play a big role. If there is a question on the Leaving Cert that is any way unexpected, there will be an outcry in the media, and they almost expect a commission to be set up to find out what went "wrong". That's not good.

Secondary school teachers come under pressure from the parents to produce points for their children, so teachers are recommending subjects that bring in high points. If it's needlework that will get them into medicine, then they will recommend needlework.

Another thing is that school leavers who are good at maths are being channelled into other courses, usually courses that require high points. The number of courses that demand honours maths in the Leaving Cert is increasing, so those courses are taking some of the honours maths students, whereas the number of students taking honours maths in the Leaving Cert has remained relatively constant at about 15%.

GMG: Should mathematicians be better at public relations?

SD: Mathematics is not very clever at selling itself as a profession. We should have a more visible presence. Not every maths student can have an academic career, but someone with a degree in mathematics is well trained scientifically. However, they still have to convince an employer that they are worth hiring. There are a lot of career options available after a degree in maths or statistics, and to help mathematics become more visible as a profession we should tell people about these options while they are thinking about their third level options.

I also think that mathematicians and statisticians have a new and important role to play in society, because a lot of mathematics is now being used irresponsibly in society. People are devising measures and using figures for their own purposes. Mathematicians have a political and social role in society that they never had before, and need to show students how to challenge these over-simplifications and trivializations. In a very hidden and subtle way, mathematics is being used to shape society.

GMG: Sean, thank you very much.

Gary McGuire,
School of Mathematical Sciences,
University College Dublin,
Dublin 4, Ireland
gary.mcguire@ucd.ie

Received on 2 December 2009.

Fuchs' Problem When Torsion-Free Abelian Rank-One Groups are Slender

PAVLOS TZERMIAS

Abstract. We combine Baer's classification in [Duke Math. J. **3** (1937), 68–122] of torsion-free abelian groups of rank one together with elementary properties of p-adic numbers to give a new solution to research problem 26 posed by Fuchs in his book on abelian groups in 1958.

Every group considered in this paper is abelian with its group law written additively. The Baer–Specker group $\Pi = \mathbb{Z}^{\aleph_0}$ is the direct product of countably many copies of $\mathbb{Z}$. Its extensive study in the literature has given rise to a wealth of interesting problems and a number of unexpected connections with diverse mathematical areas such as homology theories in algebraic topology, infinitary logic and various aspects of set theory. We refer the reader to [2], [3], [4], [6] and [11] for good accounts of some of these connections.

For a positive integer n, let e_n denote the element of Π whose n-th coordinate equals 1 and all its other coordinates equal 0. Following Łoś, a torsion-free group G is called slender if for every homomorphism $\phi : \Pi \to G$ we have $\phi(e_n) = 0$, for all but finitely many n. Specker ([10]) showed that $\mathbb{Z}$ is slender. Research problem 26 in Fuchs' book (page 184 in [5]) reads as follows:

Problem. *Find all slender groups of rank one.*

A partial answer to the problem was given by Łoś (see Theorem 47.3 in [5]). The problem was fully solved by Sąsiada ([9]) who in fact showed that:

Theorem 1. *A torsion-free group of cardinality less than $2^{\aleph_0}$ is slender if and only if it has no non-trivial divisible subgroups.*

2000 *Mathematics Subject Classification.* Primary 20K15; Secondary 20K25.
Key words and phrases. Rank-one groups, slender groups, Baer–Specker group.

Sąsiada's result is superseded by Nunke's characterization of all torsion-free groups which are slender (see [7] and [8]).

Theorem 2. *A torsion-free group is slender if and only if none of its subgroups is isomorphic to* Π*,* $\mathbb{Q}$ *or the* p*-adic integers* $\mathbb{Z}_p$ *for some prime* p*.*

The purpose of this short note is to give a new solution to the original problem of Fuchs. We claim no novelty for the general p-adic flavor of our arguments which is not uncommon in inquiries of this type but, as far as we can tell, a solution like ours has not appeared in the literature before. Let us first briefly recall Baer's classification ([1]) of torsion-free groups of rank one (see [5] for details).

