Modules With Only Finitely Many Direct Sum Decompositions up to Isomorphism

ALBERTO FACCHINI * AND DOLORS HERBERA **

Abstract. In this paper we study *almost Krull–Schmidt modules*, that is, modules with only finitely many direct sum decompositions up to isomorphism. This is a finiteness condition on modules that has nothing to do with the other finiteness conditions usually considered in the mathematical literature, like being noetherian, or AB5*, or having finite Goldie dimension, or finite dual Goldie dimension, or finite Krull dimension. Then we compute bounds on the number and the lengths of the direct sum decompositions of modules.

1. Introduction

Two direct sum decompositions $M=M_1\oplus\cdots\oplus M_t=N_1\oplus\cdots\oplus N_s$ of a module M into finitely many direct summands $M_1,\dots,M_t$, $N_1,\dots,N_s$ are *isomorphic* if $t=s$ and there is a permutation σ of $\{1,\dots,t\}$ such that $M_i\cong N_{\sigma(i)}$ for all $i=1,\dots,t$.

Considerable attention has recently been paid to the modules for which the Krull–Schmidt Theorem holds, that is, the modules with exactly one direct sum decomposition into indecomposable submodules, up to isomorphism. The aim of this paper is to draw the reader's attention to the modules that have only a finite number

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2000 *Mathematics Subject Classification.* 16D70, 16L30, 16P99.

* Partially supported by Gruppo Nazionale Strutture Algebriche e Geometriche e loro Applicazioni of Istituto Nazionale di Alta Matematica, Italy, and by Ministero dell'Università e della Ricerca Scientifica e Tecnologica (progetto di ricerca di rilevante interesse nazionale "Nuove prospettive nella teoria degli anelli, dei moduli e dei gruppi abeliani"), Italy.

** The research of the present paper was partially supported by the DGI (Spain) and the European Regional Development Fund, jointly, through project BFM2002-01390, and by the Comissionat per Universitats i Recerca de la Generalitat de Catalunya.

of direct sum decompositions into non-zero summands, up to isomorphism. This class of modules, almost overlooked until now, was already considered by Vámos in [12]. He called these modules *almost Krull–Schmidt* modules (AKS modules for short). For instance, torsion-free abelian groups of finite rank are AKS $\mathbb{Z}$-modules [11].

In the first part of the paper we derive and develop the elementary properties of AKS modules, and give a number of examples. The property of being an AKS module is completely determined by the endomorphism ring of the module, because M_R is an AKS module if and only if in its endomorphism ring $\mathrm{End}(M_R)$ the identity can be written in only finitely many ways as a sum of pair-wise orthogonal non-zero idempotents, up to isomorphism of idempotents (Theorem 2.1). We shall call AKS rings the rings with this property. Examples of AKS modules and rings are very easy to find, and this contrasts with what happens for modules with the Krull–Schmidt property, that is, for modules having a unique direct sum decomposition into indecomposables up to isomorphism. The modules with the Krull–Schmidt property turn out to be almost exceptions. On the contrary, AKS modules are very common.

The class of AKS modules is not closed for the most common closure properties, for instance, the direct sum of two AKS modules is not necessarily AKS. A very powerful technique to construct AKS modules with strange pathologies is furnished by a result due to Bergman and Dicks (Theorem 3.1). Using Bergman and Dicks' Theorem, we show that being an AKS module, which is a finiteness condition on the module, has nothing to do with the other finiteness properties usually considered in the literature, like being noetherian, or being AB5*, or having finite Goldie dimension, or finite dual Goldie dimension, or finite Krull dimension, and so on (Examples 4.2).

The last section of the paper is devoted to computing bounds on the number and the lengths of the direct sum decompositions of a module. Bergman and Dicks' Theorem is again useful to us in order to construct examples showing that almost everything can happen.

The authors thank Professor T. Y. Lam for discovering a serious error in a previous version of the paper.

Throughout, ring means associative ring with identity $1 \neq 0$. If R is a ring, we shall denote the Jacobson radical of R by $J(R)$. All modules will be unital right modules unless otherwise specified. The symbols $\mathbb{N}, \mathbb{N}^*, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ will denote the set of all non-negative integers, positive integers, integers, rational numbers and real numbers, respectively.

2. First Examples and Properties

A right R-module M_R is said to be an *almost Krull–Schmidt* module (AKS module for short) if it has only a finite number of direct sum decompositions into non-zero summands up to isomorphism [12]. In particular, an AKS module has only finitely many direct summands up to isomorphism.

If M_R is an AKS module, then any direct sum decomposition of M is finite, because if $M = \oplus_{\lambda \in \Lambda} M_\lambda$ is a direct sum of infinitely many non-zero modules M_λ, then there is a direct sum decomposition of M_R into n direct summands for every $n \geq 1$, hence M_R cannot be AKS. Moreover, any decomposition of an AKS module M can be refined into a direct sum of indecomposable submodules.

In our first result we show that the property of being an AKS module is completely determined by the endomorphism ring of the module. Recall that two idempotents e, f in a ring R are called *isomorphic* if the principal right ideals eR, fR are isomorphic right R-modules, or, equivalently, if the principal left ideals Re, Rf are isomorphic left R-modules [1, Exercise 7.2]. We say that a ring R with identity 1 is an *almost Krull–Schmidt* ring (AKS ring) if 1 can be written in only finitely many ways as a sum of pair-wise orthogonal non-zero idempotents, up to isomorphism of idempotents.

Theorem 2.1. *Let M_R be an R-module and let $E = \mathrm{End}(M_R)$ be its endomorphism ring. The following conditions are equivalent:*

(a) *The right R-module M_R is an* AKS *module.*
(b) *The ring E is an* AKS *ring.*
(c) *The right E-module E_E is an* AKS *module.*

Proof. The equivalence of (b) and (c) is trivial.

