JONSSON GROUPS, RINGS AND ALGEBRAS

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A Jonsson group G is one all of whose proper subgroups have smaller cardinality than G. Jonsson rings and Jonsson algebras are defined in a similar fashion. In this paper, we present an introductory account of Jonsson algebras in the light of pcf theory, a recent development within set theory. In section 1, we give some examples and summarize what is known about Jonsson groups and rings. In section 2, we prove the basic results on Jonsson algebras. Most of this section is self-contained, and the reader will need to know little more than some naive set theory and first-order model theory [H, HS or ChK]. Section 3 contains the elements of pcf theory, deals with the most recent results on Jonsson algebras, and summarizes the impact of additional set-theoretic axioms in this area.

1. Jonsson groups, rings, algebras and cardinals

To start matters off, we define Jonsson groups, algebras and cardinals.

Definition

1. A group G is a **Jonsson group** iff G has no proper subgroup H of the same cardinality as G, i.e. every proper subgroup of G has fewer elements than G.
2. Suppose that F is a countable set of finitary operations on a set A. The algebra $\mathbf{A} = (A, F)$ is a **Jonsson algebra** iff $\mathbf{A}$ has no proper subalgebra $\mathbf{B} = (B, F|B)$ of the same cardinality as $\mathbf{A}$.

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3. A cardinal λ is a **Jonsson cardinal** iff there is no Jonsson algebra of cardinality λ.

In writing $F|B$, we mean the family of operations in F, each restricted to B^n for the appropriate number n of arguments. Since the nature of the underlying set A is irrelevant, we shall often assume without comment that it is a cardinal, and also say that there is a Jonsson algebra on λ meaning that there is one on a set of power λ.

It seems that Jonsson algebras were identified by B. Jonsson in the fifties, [EH]. Relatively little was known about them (at least in ordinary set theory) until the early eighties. Devlin surveys the state of play up to 1973 in section 3 of his paper [D].

Every Jonsson group is a Jonsson algebra (treating the identity element as a 0-ary operation). It is obvious that every finite algebra is a Jonsson algebra. So the first natural question is whether there are any (infinite) Jonsson cardinals at all.

Example 1: Let $A = (\omega, \{m\})$, where ω (the first infinite ordinal) is the set of natural numbers and $m(x) := x - 1$ for all $x > 0$, $m(0) := 0$. The algebra A is a Jonsson algebra of cardinality $\aleph_0$ (the first infinite cardinal), so $\aleph_0$ is not a Jonsson cardinal.

Example 2 [F]: Let p be prime number and let $C(p^n)$ be the cyclic group of order p^n. Then the p-quasicyclic group,

$$C(p^\infty) = \cup_{n=1}^{\infty} C(p^n),$$

is a countably infinite abelian group all of whose proper subgroups are finite. In accordance with the convention in abelian group theory, we shall write $C(p^\infty)$ additively. It is generated by elements $c_1, c_2, \ldots, c_n, \ldots$, such that

$$pc_1 = 0, pc_2 = c_1, \ldots, pc_{n+1} = c_n, \ldots$$

and $o(c_n) = p^n$. It is easy to check that $C(p^\infty)$ is a Jonsson group of cardinality $\aleph_0$. For, if H is a proper subgroup, then there exists a least n such that c_{n+1} does not belong to H. Now $\langle c_n \rangle \leq H$. Conversely, if $h \in H$, then $h \in C(p^\infty)$, and so there exists k such that $h = kc_m$, where $(p,k) = 1$. Since $(p^m, k) = 1$, there exist r and s such that $rk + sp^m = 1$, and hence

$$c_m = (rk + sp^m)c_m = rh \in H.$$

So $c_m \in H$ and $m \leq n$, so that $h \in \langle c_n \rangle$ and $H = \langle c_n \rangle$.

The family of p-quasicyclic groups contains all the countable Jonsson abelian groups, since if G is infinite abelian with all its proper subgroups finite, then G is a p-quasicyclic group for some prime p.

