MacCool's Proof of Napoleon's Theorem

A sequel to The MacCool/West Point [1]

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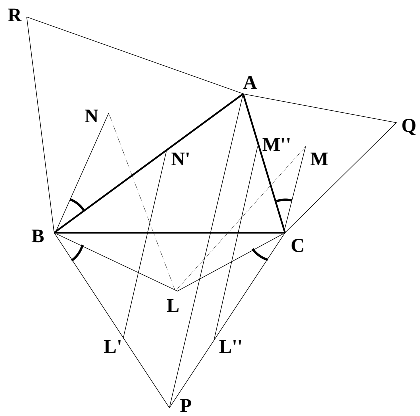
I came across this incredibly short proof in one of MacCool's notebooks. Napoleon's Theorem is one of the most often proved results in mathematics, but having scoured the World Wide Web at some length I have yet to find a proof that comes near to matching this particular one for either brevity or simplicity.

MacCool refers to equilateral triangles as *e-triangles* and he uses κ to denote the distance from a vertex of an e-triangle with unit side to its centroid. Naturally κ is a universal constant. He also treats anti-clockwise rotations as positive and clockwise rotations as negative.

Theorem 1. *If exterior e-triangles are erected on the sides of any triangle then their centroids form a fourth e-triangle.*



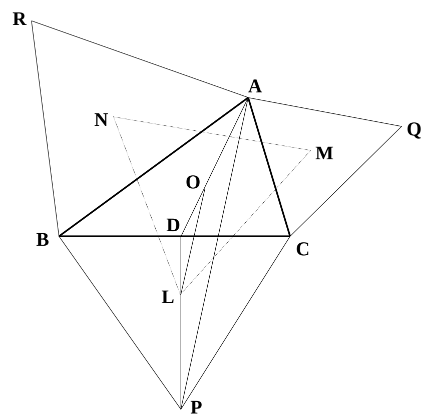
[1] Irish Math. Soc. Bulletin **57** (2006), 93–97

Proof. Let ABC be any triangle and construct the three exterior e-triangles with centroids L, M, N as shown. Rotate LN by $-30°$ about B to give $L'N'$ and LM by $+30°$ about C giving $L''M''$. Since all four marked angles are $30°$ it follows that L', N', L'', M'' will lie on BP, BA, CP, CA respectively and $\kappa = BL' : BP = BN' : BA = CL'' : CP = CM'' : CA$. Then by similarity $L'N' = \kappa AP = L''M''$ and $L'N' \parallel AP \parallel L''M''$ so $LN = LM$ and the angle between them is $30° + 30° = 60°$. Hence ΔLMN is an e-triangle. □

Theorem 1 is the classical Napoleon theorem. MacCool refers to the resultant e-triangle as the *outer triangle* to distinguish it from the *inner triangle* whose vertices are the centroids of the internally erected e-triangles.

The proof shows that each side of the outer triangle is equal to κAP. Since it could equally well have used BQ or CR instead this means $AP = BQ = CR$. The common length of these three lines is central to the next result. Also required is the fact that the centroid lies one third of the way along any median. This important property is easily deduced by observing that the medians of any triangle dissect it into six pieces of equal area.

Theorem 2. *The centroids of the outer triangle and the original triangle are coincident.*



Proof. Let D be the mid point of BC, O be the centroid of ΔABC, and L be the centroid of ΔBPC. Then $DA = 3DO$ and $DP = 3DL$ so ΔDLO and ΔDPA are similar, giving $AP \parallel OL$ and $AP = 3\,OL$.

Likewise $BQ = 3\,OM$ and $CR = 3\,ON$. Since $AP = BQ = CR$ the distances from O to the vertices of ΔLMN are equal. As ΔLMN is equilateral O must be its centroid. □

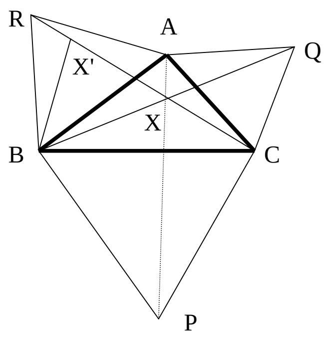
Next MacCool fixes ΔBPC and allows A to vary continuously throughout the plane. He notes that the proofs of these two theorems still apply whenever A drops below the level of BC, in effect making the angle at A reflexive and the angles at B and C negative. Essentially this is because the three e-triangles always retain their original orientation. For the orientation of an e-triangle to change under continuous deformation its area must first become zero which means that it must shrink to a point, but for the e-triangles in question this can only happen at B or C. So long as A avoids those two points no orientational changes to the e-triangles can occur.