Let $\mathrm{add}(M_R)$ be the full subcategory of Mod-R whose objects are all modules isomorphic to direct summands of direct sums M^n of finitely many copies of M, and let proj-E be the full subcategory of Mod-E whose objects are all finitely generated projective right E-modules. The functors $\mathrm{Hom}_R(M_R, -)\colon$ Mod-$R \to$ Mod-E and $-\otimes_E M_R\colon$ Mod-$E \to$ Mod-R induce an equivalence between the categories $\mathrm{add}(M_R)$ and proj-E [5, Theorem 4.7]. In this equivalence, for every idempotent $e \in E$, the object eE of proj-E corresponds to the

direct summand eM of M_R. It follows that M_R has only a finite number of direct sum decompositions up to isomorphism if and only if E_E has only a finite number of direct sum decompositions up to isomorphism.
This proves the equivalence of (a) and (c). □

One of the most important properties as far as direct sum decompositions are concerned is the so-called exchange property. For the terminology relative to the exchange property and the definition of exchange ring we refer the reader to Warfield's original paper [13] or to the monograph [5, Chapter 2].

Theorem 2.2. *An exchange ring is an* AKS *ring if and only if it is semiperfect.*

Proof. The fact that every semiperfect ring is AKS follows immediately from [5, Theorem 2.12 and Proposition 3.14]. Conversely, let R be an AKS exchange ring. As R_R is an AKS module, it must be a finite direct sum of indecomposable modules $R_R = \oplus_{i=1}^n e_i R$. Since R_R is an exchange module, the indecomposable modules $e_i R$ must have the exchange property, so that their endomorphism rings $e_i R e_i$ must be local. This proves that R is semiperfect [5, Theorem 3.6]. □

From Theorem 2.2 one immediately obtains that AKS von Neumann regular rings are exactly semisimple artinian rings. Similarly, AKS strongly π-regular rings are semiperfect rings.

Recall that a ring R is F*-semiperfect* if $R/J(R)$ is von Neumann regular and idempotents can be lifted modulo $J(R)$. F-semiperfect rings are exchange rings. Therefore an F-semiperfect ring is AKS if and only if it is semiperfect.

By Theorems 2.1 and 2.2, an injective (quasi-injective, pure-injective, continuous) module is AKS if and only if it is a direct sum of finitely many indecomposable modules.

Let A be a commutative additive monoid. The monoid A is said to be *reduced* if $a + b = 0$ implies $a = b = 0$ for every $a, b \in A$, that is, if no non-zero element a of A has an additive inverse $-a$ in A.

If a and b are two elements of A, define $a \leq b$ if $b = a + c$ for some $c \in A$. The relation $\leq$ is reflexive, transitive and invariant under translations (that is, for any $d \in A$, $a \leq b$ implies $a+d \leq b+d$). Thus $\leq$ is a pre-order on A, usually called the *algebraic pre-order* of A. An element u of A is an *order-unit* if it is $\neq 0$ and for any $a \in A$ there exists an integer $n \geq 0$ such that $a \leq nu$. The pre-order relation on A allows us to define *intervals*, that is, if $a, b \in A$ and $a \leq b$, we can consider the interval $[a, b]$, i.e., the set $[a, b] = \{c \in A \mid a \leq c \leq b\}$.

We shall now consider the category of commutative monoids with order-unit, which is defined as follows. Its objects are the pairs (A, u), where A is a commutative monoid and $u \in A$ is an order-unit. The morphisms $f\colon (A, u) \to (A', u')$ are the monoid homomorphisms $f\colon A \to A'$ such that $f(u) = u'$.

Let M_R be a right R-module. We shall denote by $\langle N_R \rangle$ the isomorphism class of any object N_R of $\mathrm{add}(M_R)$. Let $V(M_R)$ be the set of isomorphism classes of all objects of $\mathrm{add}(M_R)$; there is a natural addition on $V(M_R)$ defined as $\langle N_1 \rangle + \langle N_2 \rangle = \langle N_1 \oplus N_2 \rangle$ for any elements $\langle N_1 \rangle, \langle N_2 \rangle \in V(M_R)$. This addition gives $V(M_R)$ a monoid structure, and $\langle M_R \rangle$ is an order-unit in $V(M_R)$.

Notice that $\langle N_1 \rangle \leq \langle N_2 \rangle$ means that N_1 is isomorphic to a direct summand of N_2. Therefore $V(M_R)$ is a reduced monoid, and if a module M_R is AKS, then the interval $[0, \langle M_R \rangle]$ in $V(M_R)$ is finite.

For any ring R, there is a duality between the full subcategory proj-R of Mod-R whose objects are all finitely generated projective right R-modules and the full subcategory R-proj of R-Mod whose objects are all finitely generated projective left R-modules. The duality, defined by $P_R \mapsto \mathrm{Hom}(P_R, R_R)$, induces an isomorphism of commutative monoids with order-unit

$$(V(R_R), \langle R_R \rangle) \to (V({}_RR), \langle {}_RR \rangle).$$

Therefore there is no ambiguity in denoting the monoid $V(R_R) \cong V({}_RR)$ simply by $V(R)$.

Let M_R denote a right R-module and E its endomorphism ring. The category equivalence between $\mathrm{add}(M_R)$ and proj-E (proof of Theorem 2.1) induces an isomorphism of commutative monoids with order-unit $(V(M_R), \langle M_R \rangle) \to (V(E), \langle E \rangle)$.

Lemma 2.3. *Let I be a two-sided ideal in a ring R.*

(a) *If $I \subseteq J(R)$ and R/I is an* AKS *ring, then R also is an* AKS *ring.*

(b) *If idempotents lift modulo I and R is an* AKS *ring, then R/I also is an* AKS *ring.*

Proof. The canonical projection $\pi\colon R \to R/I$ induces a morphism of monoids with order-unit $V(\pi)\colon (V(R), \langle R\rangle) \to (V(R/I), \langle R/I\rangle)$.

(a) If $I \subseteq J(R)$, then $V(\pi)$ is an injective monoid morphism, because if P, Q are finitely generated projective right R-modules and $P/PI \cong Q/QI$ are isomorphic, then the projective covers P, Q of $P/PI \cong Q/QI$ are isomorphic. Thus if $\langle R/I\rangle$ can be written as a sum of elements in $V(R/I)$ in only finitely many ways, then, a fortiori, $\langle R\rangle$ can be written as a sum of elements in $V(R)$ in only finitely many ways. In other words, R/I AKS implies R AKS.