Consider a torsion-free group G and a non-zero element $g \in G$. For a prime number p, define the height $H_p(g)$ of g at p to be the maximum integer $k \in \mathbb{N}$ such that the equation $p^k x = g$ is solvable in G (if no maximum k exists, we set $H_p(g) = \infty$). The height $H(g)$ of g is the infinite tuple

$$H(g) = (H_{p_1}(g), H_{p_2}(g), \ldots, H_{p_n}(g), \ldots),$$

where p_1, p_2, …, p_n, … is the increasing sequence of prime numbers.

Consider the set H of infinite tuples $(k_1, k_2, \ldots, k_n, \ldots)$, where $k_n \in \mathbb{N} \cup \{\infty\}$ for all n. Define a partial order $\leq$ on H as follows:

$$(k_1, k_2, \ldots, k_n, \ldots) \leq (m_1, m_2, \ldots, m_n, \ldots)$$

if $k_n \leq m_n$ for all n (where it is understood that $s \leq \infty$ for all $s \in \mathbb{N} \cup \{\infty\}$).

We also define an equivalence relation $\sim$ on H as follows:

$$(k_1, k_2, \ldots, k_n, \ldots) \sim (m_1, m_2, \ldots, m_n, \ldots)$$

provided that $k_n = m_n$ for all but finitely many n and that $k_n \neq m_n$ can happen only if neither k_n nor m_n equals ∞. An equivalence class of H under $\sim$ is called a type and the set of all types is denoted by T. It is easy to see that the partial order $\leq$ on H defined above induces a partial order (also denoted by $\leq$) on the set of types T.

Now suppose that G is of rank one and let g, g' be any two non-zero elements of G. It is not difficult to show that $H(g) \sim H(g')$. Therefore, all non-zero elements of G are of the same type, which we call the type of G and denote by $T(G)$. Every torsion-free group of rank one is isomorphic to a subgroup of $\mathbb{Q}$ and Baer showed in [1] that the set of isomorphism classes of torsion-free groups of rank one is parametrized by T via the bijective correspondence given by

$G \mapsto T(G)$. In addition, a torsion-free group G_1 of rank one is isomorphic to a subgroup of a torsion-free group G_2 of rank one if and only if $T(G_1) \leq T(G_2)$.

We now give a new solution to the original problem of Fuchs:

Theorem 3. *A torsion-free group of rank one is slender if and only if it is not isomorphic to* $\mathbb{Q}$.

Proof. One direction is standard: Let Σ denote the direct sum of countably many copies of $\mathbb{Z}$. Since $\mathbb{Q}$ is an injective $\mathbb{Z}$-module, the homomorphism $\Sigma \to \mathbb{Q}$ defined by

$$(x_1, x_2, \ldots, x_n, \ldots) \mapsto \sum_{n=1}^{\infty} x_n$$

can be lifted to a homomorphism $\phi : \Pi \to \mathbb{Q}$ such that $\phi(e_n) \neq 0$, for all n. Therefore, $\mathbb{Q}$ is not slender.

To prove the converse, let G be a torsion-free group of rank one which is not isomorphic to $\mathbb{Q}$. From the discussion of types given before, it follows that $T(G)$ can be represented by an infinite tuple $(k_1, k_2, \ldots, k_n, \ldots)$, where $k_n \neq \infty$ for at least one $n \in \mathbb{N}$. It is easy to see that this tuple is equivalent to $(k_1, k_2, \ldots, k_{n-1}, 0, k_{n+1}, \ldots)$ and that

$$(k_1, k_2, \ldots, k_{n-1}, 0, k_{n+1}, \ldots) \leq (\infty, \infty, \ldots, \infty, 0, \infty, \ldots).$$

As the latter tuple represents the type of the group $\mathbb{Z}_{(p)}$ of rational numbers with denominators not divisible by the prime $p = p_n$, we may assume that G is a subgroup of $\mathbb{Z}_{(p)}$. Since subgroups of slender groups are obviously slender, it suffices to show that $\mathbb{Z}_{(p)}$ is slender.