However one subtle change does take place as A drops below BC in that the orientation of ΔABC itself changes. When that happens the e-triangles become internal rather than external. This has the following consequence.

Theorem 3. *The inner triangle is an e-triangle whose centroid coincides with the centroid of the original triangle.*

The next result gives an alternative proof that $AP = BQ = CR$. Only the "external" proof is given since the "internal" case is handled by exactly the same proof with the assumption that A lies below rather than above BC.

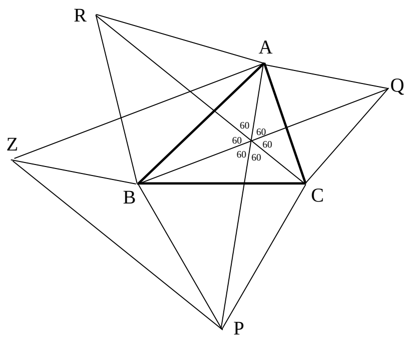
Theorem 4. *Suppose external (internal) e-triangles are erected on the sides of a given triangle. Then the three lines joining each vertex of the given triangle to the remote vertex of the opposite e-triangle are equal in length, concurrent, and cut one another at angles of* $60°$.



Proof. Let ΔABC be given and CBP, ACQ, BAR be the external e-triangles. Clearly ΔABQ is a $+60°$ rotation of ΔARC about A, ΔBCR is a $+60°$ rotation of ΔBPA about B and ΔCAP is a $+60°$ rotation of ΔCQB about C. It follows that $AP = BQ = CR$ and all angles of intersection are $60°$. To prove concurrency assume BQ and CR cut at X and construct BX' by rotating BX through $+60°$ about B as shown. Since $\angle BXR = 60°$ and $BX = BX'$ it follows that X' must lie on CR. However a rotation of the line $CX'R$ through $-60°$ about B will map $C \mapsto P$, $R \mapsto A$, and $X' \mapsto X$. Therefore A, X, and P are collinear which means that AP, BQ, CR must be concurrent. □

MacCool next studies the areas of the various triangles. He uses (UVW) to denote the *algebraic* area of ΔUVW. In other words (UVW) is equal to the area of ΔUVW when the orientation of ΔUVW is positive, and minus that value whenever the orientation is negative.

Lemma 5. *In the diagram below BPC, ACQ, and ARB are e-triangles whose mean area is Ω, and Z is constructed so that $AZBQ$ is a parallelogram. Then AZP is also an e-triangle and* $2(AZP) = 3\Omega + 3(ABC)$.



Proof. As $AZBQ$ is a parallelogram $\angle ZAP$ is alternate to an angle of $60°$ so it too is $60°$. Also $AP = BQ = AZ$ so AZP must be an e-triangle. Clearly

$(AZP) = (ABP) + (BZP) + (AZB)$ by tesselation

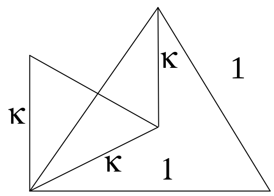
$\qquad = (ABP) + (APC) + (ABQ)$

as $(APC) = (BZP)$ and $(ABQ) = (AZB)$.

Now $(BCR) = (ABP)$ and $(BCQ) = (APC)$ and $(ARC) = (ABQ)$ therefore

$$\begin{aligned}2(AZP) &= (ABP)+(APC)+(ABQ)+(BCR)+(BCQ)+(ARC)\\ &= 3\Omega + 3(ABC).\end{aligned}$$ □

The diagram below shows two e-triangles, one with unit side and the other with side κ. Although I have found no evidence that MacCool was familiar with Pythagoras, he inferred from this diagram that $3\kappa^2 = 1$ and he deduced that the areas of the inner and outer triangles were one third the area of an e-triangle of side AP.



The area of the smaller equilateral triangle is clearly κ^2 that of the larger, from which it follows that κ must satisfy the equation :

$$3\kappa^2 = 1$$

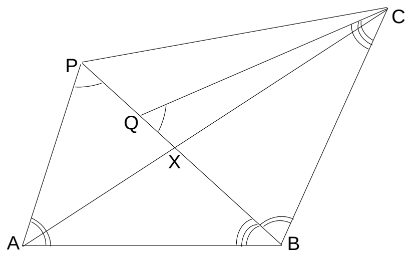
Theorem 6. *The mean area of the three e-triangles plus (minus) the area of the original triangle equals twice the area of the outer (inner) triangle.*

Proof. Let Δ be the area of the outer triangle. As explained on the previous page $(AZP) = 3\Delta$. Applying Lemma 5 now yields $2\Delta = \Omega + (ABC)$. Alternatively, if Δ is the area of the inner triangle this equation still holds, but there is a caveat. The orientations of ΔAZP and the inner triangle don't change as long as A avoids the point P where the latter shrinks to a point, but ΔABC has changed its orientation and so the value of (ABC) is now negative. Hence rewriting the equation in positive terms, $2\Delta = \Omega - (ACB)$. □

Corollary 7. *The area of the outer triangle is that of the inner triangle plus that of the original one.*

Finally MacCool presents a generalisation of Theorem 1.