(b) If idempotents lift modulo I, the morphism $V(\pi)$ induces a surjective mapping of the interval $[0, \langle R\rangle]$ in $V(R)$ onto the interval $[0, \langle R/I\rangle]$ in $V(R/I)$. It follows that the number of direct sum decompositions of R up to isomorphism is greater or equal to the number of direct sum decompositions of R/I up to isomorphism. □

Examples 2.4. (1) Every torsion-free abelian group of finite rank is an AKS $\mathbb{Z}$-module [11].

(2) Every ring with finitely many idempotents is an AKS ring. In particular, every integral domain is an AKS ring.

(3) Two idempotents of a commutative ring R are isomorphic if and only if they are equal. Moreover, as over a commutative ring the product of idempotents is idempotent, two direct sum decompositions of R have a common refinement. Therefore a commutative ring R is AKS if and only if its identity 1 is the sum of finitely many primitive (i.e., indecomposable) idempotents, that is, if and only if R is the direct product of finitely many rings each of which has no nontrivial idempotents. Thus if R is a commutative AKS, the module R_R has a unique direct sum decompositions into indecomposables.

(4) A ring R is *semilocal* if $R/J(R)$ is semisimple artinian. Since semisimple artinian rings are trivially AKS, every semilocal ring is an AKS ring by Lemma 2.3(a). In particular, right artinian rings are AKS rings. If M is a linearly compact module, then $\mathrm{End}(M)$ is semilocal [10, Corollary 5], so that M is an AKS module by Theorem 2.1. In particular, artinian modules are AKS.

(5) The Jordan–Zassenhaus Theorem states that, if R is a Dedekind domain whose field of quotients Q is a global field and S is an R-order in a semisimple Q-algebra, then every S-lattice, that is, every right S-module that is finitely generated and projective as an R-module, is an AKS S-module [8].

3. Construction of AKS Rings via Bergman and Dicks' Theorem

The following wonderful result, due to Bergman and Dicks [3, p. 315], will be one of our main tools to construct examples of AKS modules.

Theorem 3.1. *Let k be a field, and let A be a commutative reduced monoid with order-unit u. Then there exists a right and left hereditary k-algebra R such that (A, u) and $(V(R), \langle R \rangle)$ are isomorphic monoids with order-unit.*

The history of this result is the following. In [2, Theorems 6.2 and 6.4], Bergman proved Theorem 3.1 for *finitely generated* reduced monoids with order-unit. Bergman's method and the fact that taking the semigroup $V(R)$ commutes with colimits yield that if (A, u) is a commutative reduced monoid with order-unit, then there exists a k-algebra R, isomorphic to a colimit of hereditary k-algebras, such that $(V(R), \langle R \rangle)$ is isomorphic to (A, u). In [3], Bergman and Dicks proved that certain classes of colimits of hereditary algebras are also hereditary [3, Corollary 3.2], and their results imply that the algebra obtained through Bergman's coproduct constructions is also hereditary.

As a consequence of Theorem 3.1 we obtain:

Corollary 3.2. *Let k be a field, and let A be a commutative reduced monoid with order-unit u, with the property that u can be written as a sum of non-zero elements of A in only finitely many different ways. Then there exists a hereditary* AKS *k-algebra R such that (A, u) and $(V(R), \langle R \rangle)$ are isomorphic monoids with order-unit.*

We have already remarked that AKS modules have only finitely many direct summands up to isomorphism. The converse does not hold, that is, there exist modules with only finitely many direct summands up to isomorphism that are not AKS, as the following example shows.

Example 3.3. Let A be any commutative reduced finite non-zero monoid. Then A has order-units, for instance the sum of all elements of A, which cannot be zero because A is reduced. Moreover, every element of A has a multiple that is idempotent, because every element of A generates a cyclic finite submonoid of A, which contains a finite subgroup, whose identity is idempotent. Thus A has an idempotent order-unit. By Theorem 3.1, there exists a ring R with $V(R) \cong A$

and $R_R \oplus R_R \cong R_R$. Then $R_R \cong R_R^n$ for every n is not AKS, but has only finitely many direct summands up to isomorphism, because A is finite.

Notice that the fact that a module M_R has only finitely many direct summands up to isomorphism is reflected in the monoid with order-unit $(V(M_R), \langle M_R \rangle)$ by the fact that the interval $[0, \langle M_R \rangle]$ is finite. There is no relation between the fact that the commutative monoid A is finitely generated and the fact that the interval $[0, u]$ is finite, as the following two examples show.

Example 3.4. (a) *Example of a finitely generated commutative reduced monoid A with order-unit u but with $[0, u]$ infinite.*

Let $\sim$ be the congruence on the additive monoid $\mathbb{N} \times \mathbb{N}$ defined by

$$(x, y) \sim (x', y') \quad \text{if} \quad \begin{cases} x = x' \text{ and } y = y' \\ \text{or} \\ x, y, x', y' \text{ are all } \geq 1 \text{ and } x + y' = y + x'. \end{cases}$$

Let $A = \mathbb{N} \times \mathbb{N}/\sim$ be the quotient monoid and $u = [(1, 1)]$ be the congruence class of $(1, 1)$. Notice that $u = [(n, n)]$ for every $n \geq 1$, that $[(n, 0)] = \{(n, 0)\}$ and that $[(0, n)] = \{(0, n)\}$. Thus $u = [(n, 0)] + [(0, n)]$ in infinitely many different ways. The element u is an order-unit in A because for every $[(x, y)] \in A$ one has that $[(x, y)] + [(y + 1, x + 1)] = [(x + y + 1, y + x + 1)] = u$. Moreover the monoid A is reduced, because if $[(x, y)] + [(x', y')] = [(0, 0)]$, then $(x, y) + (x', y') \sim (0, 0)$, so that $x + x' = 0$ and $y + y' = 0$, from which $x = x' = y = y' = 0$. Thus A has the required properties.

