ON A COMMENT OF DOUGLAS CONCERNING WIDOM'S THEOREM

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§1 Introduction.

Douglas, after presenting an adaptation of Widom's proof [2] that every Toeplitz operator has connected spectrum, comments, *"Despite the elegance of the preceding proof of connectedness, we view it as not completely satisfactory for two reasons: First, the proof gives us no hint as to why the result is true. Second, the proof seems to depend on showing that the set of some kind of singularities for a function of two complex variables is connected ... "* [1, p.196]. The purpose of this note is to demonstrate that one of the variables referred to by Douglas can be effectively suppressed by extensive use of the F. & M. Riesz Theorem; the modified proof is, I believe, somewhat cleaner.

§2 Preliminary concepts.

The items in this section are well-known and are covered in [1].

Notation. *The unit circle is denoted by* $\mathbf{T}$. *We consider the spaces* $L^p = L^p(\mathbf{T})$ *for* $p = 1, 2, \infty$, *where the measure is Lebesgue measure and the vectors are treated as functions defined almost everywhere. The functions* $e_n : z \to z^n$ $(n \in \mathbf{Z})$ *form an orthonormal basis for the Hilbert space* L^2. *The Hardy spaces* H^p *are* $H^p = \{ f \in L^p : \int_{\mathbf{T}} f e_n = 0 \ \ \forall n > 0 \}$, *(*$p = 1, 2, \infty$*).* P *will denote the orthogonal projection from* L^2 *onto* H^2. *Note that* L^∞ *and* H^∞ *are Banach algebras, that* $L^\infty \subset L^2 \subset L^1$ *and* $H^\infty \subset H^2 \subset H^1$ *and that* $L^\infty L^2 = L^2$. *For* $\phi \in L^\infty$, $\sigma(\phi)$ *will denote the spectrum,* $\{\lambda \in \mathbf{C} : \phi - \lambda \text{ not invertible in } L^\infty\}$, *of* ϕ *in* L^∞*; note that this is the same as the essential range of* ϕ, *namely, the set of all* $\lambda \in \mathbf{C}$

such that, for every $\epsilon > 0$, the set $\{z \in \mathbf{T} : |\phi(z) - \lambda| < \epsilon\}$ has positive measure. For $T \in \mathcal{B}(H^2)$, the algebra of bounded linear operators on H^2, $\sigma(T)$ will denote the spectrum of T in $\mathcal{B}(H^2)$.

Definition. *For each $\phi \in L^\infty$ we define the Toeplitz operator $T_\phi \in \mathcal{B}(H^2)$ by $T_\phi f = P(\phi f)$ for each $f \in H^2$.*

Proposition 1. *Suppose $f \in L^1$ and $\int_\mathbf{T} fe_n = 0$ for all $n \in \mathbf{Z}$. Then $f = 0$.*

Proposition 2. *Suppose $f, g \in H^2$. Then $fg \in H^1$.*

Proposition 3. *Suppose $\phi \in L^\infty$. Then $T_{\overline{\phi}} = T_\phi^*$.*

Proposition 4. *Suppose $\phi \in L^\infty$. Then $\sigma(\phi) \subseteq \sigma(T_\phi)$. (This implies, of course, that, if T_ϕ is invertible, then so is ϕ. However, it is worth noting that the inverse of T_ϕ is not, except in very special cases, equal to $T_{\phi^{-1}}$.)*

F. & M. Riesz Theorem. *Suppose $f \in H^2$. If $f \neq 0$ then the set of zeroes of f has zero measure. (It follows immediately from this that if $\phi \in H^\infty$ and the essential range of ϕ is countable, then ϕ is essentially constant.)*

§3 The connectedness.

Proposition 5. *Suppose Γ is a simple closed integration path and K is a compact subset of the complex plane with $K \cap \Gamma = \emptyset$. Let $\phi \in L^\infty$ be such that $\sigma(\phi) = K$. Then Γ fails to separate K if and only if*

$$\exp\left(P\int_\Gamma \frac{d\mu}{\phi - \mu}\right) = e_0.$$

Proof: Γ fails to separate $\sigma(\phi)$ if and only if the winding number function in L^∞

$$w(\Gamma, \phi) = \frac{1}{2\pi i}\int_\Gamma \frac{d\mu}{\mu - \phi}$$

is (essentially) constant. Since $w(\Gamma, \phi)$ has only integer values, the F. & M. Riesz Theorem ensures that this happens if and only if $w(\Gamma, \phi)$ is in H^∞, i.e., if and only if

$$P\int_\Gamma \frac{d\mu}{\mu - \phi} = \int_\Gamma \frac{d\mu}{\mu - \phi}.$$

Since $\exp\left(\int_\Gamma \frac{d\mu}{\phi-\mu}\right) = e_0$, the result follows by invoking the F. & M. Riesz Theorem again. □