Lemma 8. *Let A, B, C be non-collinear and X any point between A and C. Construct P and Q on BX such that $\angle PAB = \angle XBC$ and $\angle QCB = \angle XBA$. Then the triangles PAB and QBC are directly* *similar, moreover P and Q coincide if and only if $AX : XC = AB^2 : BC^2$.*



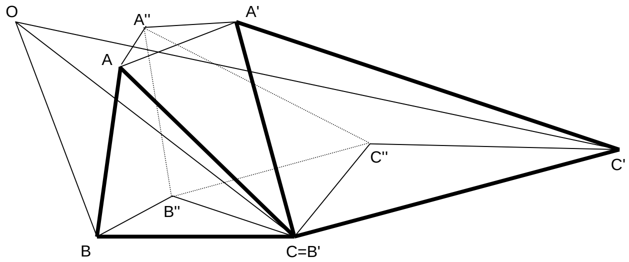
Proof. Clearly ΔPAB and ΔQBC are directly similar. Suppose $BC = \lambda AB$ and $XC = \mu AX$. Then $(QBC) = \lambda^2(PAB)$ whereas $(PBC) = \mu(PAB)$. If P and Q coincide then clearly $\mu = \lambda^2$. Conversely if $\mu = \lambda^2$ then $(PBC) = (QBC)$ so $(PQC) = 0$ which implies $P = Q$. □

Note that if AB and BC have equal length then ΔPAB and ΔPBC are similar (but not directly similar) for all points P on the bisector of $\angle ABC$. Also the lines AB and BC (extended) divide the plane into four zones, and if a point O exists such that ΔOAB and ΔOBC are directly similar then O must lie in the zone that includes the line segment AC. This leads to a key result.

Corollary 9. *If the points A, B, C are non-collinear then there exists a unique point O such that the triangles OAB and OBC are directly similar.*

Theorem 10 (Generalised Napoleon)**.** *Let ABC and $A'B'C'$ be directly similar triangles with a common vertex $C = B'$. Suppose A'', B'', C'' are chosen such that the triangles $AA'A''$, $BB'B''$, $CC'C''$ are directly similar. Then so too are the triangles $A''B''C''$ and ABC.*

Proof. There are 3 separate cases. First if B' is midway between B and C' then $ABB'A'$ is a parallelogram and the result follows easily. Otherwise if B, B', C' are collinear take O to be the point where AA' cuts BB'. Then $\Delta A'B'C'$ is a dilation of ΔABC and it is clear that $\Delta A''B''C''$ may be obtained from ΔABC by a rotation



of $\angle AOA''(= \angle BOB'' = \angle COC'')$ about O followed by a dilation of size OA''/OA. So once again the result holds. Finally if B, B', C' aren't collinear apply Corollary 9 to $\Delta BB'C'$ (aka BCC') giving the point O such that OBB' and OCC' are directly similar. Let $\theta = \angle BOB' = \angle COC'$ and $\lambda = OB'{:}OB = OC'{:}OC$. Let τ be the transformation that first rotates through the angle θ about O and then dilates by the scaling factor λ. Clearly τ preserves directly similar figures and maps $B \mapsto B', C \mapsto C'$ so as ABC and $A'B'C'$ are directly similar it must also map $A \mapsto A'$. Thus $\angle AOA' = \theta$ and $OA'{:}OA = \lambda$ from which it follows that $\Delta OAA'$ is directly similar to both $\Delta OBB'$ and $\Delta OCC'$. Then $OAA''A'$, $OBB''B'$, $OCC''C'$ are directly similar quadrilaterals so OAA'', OBB'', OCC'' are directly similar triangles. Thus $OA'' : OA = OB'' : OB = OC'' : OC = \mu$ and $\angle AOA'' = \angle BOB'' = \angle COC'' = \phi$ for some μ and ϕ. That means the quadrilateral $OA''B''C''$ may be obtained from $OABC$ by rotating it through ϕ about O and dilating the result by the scaling factor μ. Therefore $\Delta A''B''C''$ and ΔABC are directly similar. $\square$

The wheel has come full circle. To derive Napoleon's Theorem from this result take ΔABC to be equilateral and choose A'' so that $\Delta AA'A''$ is isosceles with base AA' and base angles of 30°.

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