(b) *Example of a commutative reduced monoid A with order-unit u, u indecomposable in A, but A not finitely generated as a commutative monoid.*

It suffices to consider the submonoid

$$A = \{(0, 0)\} \cup \{\, (x, y) \in \mathbb{R} \times \mathbb{R} \mid x \geq 1, y \geq 1 \,\}$$

of the additive abelian group $\mathbb{R} \times \mathbb{R}$ with $u = (1, 1)$.

If R is a semiperfect ring (that is, R is the endomorphism ring of a finite direct sum of modules each of which has a local endomorphism ring [5, Proposition 3.14]), then the monoid $V(R)$ is isomorphic to $\mathbb{N}^t$, where t is the number of non-isomorphic simple R-modules. More generally, if R is a semilocal ring, then $V(R)$ is isomorphic to a full affine submonoid of $\mathbb{N}^t$ [6]. For an AKS ring R, the monoid $V(R)$ is not necessarily torsion-free. For instance, let k, n, u be three positive integers with $u < k$, and let $\sim_{k,n}$ denote the congruence on the additive monoid $\mathbb{N}$ defined by

$$x \sim_{k,n} y \text{ if } \begin{cases} x = y \\ \text{or} \\ x \geq k, y \geq k \text{ and } x \equiv y \pmod{n} \end{cases}$$

for every $x, y \in \mathbb{N}$. Let A be the monoid $\mathbb{N}/\sim_{k,n}$. By Corollary 3.2, there exists an AKS ring R with $(V(R), \langle R\rangle) \cong (A, \overline{u})$. In particular, $V(R)$ is not torsion-free.

4. Closure Properties. AKS versus other Finiteness Conditions

In this section we analyze the class of AKS modules with respect to the most usual closure properties of classes of modules, and we compare the property of being an AKS module with other finiteness conditions for modules, like being noetherian, AB5*, of finite Goldie dimension, or of finite dual Goldie dimension. Clearly, a direct summand of an AKS module is an AKS module. In Example 4.1(1) we show that a direct sum of two AKS modules is not necessarily an AKS module.

Examples 4.1. (1) *Example of a ring R such that R_R is* AKS *and $(R \oplus R)_R$ is not* AKS.

(2) *Example of a finitely generated module over a Dedekind domain that is not an* AKS *module.*

(3) *Example of a right and left noetherian ring that is not* AKS.

Let R be a Dedekind domain with field of fractions Q. Recall that a *fractional ideal* of R is an R-submodule I of Q for which there exists a non-zero $r \in R$ with $rI \subseteq R$. The *class group* of R is the group whose elements are the isomorphism classes $\langle I\rangle$ with I non-zero fractional ideal of R. Every abelian group is the class group of a Dedekind domain [4]. Let R be a Dedekind domain with infinite class group. Since R is an integral domain, the module R_R is AKS. Every ideal of a Dedekind domain R is isomorphic to a direct summand of $R \oplus R$. Since R has infinitely many non-isomorphic ideals, the R-module $(R \oplus R)_R$ is not AKS. This gives the example required in (1) and (2). For (3) it suffices to consider the ring $M_2(R)$ of 2×2 matrices with entries in R, and apply Theorem 2.1.

For a different example of (1), let A be the submonoid of the additive group $\mathbb{R}\times\mathbb{R}$ whose elements are $(0,0)$ and all the elements $(x,y)\in\mathbb{R}\times\mathbb{R}$ with $x\geq 0$, $y\geq 0$ and $x+y\geq 2$, and set $u=(1,1)$. Now apply Theorem 3.1 to (A,u).

Compare Example 4.1(2) with the fact that every finitely generated module over a principal ideal domain is an AKS module (every finitely generated module over a principal ideal domain is a direct sum of indecomposable modules in a unique way up to isomorphism).

By Theorem 2.1 and Example 4.1, there exist AKS rings R for which the ring $M_2(R)$ of 2×2 matrices with entries in R is not AKS. Notice that if R is a Dedekind domain like those considered in Examples 4.1, that is a Dedekind domain with infinite class group, and Q is the field of fractions of R, then $(Q\oplus Q)_R$ is an AKS R-module, but its submodule $(R\oplus R)_R$ is not AKS. Thus a submodule of an AKS module is not necessarily an AKS module. This example also shows that a subring of an AKS ring is not an AKS ring (consider the rings $M_2(R)\subseteq M_2(Q)$). On the contrary, every subring of a commutative AKS ring, i.e., a commutative ring with only finitely many idempotents, is still an AKS ring.

We conclude this section by comparing the most common finiteness conditions for modules with the property of being an AKS module.

Examples 4.2. (1) *There exist noetherian modules that are not* AKS. *There exist* AKS *modules that are not finitely generated; in particular, they are not noetherian.*

See Example 4.1(2) for a noetherian module that is not AKS. Any infinitely generated indecomposable module, e.g., $\mathbb{Q}_{\mathbb{Z}}$, is an example of an infinitely generated AKS module.

(2) *There exist modules of finite Goldie dimension that are not* AKS *and* AKS *modules that are not of finite Goldie dimension.*

Example 4.1(2) shows that there exist modules of finite Goldie dimension that are not AKS. Any non-commutative integral domain that is not a right Goldie ring is an example of an AKS right module that is not of finite Goldie dimension.

(3) *There exist modules of finite dual Goldie dimension that are not* AKS *and* AKS *modules that are not of finite dual Goldie dimension.*

Every commutative integral domain E is isomorphic to the endomorphism ring of a *local* module M_R, that is, a cyclic module M_R of dual Goldie dimension 1 [10, Example 10(1)]. As in Examples 4.1, let E be a Dedekind domain with infinite class group. If M_R is a local module with $\mathrm{End}(M_R) \cong E$, then the module $M_R \oplus M_R$ has dual Goldie dimension 2, but its endomorphism ring, which is isomorphic to the ring $M_2(E)$ of 2×2 matrices with entries in E, is not an AKS ring. Therefore $M_R \oplus M_R$ is not an AKS module. (Notice that every module of dual Goldie dimension 1 is indecomposable, hence it is an AKS module.)

Conversely, the abelian group $\mathbb{Z}$ is an AKS $\mathbb{Z}$-module of infinite dual Goldie dimension.