Proposition 6. *Suppose $\phi \in L^\infty$ and T_ϕ is invertible in $\mathcal{B}(H^2)$. If $f \in H^2$ satisfies $T_\phi f = e_0$, then $f^{-1} \in H^2$ and $T_{\phi^{-1}} f^{-1} = e_0$.*

Proof: Firstly, since T_ϕ is invertible, Propositions 3 and 4 ensure that ϕ and $\overline{\phi}$ are invertible in L^∞ and that $T_{\overline{\phi}} = T_\phi^*$ is invertible in $\mathcal{B}(H^2)$. In particular, there is exactly one vector mapped to e_0 by $T_{\overline{\phi}}$, so that $\dim(\overline{\phi}H^2 \cap \overline{H^2}) = 1$. It follows that the space $H^2 \cap \overline{\phi}^{-1}\overline{H^2}$ has dimension 1 and then also that

$$\dim(\phi^{-1}H^2 \cap \overline{H^2}) = 1.$$

We deduce that $T_{\phi^{-1}}$ is injective and that there exists $g \in H^2$ such that $T_{\phi^{-1}} g = e_0$. Then there exist $u, v \in (H^2)^\perp$ such that $\phi f = e_0 + u$ and $\phi^{-1} g = e_0 + v$. By multiplication we have $fg = e_0 + u + v + uv$, whence $u + v + uv \in H^1$ by Proposition 2. Since $u, v \in (H^2)^\perp$, an easy calculation using Proposition 1 shows that $u + v + uv = 0$ and hence that $fg = e_0$. □

Widom's Theorem. *Suppose $\phi \in L^\infty$; then $\sigma(T_\phi)$ is connected.*

Proof: Consider the function $f : \mathbf{C}\backslash\sigma(T_\phi) \to H^2$ given by the equations $f(\lambda) = (\lambda - T_\phi)^{-1} e_0$. Then f is differentiable and we have $P[(\lambda - \phi) f'(\lambda) + f(\lambda)] = 0$. But Proposition 6 gives also the equation $P[1/((\lambda - \phi) f(\lambda))] = e_0$. Multiplying, we get the differential equation

$$f'(\lambda) = f(\lambda) P\left(\frac{1}{\phi - \lambda}\right).$$

Note that any non-zero solution of this equation is a multiple of f by a non-zero function independent of λ. So, using the F. & M. Riesz Theorem again, we solve to get, for any fixed α in each connected component of $\mathbf{C}\backslash\sigma(T_\phi)$ and for each λ in that component,

$$f(\lambda) = f(\alpha) \exp\left(P \int_\Gamma \frac{d\mu}{\phi - \mu}\right)$$

where Γ is **any** simple integration arc in the component going from α to λ. If Γ is closed, the condition

$$\exp\left(P\int_\Gamma \frac{d\mu}{\phi-\mu}\right) = e_0$$

of Proposition 5 holds, so no such Γ separates $\sigma(\phi)$ and connectedness of $\sigma(T_\phi)$ will follow if we can show that $\sigma(T_\phi)$ is exterior to such a Γ whenever $\sigma(\phi)$ is. Suppose, then, that Γ is a simple closed integration path in $\mathbf{C}\backslash\sigma(T_\phi)$ and that $\sigma(\phi)$ is exterior to Γ. Then the solution to the differential equation gives a unique analytic continuation of f to the interior of Γ, so that, setting Q to be the associated spectral idempotent for T_ϕ, we have

$$Qe_0 = \frac{1}{2\pi i}\int_\Gamma (\mu - T_\phi)^{-1} e_0 \, d\mu = \frac{1}{2\pi i}\int_\Gamma f(\mu)\, d\mu = 0.$$

Now $(\lambda - T_\phi)(e_n f(\lambda)) = e_n + \sum_{i=0}^{n-1}\beta_i e_i$ for each $\lambda \in \Gamma$ and some related scalars β_i; assuming inductively that $Qe_i = 0$ for $i < n$, it follows, since Q commutes with T_ϕ, that

$$Q(e_n f(\lambda)) = (\lambda - T_\phi)^{-1} Q e_n,$$

and integration around Γ gives $Qe_n = 0$. So $Q = 0$ by induction, whence no part of $\sigma(T_\phi)$ is interior to Γ and the theorem is proved. □

References

[1] Douglas, R. G., *Banach Algebra techniques in Operator Theory*, Academic Press, New York & London, 1972.

[2] Widom, H., *On the spectrum of a Toeplitz operator*, Pacific. J. Math., **14** (1964), 365–375.

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