In 1979, Ol'shanskii, [O], proved the existence of an infinite non-abelian group all of whose proper subgroups are finite, solving Schmidt's problem[2].

Jonsson rings are rings all of whose proper subrings have smaller cardinality.

Example 3 [L]: From the p-quasicyclic group $C(p^\infty)$, it is easy to construct a countable Jonsson ring by defining all products to be zero. Since every proper subring of $C(p^\infty)$ is also a proper subgroup of $C(p^\infty)$, it must be finite.

We know all about countable Jonsson rings. Laffey classified them in a slightly different terminology, proving in his paper [L]:

Theorem. *If R is a countable Jonsson ring, then either*
(i) $R^2 = \{0\}$ *and* $R = C(p^\infty)$ *for some prime* p,
or
(ii) $R = G_{p,q}$ *for some primes* p *and* q, *where*

$$G_{p,q} = \cup_{n=0}^{\infty} \mathrm{GF}(p^{q^n})$$

and $\mathrm{GF}(p^{q^n})$ *is the finite field of order* p^{q^n}.

In the early sixties, Kurosh conjectured that uncountable Jonsson groups exist. This conjecture was settled positively by Shelah in 1980. The group that Shelah built had even stronger properties:

Theorem. [Sh80] *There is a Jonsson group S of cardinality $\aleph_1$. This group is simple.*

[2] Professor Wilfrid Hodges kindly supplied this reference.

It follows that $\aleph_1$ is not a Jonsson cardinal. We shall derive a more general version of this consequence in section 2. Shelah noted too that his group S has no maximal proper subgroup. In particular, the operation of taking the Frattini subgroup does not commute with direct products. For if $\tau(S)$ is the Frattini subgroup of S, i.e. the intersection of all the maximal subgroups of S, then $\tau(S) = S$, but $\tau(S \times S) = \{(a,a) : a \in S\}$.

To close this section, let us note a rather surprising connection between Jonsson semigroups and Jonsson groups.

Theorem. [McK] *If a semigroup G of cardinality λ is a Jonsson semigroup, then either G is a group, or else λ has countable cofinality and $(\exists \kappa < \lambda)$ $(\lambda < 2^\kappa)$.*

In particular, if λ has uncountable cofinality or if $\kappa < \lambda$ implies $2^\kappa \leq \lambda$, then every Jonsson semigroup of power λ is a Jonsson group. This means for example that if one wishes to construct a Jonsson group of power λ, where λ has uncountable cofinality (e.g. λ a successor cardinal), then it is enough (or as hard) to build a Jonsson semigroup.

Within ordinary set theory, there seems to be no easy way to climb from Jonsson groups of power $\aleph_1$ to ones of power $\aleph_2$. The theory of Jonsson algebras is in this respect a good deal smoother.

2. Jonsson algebras

In this section we prove the easiest results on Jonsson algebras in ordinary set theory. Although some of these can be established using combinatorial arguments and historically were first obtained in this way, one can abbreviate the arguments by employing elementary submodels. A Jonsson model is a model $\mathbf{A} = (A, R, F)$ where R and F are countable sets of finitary relations and operations on the set A such that every elementary submodel of $\mathbf{A}$ has smaller cardinality. So every Jonsson algebra is a Jonsson model, since it has no relations and its elementary submodels are therefore subalgebras. The reader can find all relevant basic information about elementary submodels in the appendix [pp.165-176] of the monograph by Heindorf and Shapiro, [HS], or in the standard texts [H, ChK].

Proposition 1. *There is a Jonsson algebra of power λ iff there is a Jonsson model of power λ.*
Proof: For the non-trivial direction, add Skolem functions to the Jonsson model; the Skolem hull is the required Jonsson algebra. This is a standard and very useful technique for building algebras from models. We give a brief sketch. Fix a well-order $\leq$ of the universe A of the Jonsson model $\mathbf{A} = (A, R, F)$. For each formula $\psi(x_1, \ldots, x_n, y)$ in the language $L = R \cup F$ of the model, define a new n-ary operation f_ψ on A^n as follows: $f_\psi(a_1, \ldots, a_n)$ is the $\leq$-least element $b \in A$ such that $\psi(a_1, \ldots, a_n, b)$ is true in $\mathbf{A}$ if such an element exists, otherwise $f_\psi(a_1, \ldots, a_n)$ is any element of A. Repeat this process for the new (countable) language