We remark that a ring of finite dual Goldie dimension is semilocal [5, Proposition 2.43], hence it is an AKS ring (Example 2.4(4)). Conversely, there are AKS rings that are not semilocal, for instance, the ring $\mathbb{Z}$.

(4) *There exist modules of finite Krull dimension that are not* AKS *and* AKS *modules which fail to have Krull dimension.*

The Dedekind domains that are not fields have Krull dimension 1. Let R be the Dedekind domain of Examples 4.1(1) and (2). Then $M_R = (R \oplus R)_R$ has Krull dimension 1 and is not AKS. Conversely, if R is a valuation domain whose value group is divisible, then R turns out to be an AKS ring which fails to have Krull dimension.

Notice that if M is a module of Krull dimension ≤ 0, then M is artinian, hence it is an AKS module (Example 2.4(4)).

(5) *There exist rings R that are not* AKS *and whose $V(R)$ is finite, and* AKS *rings R whose $V(R)$ is not finitely generated.*

For an example of the first type of rings, see Example 3.3. For an example of the second type of rings, apply Theorem 3.1 to the monoid with order-unit $(A, (1,1))$ of Example 3.4(b). The ring R obtained in this way has no non-trivial idempotents, hence it is an AKS ring, but $V(R) \cong A$ is not finitely generated as a commutative monoid.

(6) *There exist* AB5* *modules that are not* AKS *and* AKS *modules that are not* AB5*.

An infinite direct sum of pair-wise non-isomorphic simple modules is an example of an AB5* module that is not AKS. The $\mathbb{Z}$-module $\mathbb{Q}/\mathbb{Z}$ is another example with these properties. Any integral domain that is not local is an example of a ring that is AKS and not AB5*.

5. Bounds on the Number of Direct Summands

In this section we shall consider the following problem. Suppose we know the direct sum decompositions of two R-modules M and N. What can we say about the direct sum decompositions of the R-modules $M \oplus N$ and M^n $(n \geq 1)$? The answer is, as we shall see in this section, that we can say very little. Practically everything can happen.

Firstly, we consider the direct sum decompositions of $M \oplus N$. For every R-module M, let $\nu(M)$ denote the number, up to isomorphism, of decompositions of M as direct sums of indecomposables. Thus $\nu(M)$ is either a non-negative integer or $+\infty$. Moreover, if M is an AKS module, then $\nu(M)$ is a positive integer. Suppose we know $\nu(M)$ and $\nu(N)$ for two R-modules M and N. What can we say about $\nu(M \oplus N)$? If M and N have no isomorphic non-zero direct summands (that is, if for every R-module P that is isomorphic both to a direct summand of M and to a direct summand of N one has that $P = 0$), then $\nu(M \oplus N) \geq \nu(M)\nu(N)$. The next example shows that the inequality $\nu(M \oplus N) \geq \nu(M)\nu(N)$ may not hold if M and N have isomorphic non-zero direct summands.

Example 5.1. Let R be a ring whose monoid $V(R)$ is isomorphic to the submonoid A of the additive monoid $\mathbb{N} \times \mathbb{N}$ generated by $\{(2,0),(1,1),(0,2)\}$ (Theorem 3.1). Let $M = N$ be the R-module corresponding to the element $(2,2)$ of A. Since $(2,2) = (2,0) + (0,2) = 2(1,1)$ are the only sum decompositions of $(2,2)$ up to the order, it follows that M and N have only two decompositions, up to isomorphism, as direct sums of indecomposables. Thus $\nu(M) = \nu(N) = 2$. But $M \oplus N$ corresponds to the element $(4,4)$ of A, and $(4,4) = 2(2,0)+2(0,2) = 4(1,1) = (2,0)+(0,2)+2(1,1)$. Therefore $\nu(M \oplus N) = 3$.

To be able to construct examples with a more complicated behaviour it is better to use constructions of monoids through generators and relations. The next lemma, essentially taken from [7, Lemma 2.7], will be helpful in controlling such monoids.

First we recall that a submonoid B of a monoid C is said to be a *full submonoid* of C if for any $x \in B$ and any $y \in C$, $x + y \in B$ implies $y \in B$ [7]. If $f\colon B \to C$ is a monoid monomorphism such that $f(B)$ is a full submonoid of C, then f is said to be a *full embedding*. Observe that if C is cancellative, f is a full embedding if and only if f is an embedding of pre-ordered monoids. In the particular case that $C = \mathbb{N}^r$, f is said to be a *full affine embedding*, and $f(B)$ a *full affine monoid*.

Lemma 5.2. *Let $f\colon B \to C$ be a full embedding of cancellative monoids, and let $J \in B$ be such that $nJ \neq 0$ for every positive integer n. Let B' be the monoid obtained by adjoining to B two elements p, q and the one relation $p + q = J$. Let $g\colon B \to B'$ be the canonical homomorphism. Then:*

(a) *g is a full embedding.*

(b) *There exists a full embedding $f'\colon B' \to C \times C$ defined by $f'(g(b)) = (f(b), f(b))$ for every $b \in B$, $f'(p) = (f(J), 0)$ and $f'(q) = (0, f(J))$.*

(c) *Let $b \in B$ and $x, y \in B' \setminus g(B)$. Then $g(b) = x + y$ if and only if there exist $b_1, b_2 \in B$ and $n > 0$ such that $b = b_1 + b_2 + nJ$ and either $x = g(b_1) + np$ and $y = g(b_2) + nq$, or $x = g(b_1) + nq$ and $y = g(b_2) + np$. In particular, $g(b) = x + y$ for $x, y \in B' \setminus g(B)$ if and only if $J \leq b$.*

(d) *If C is reduced, then p and q are indecomposable elements of B'.*

Proof. Statement (b) and the fact that g is injective is proved in [7, Lemma 2.7]. In order to show that g is a full embedding, let $b, b_0 \in B$ and $x \in B'$ be such that $g(b) + x = g(b_0)$. Then $f'(g(b) + x) = (f(b), f(b)) + f'(x) = (f(b), f(b)) + (c_1, c_2) = (f(b_0), f(b_0))$. As C is cancellative and f is a full embedding, $c_1 = c_2 \in f(B)$. Hence, as f' is injective, $x \in g(B)$.