Let $i : \mathbb{Z}_{(p)} \to \mathbb{Z}_p$ denote the natural embedding of $\mathbb{Z}_{(p)}$ in the ring of p-adic integers. We will use the well-known fact that for $y \in \mathbb{Z}_{(p)}$ the p-adic expansion of $i(y)$ is ultimately periodic, i.e. when we write

$$i(y) = \sum_{j=0}^{\infty} a_j p^j$$

then there exist positive integers N and r such that $a_{j+r} = a_j$ for all $j \geq N$. Now let $\phi : \Pi \to \mathbb{Z}_{(p)}$ be a homomorphism such that $\phi(e_n) \neq 0$ for infinitely many n. Without loss of generality, we may assume that $\phi(e_n) \neq 0$ for all n. We can therefore find, for each n, an integer b_n such that $\phi(b_n e_n)$ is a positive integer. Let $y_n = \phi(b_n e_n)$. Then $i(y_n)$ has a finite non-zero p-adic expansion for all n. Now consider $x = (p^{c_1}b_1, p^{c_2}b_2, \ldots) \in \Pi$, where the sequence of natural numbers c_1, c_2, … has been chosen to satisfy

$$p^{c_{n+1}} > p^n \ (p^{c_n}y_n + p^{c_{n-1}}y_{n-1} + \ldots + p^{c_1}y_1)$$

for all n. Let $y = \phi(x)$. Note that for all n we get

$$y = p^{c_1}y_1 + p^{c_2}y_2 + \ldots + p^{c_n}y_n + \phi(0, 0, \ldots, 0, p^{c_{n+1}}b_{n+1}, p^{c_{n+2}}b_{n+2}, \ldots).$$

We clearly have $c_{n+1} \leq c_{n+2} \leq \ldots$, so

$$(0, 0, \ldots, p^{c_{n+1}}b_{n+1}, p^{c_{n+2}}b_{n+2}, \ldots)$$

is divisible by $p^{c_{n+1}}$ in Π. Hence,

$$y \equiv p^{c_1}y_1 + p^{c_2}y_2 + \ldots + p^{c_n}y_n \pmod{p^{c_{n+1}}}.$$

for all n. By our assumption on c_1, c_2, …, we see that the p-adic expansion of $i(y)$ is obtained by writing down the non-zero p-adic expansion of $p^{c_1}i(y_1)$, followed by at least 1 zero digit, followed by the non-zero p-adic expansion of $p^{c_2}i(y_2)$, followed by at least 2 zero digits, etc. Hence, the p-adic expansion of $i(y)$ contains arbitrarily large blocks of zero digits followed by non-zero blocks of digits and this contradicts the fact that the p-adic expansion of $i(y)$ is ultimately periodic. □

Acknowledgement. I thank Yasuyuki Kachi for asking me a question which, unbeknownst to me at the time, was equivalent to the slenderness of $\mathbb{Z}$ and provided the motivation for this work.

REFERENCES

[1] R. Baer, Abelian groups without elements of finite order, *Duke Math. J.* **3** (1937), 68–122.
[2] A. Blass and J. Irwin, Free subgroups of the Baer–Specker group, *Comm. in Algebra* **29** (2001), no. 12, 5769–5794.
[3] E. Coleman, The Baer–Specker group, *Ir. Math. Soc. Bull.* **40** (1998), 9–23.
[4] P. Eklof and A. Mekler, *Almost free modules. Set-theoretic methods*, Revised Edition, North Holland Publishing Co., Amsterdam, 2002.
[5] L. Fuchs, *Abelian groups*, Publishing House of the Hungarian Academy of Sciences, Budapest, 1958.
[6] W. Massey, How to give an exposition of the Čech–Alexander–Spanier type homology theory, *Amer. Math. Monthly* **85** (1978), no. 2, 75–83.
[7] R. Nunke, Slender groups, *Bull. Amer. Math. Soc.* **67** (1961), 274–275.
[8] R. Nunke, Slender groups, *Acta Sci. Math. (Szeged)* **23** (1962), 67–73.
[9] E. Sąsiada, Proof that every countable and reduced torsion-free Abelian group is slender, *Bull. de l'Acad. Polonaise des Sciences*, Série des Sciences Mathématiques, Astronomiques et Physiques **7** (1959), 143–144.

[10] E. Specker, Additive Gruppen von Folgen ganzer Zahlen, *Portugaliae Mathematica* **9** (1950), 131–140.
[11] E. Zeeman, On direct sums of free cycles, *J. London Math. Soc.* **30** (1955), 195–212.

Pavlos Tzermias,
Department of Mathematics,
University of Tennessee,
104 Aconda Court,
1534 Cumberland Avenue,
Knoxville, TN 37996-0612, USA
tzermias@math.utk.edu

Received on 15 December 2008.