To show (c) it suffices to prove the only if part. Notice that, by the definition of B', its elements can be written either in the form $g(b_0) + np$ or $g(b_0) + mq$. Let $b \in B$ be such that $g(b) = x + y$ for $x, y \in B' \setminus g(B)$. Assume that there exist $b_1 \in B$ and $n > 0$ such that $x = g(b_1) + np$. Then

$$f'(g(b)) = (f(b), f(b)) = f'(x) + f'(y) = \\ (f(b_1), f(b_1)) + (nf(J), 0) + (c_1, c_2).$$

As f is a full embedding and C is cancellative, $c_1 = f(b_2)$ and $c_2 = f(b_2) + nf(J)$ for some $b_2 \in B$. Since f' is injective, $y = g(b_2) + nq$. An analogous argument works if $x = g(b_1) + nq$.

It remains to show (d). Assume that there exist $x \neq 0$ and y elements of B' such that $x + y = p$. Then $f'(p) = (f(J), 0) =$

$f'(x)+f'(y)$. As C is reduced, $f'(x)=(c_1,0)$ and $f'(y)=(c_2,0)$ for suitable $c_1,c_2\in C$. Now $x=g(b)+np+mq$ for suitable $b\in B$ and $n,m\geq 0$, so that $(c_1,0)=f'(x)=f'(g(b))+f'(np)+f'(mq)=(f(b)+nf(J),f(b)+mf(J))$. Since C is reduced, $f(b)=mf(J)=0$. As f is injective, $b=0$ and $m=0$. Thus $x=np$ and $n>0$, so $p=np+y$. Thus $0=(n-1)p+y$, from which $n=1$ and $y=0$. This shows that p is indecomposable. By symmetry, q also is indecomposable. □

Theorem 5.3. *Let m,n,p be positive integers with $p\geq mn$. Then there exist two finitely generated projective modules M,N over a suitable ring R with no isomorphic non-zero direct summands such that $\nu(M)=m$, $\nu(N)=n$ and $\nu(M\oplus N)=p$.*

Proof. Let B_0 be the free commutative monoid freely generated by I and J. Consider the commutative monoid A obtained from B_0 in $r=m+n+(p-mn)$ steps by subsequently adding the generators x_{1i},x_{2i} and the relation $I=x_{1i}+x_{2i}$ $(i=1,2,\dots,m)$, then the generators y_{1j},y_{2j} and the relation $J=y_{1j}+y_{2j}$ $(j=1,2,\dots,n)$, and, finally, the generators z_{1k},z_{2k} with the relation

$$I+J=z_{1k}+z_{2k}\qquad (k=mn+1,mn+2,\dots,p.)$$

Let $f_0\colon B_0\to\mathbb{N}\times\mathbb{N}$ be the isomorphism of monoids defined by $f_0(I)=(1,0)$ and $f_0(J)=(0,1)$. By applying Lemma 5.2 r times, it follows that f_0 can be extended to a full affine embedding $f\colon A\to\mathbb{N}^{2^{r+1}}$. In particular, A is a reduced monoid. Since $f_0(I+J)=(1,1)$, $f(I+J)=(1,\dots,1)$. Therefore $I+J$ is an order-unit in A.

By Theorem 3.1 there exists a right and left hereditary ring R such that $(A,I+J)$ and $(V(R),\langle R\rangle)$ are isomorphic monoids with order-unit.

Let M and N be the projective R-modules corresponding to the elements I and J of A. The direct sum decompositions, up to isomorphism, of M, N and $M\oplus N$, correspond to the decomposition in A of I, J and $I+J$ respectively. It follows from Lemma 5.2(d) that the elements x_{ti}'s, y_{tj}'s and z_{tk}'s are indecomposable. It remains to check that I, J and $I+J$ have no other decompositions than those ones imposed through the relations.

The elements I and J are uncomparable in B_0, therefore by Lemma 5.2(a) they remain uncomparable at each step of the construction. Moreover, at each step, I and J are strictly smaller that

$I+J$. Then, by Lemma 5.2(c), the only decompositions of I and J into indecomposable elements are those given in the relations. In a similar way, it is possible to prove that $I+J$ has exactly p decompositions into indecomposable elements of A. □

Remark. The monoid A in the proof of Theorem 5.3 is full affine. We proved in [6] that this is exactly the class of monoids that can be realized as $V(R)$ for a semilocal ring R. Therefore the ring R in the statement of the theorem can be taken semilocal.

Now let M be an arbitrary right module. For every $n \in \mathbb{N}$ put

$$X_n = \{t \in \mathbb{N} \mid M^n \text{ is the direct sum of } t \text{ indecomposable modules}\}.$$

Obviously, the cardinality of X_n is $\leq \nu(M^n)$.

If X, X' are subsets of $\mathbb{N}$, we shall write $X+X'$ to denote the set of all $t+t'$ with t in X and t' in X'. Notice that $X+X' = X'+X$ and $X+\emptyset = \emptyset$ for every $X, X' \subseteq \mathbb{N}$. The following properties for the subsets X_n of $\mathbb{N}$ are easily verified:

(a) $X_0 = \{0\}$.
(b) Either $0 \notin X_n$ for every $n \neq 0$, or $X_n = \{0\}$ for every $n \in \mathbb{N}$.
(c) $1 \notin X_n$ for every $n \neq 1$.
(d) Either $1 \notin X_1$ or $X_1 = \{1\}$.
(e) $X_n + X_m \subseteq X_{n+m}$ for every $n, m \in \mathbb{N}$.

The two cases in (b) correspond to the two cases $M \neq 0$ or $M = 0$, and the two cases in (d) correspond to the two cases M decomposable or M indecomposable.

Theorem 5.4. *For every $n \in \mathbb{N}$ let X_n be a subset of $\mathbb{N}$. Suppose that the family $\{X_n\}_{n\in\mathbb{N}}$ satisfies conditions* (a) *to* (e) *above. Then there exists a finitely generated projective right module M over a suitable ring R with the property that for every $n, t \in \mathbb{N}$ one has that $t \in X_n$ if and only if M^n is a direct sum of t indecomposable modules.*

Notice that for every assignment $n \mapsto X_n$ satisfying properties (a) and (e), the set $S = \{\, n \in \mathbb{N} \mid X_n \neq \emptyset \,\}$ turns out to be a submonoid of the additive monoid $\mathbb{N}$.

To prove the theorem, we will construct a suitable reduced monoid with order-unit and then the result will follow from Theorem 3.1. For the construction, in general, we shall need to use Lemma 5.2 an infinite number of times. To this aim it will be useful to have in mind the following easy, but technical, lemma.

Lemma 5.5. *For every monoid C, let $\Delta\colon C \to C \times C$ be the monoid homomorphism defined by $\Delta(c) = (c, c)$. Let $\{B_n\}_{n\in\mathbb{N}}$ be a family of monoids, and for any $n \geq 0$ let $g_n\colon B_n \to B_{n+1}$ and $f_n\colon B_n \to C^{2^n}$ be full embeddings. Assume that for any $n \geq 0$, $f_{n+1} \circ g_n = \Delta \circ f_n$. Then the induced map $f\colon \varinjlim B_n \to \varinjlim C^{2^n}$ is a full embedding.*

Here is a sketch of the proof of Theorem 5.4. If $X_n = \{0\}$ for every $n \in \mathbb{N}$, it suffices to take $M = 0$. Thus we may suppose that $0 \notin X_n$ for every $n \neq 0$.

Consider the submonoid $S = \{\, n \in \mathbb{N} \mid X_n \neq \emptyset \,\}$ of $\mathbb{N}$. Let A be the commutative additive monoid having as generators

(G1) I,

(G2) $x_{(n,t,i)}$, where n, t, i are integers, $n \geq 1$, $t \in X_n$ and $i \in \{1, 2, \dots, t\}$,

(G3) $y_{(n,i_1,i_2,\dots,i_m)}$, where $n \in \mathbb{N} \setminus S$, $m \geq 1$, and the $i_1, i_2, \dots, i_m$ are equal to 0 or 1,

subject to the relations

(R1) $nI = x_{(n,t,1)} + \cdots + x_{(n,t,t)}$ for every $n \geq 1$ and every $t \in X_n$,

(R2) $nI = y_{(n,0)} + y_{(n,1)}$ for every $n \in \mathbb{N} \setminus S$,

(R3) $y_{(n,i_1,i_2,\dots,i_m)} = y_{(n,i_1,i_2,\dots,i_m,0)} + y_{(n,i_1,i_2,\dots,i_m,1)}$ for every $n \in \mathbb{N} \setminus S$, every $m \geq 1$ and every $i_1, i_2, \dots, i_m$.

Note that if $S = \mathbb{N}$, that is, if $X_n \neq \emptyset$ for every $n \in \mathbb{N}$, then there are only the generators in (G1) and (G2) subject to the relations (R1).

Let B_0 be the free monoid generated by I, and let $f_0\colon B_0 \to \mathbb{N}$ be the monoid isomorphism defined by $f_0(I) = 1$. By suitably ordering the generators (G2) and (G3) and the relations (R1), (R2), (R3), it follows from Lemma 5.2 that we may construct a sequence of monoids B_r $(r > 0)$ and full embeddings $g_r\colon B_r \to B_{r+1}$ and $f_r : B_r \to \mathbb{N}^{2^r}$ $(r \geq 0)$ such that $\varinjlim B_r = A$. By Lemma 5.5, A is a full submonoid of $\varinjlim \mathbb{N}^{2^r}$.

Now we apply Bergman and Dicks' Theorem 3.1 to the reduced monoid with order-unit (A, I). Arguing as in the previous proofs in this section it is possible to see that the module $M = R_R$ corresponding to I has the required properties.

Remark. Let k be a field. A k-algebra T is said to be *ultramatricial* if it is the colimit of finite direct products of matrices over k. Ultramatricial algebras are von Neumann regular [9, p. 219].

If (A,u) is a monoid with order-unit that is a countable colimit of monoids with order-unit of the form $(\mathbb{N}^r,u)$, it follows from Elliott's Theorem [9, Theorem 15.24] that there exists an ultramatricial k-algebra T such that $(V(T),\langle T\rangle)$ is isomorphic to (A,u). Moreover $V(T)$ of any ultramatricial algebra T is of this type.

For example, the monoid $C=\varinjlim \mathbb{N}^{2^r}$, with the image of 1 as order-unit and taking as maps $\mathbb{N}^{2^r}\to\mathbb{N}^{2^{r+1}}$ the maps defined by $c\mapsto(c,c)$, is the monoid $V(T)$ of the ultramatricial algebra $T=\varinjlim M_{2^r}(k)$, taking as morphisms $M_{2^r}(k)\to M_{2^{r+1}}(k)$ the morphisms defined by $\alpha\mapsto\left(\begin{smallmatrix}\alpha&0\\0&\alpha\end{smallmatrix}\right)$ for any $\alpha\in M_{2^r}(k)$.

Let $A=\varinjlim B_r$ be the submonoid that appears in the proof of Theorem 5.4. By construction, each B_r is a full affine submonoid of $\mathbb{N}^{2^r}$ and A is a full submonoid of C. By [6], (B_r,I) can be realized as $(V(R_r),\langle R_r\rangle)$ for some semilocal ring R_r. The techniques used in [6] can be used to construct a countable directed system of semilocal rings, such that if $R=\varinjlim R_r$ and $J(R)$ denotes the Jacobson radical of R, then $(A,I)\cong(V(R),\langle R\rangle)$ and $R/J(R)\cong\varinjlim M_{2^r}(K)$. Therefore the ring in the statement of Theorem 5.4 can be taken von Neumann regular modulo its Jacobson radical.

We have already remarked that for every module M the set S of the natural numbers n for which M^n is a direct sum of indecomposables is a submonoid of the additive monoid $\mathbb{N}$. From Theorem 5.4 we get the following corollary.

Corollary 5.6. *Let S be a submonoid of the additive monoid $\mathbb{N}$. Then there exists a finitely generated right module M over a suitable ring R such that for every $n\in\mathbb{N}$ the module M^n is a direct sum of indecomposable modules if and only if $n\in S$.*

Proof. For every $n\in\mathbb{N}$ let $X_n=\{n\}$ if $n\in S$ and $X_n=\emptyset$ if $n\notin S$. Then the assignment $n\mapsto X_n$ satisfies the hypotheses of Theorem 5.4, so that there exists a finitely generated right module M over a suitable ring R with the property that for every $n,t\in\mathbb{N}$ one has that $t\in X_n$ if and only if M^n is a direct sum of t indecomposable modules. Thus M has the properties required in the statement of the corollary. □

Finally, we consider a further mapping that measures the direct sum decompositions of M^n. Let $M \neq 0$ be a module. Define a function $\mu_M\colon \mathbb{N}^* \to \mathbb{N}^* \cup \{+\infty\}$ by

$$\mu_M(n) = \sup\{\, t \mid M^n = M_1 \oplus \cdots \oplus M_t \text{ for suitable non-zero modules } M_1, \dots, M_t \,\}.$$

Obviously, $\mu_M(n) + \mu_M(m) \leq \mu_M(n+m)$ for every n and m.

Notice that there is a relation between the function μ_M and the sets X_n considered in Theorem 5.4, because if $\mu_M(n) \in \mathbb{N}^*$, then $X_n \neq \emptyset$ and $\sup X_n = \mu_M(n)$. But if $\mu_M(n) = +\infty$, then we can say very little, because either $X_n = \emptyset$, or $X_n \neq \emptyset$ and $\sup X_n = +\infty$, or $X_n \neq \emptyset$ and $\sup X_n$ can be any integer ≥ 2.

From Theorem 5.4 we get the following characterization of the functions μ_M that arise in this way.

Corollary 5.7. . *Let $\mu\colon \mathbb{N}^* \to \mathbb{N}^* \cup \{+\infty\}$ be a function such that $\mu(n) + \mu(m) \leq \mu(n+m)$ for every $n, m \in \mathbb{N}^*$. Then there exists a module $M \neq 0$ over a suitable ring R such that $\mu = \mu_M$.*

Proof. Let $\mu\colon \mathbb{N}^* \to \mathbb{N}^* \cup \{+\infty\}$ be a function with $\mu(n) + \mu(m) \leq \mu(n+m)$ for every $n, m \in \mathbb{N}^*$. Define the sets $X_n \subseteq \mathbb{N}$ for every $n \in \mathbb{N}$ as follows: $X_0 = \{0\}$, $X_n = \{\, t \in \mathbb{N} \mid 2 \leq t \leq \mu(n) \,\}$ for every $n \geq 2$, and $X_1 = \{1\}$ if $\mu(1) = 1$ or $X_1 = \{\, t \in \mathbb{N} \mid 2 \leq t \leq \mu(1) \,\}$ if $\mu(1) > 1$. Then the assignment $n \mapsto X_n$ satisfies the hypotheses of Theorem 5.4. The module M satisfying the thesis of Theorem 5.4 has the property that $\mu_M = \mu$. □

If M^n is an AKS module, then $\mu_M(n) \in \mathbb{N}^*$ and X_n is a finite non-empty set whose largest element is $\mu_M(n)$. In particular, suppose that M_R is a module whose endomorphism ring $\operatorname{End}(M_R)$ is semilocal. Then $\operatorname{End}(M_R)$ has finite dual Goldie dimension d. It follows that $\operatorname{End}(M_R^n)$ has finite dual Goldie dimension nd for every n, so that $\mu_M(n) \leq nd$ for every $n \in \mathbb{N}$.

REFERENCES

[1] F. W. Anderson and K. R. Fuller, "Rings and categories of modules", Second Edition, Springer-Verlag, New York, 1992.

[2] G. M. Bergman, *Coproducts and some universal ring constructions,* Trans. Amer. Math. Soc. **200** (1974), 33–88.

[3] G. M. Bergman and W. Dicks, *Universal derivations and universal ring constructions,* Pacific J. Math. **79** (1978), 293–337.

[4] L. Claborn, *Every abelian group is the class group of a Dedekind domain*, Pacific J. Math. **18** (1966), 219–222.

[5] A. Facchini, "Module Theory. Endomorphism rings and direct sum decompositions in some classes of modules", Progress in Math. **167**, Birkhäuser Verlag, Basel, 1998.

[6] A. Facchini and D. Herbera, K_0 *of a Semilocal Ring*, J. Algebra **225** (2000), 47–69.

[7] A. Facchini and D. Herbera, *Projective modules over semilocal rings*, in "Algebra and its Applications", D. V. Huynh, S. K. Jain, S. R. López-Permouth Eds., Contemporary Math. **259**, Amer. Math. Soc., Providence, 2000, pp. 181-198.

[8] L. Fuchs and P. Vámos, *The Jordan–Zassenhaus Theorem and Direct Decompositions*, J. Algebra **230** (2000), 730–748.

[9] K. R. Goodearl, *Von Neumann Regular Rings*, Second Edition, Krieger Publishing Company, Malabar, 1991.

[10] D. Herbera and A. Shamsuddin, *Modules with semi-local endomorphism ring,* Proc. Amer. Math. Soc. **123** (1995), 3593–3600.

[11] E. L. Lady, *Summands of finite rank torsion-free Abelian groups*, J. Algebra **32** (1974), 51–52.

[12] P. Vámos, *The Holy Grail of Algebra: Seeking Complete Sets of Invariants*, in "Abelian Groups and Modules", A. Facchini and C. Menini Eds., Math. and Its Appl. **343**, Kluwer, Dordrecht, 1995, pp. 475–483.

[13] R. B. Warfield, Jr., *Exchange rings and decompositions of modules,* Math. Ann. **199** (1972), 31–36.

Alberto Facchini,
Dipartimento di Matematica Pura e Applicata,
Università di Padova,
35131 Padova, Italy,
facchini@math.unipd.it

Dolors Herbera,
Departament de Matemàtiques,
Universitat Autònoma de Barcelona,
08193 Bellaterra (Barcelona), Spain
dolors@mat.uab.es

Received on 16 October 2001 and in revised form on 11 July 2003.