$$L_1 = L \cup \{f_\psi : \psi \text{ is an } L\text{-formula}\},$$

and so on (countably many times) to get the expanded (still countable) language

$$L^* = \cup L_n.$$

The required algebra is

$$\mathbf{A}^{sk} = (A, \{f_\psi : \psi \text{ is an } L^*\text{-formula}\}),$$

since any proper subalgebra of $\mathbf{A}^{sk}$ will give rise to a proper elementary submodel of the Jonsson model $\mathbf{A}$ of the same cardinality. ■

We combine Proposition 1 with some set theory to obtain a necessary and sufficient criterion for the existence of Jonsson algebras.

Recall that for a cardinal θ, the collection of sets which are hereditarily of cardinality less than θ is denoted $H(\theta)$: a set x belongs to $H(\theta)$ iff $|x| < \theta$ and if $y \in x$, then $|y| < \theta$, and so on. For reference, we summarize the main features of $H(\theta)$ in a theorem:

Theorem. *If θ is a regular uncountable cardinal, then $H(\theta)$ is a transitive model of ZFC with the possible exception of the power set axiom. If α is an ordinal, then $\alpha < \theta$ iff $\alpha \in H(\theta)$.*

Intuitively, $H(\theta)$ is a reasonably small universe of most of the axioms of ordinary set theory. The main properties of elementary submodels of $H(\theta)$ are given in detail in [EM, pp.151-152].

Lemma 2. [BM] *Suppose that λ is an infinite cardinal. There is a Jonsson algebra on λ iff:*
for some (all) regular cardinal(s) $\theta > \lambda$ and for all elementary submodels $\mathbf{M} \leq H(\theta)$:
$(*)$ if $\lambda \in M$ and $|\lambda \cap M| = \lambda$, then $\lambda \subseteq M$.
Proof: For the forward direction, note that since $\mathbf{M} \leq H(\theta)$, there is a Jonsson algebra $\mathbf{A} \in M$ on λ, say $A = (\lambda, \{f_n : n \in \omega\})$. Let $B = M \cap \lambda$ (by hypothesis unbounded in λ). So B has cardinality λ (λ is regular), and $\mathbf{B} = (B, \{f_n|B : n \in \omega\})$ is a subalgebra of $\mathbf{A}$. However, $\mathbf{A}$ is Jonsson, hence $B = A = \lambda$, i.e. $M \cap \lambda = \lambda$, and so $\lambda \subseteq M$.

For the reverse implication, fix $\mathbf{M} \leq H(\theta)$, $|M| = \lambda$, $\lambda \subseteq M$. Let $h : \lambda \to M$ be a bijection. Then $\mathbf{M}^+ = (M, \in, h)$ is a Jonsson model of power λ. For, if $\mathbf{N} \leq \mathbf{M}^+$ is an elementary submodel of power λ, then $\lambda \in N$ (by elementarity, since λ is the least ordinal which does not belong to dom(h)). Also $|N \cap \lambda| = \lambda$, and hence by $(*)$ $\lambda \subseteq N$, since $\mathbf{N} \leq H(\theta)$. Therefore range$(h) \subseteq N$. But range$(h) = M$, so $N = M^+$, and $\mathbf{M}^+$ is a Jonsson model of power λ. We appeal now to Proposition 1 to complete the proof. ■

Theorem 3. [Sh, BM] *If there is a Jonsson algebra on λ, then there is one on λ^+, where λ^+ is the least cardinal greater than λ.*
Proof: We use the lemma. Suppose that $\mathbf{M} \leq H(\lambda^{++})$, $\lambda^+ \in M$, $|M \cap \lambda^+| = \lambda^+$. We must show $\lambda^+ \subseteq M$. Suppose that $\beta < \lambda^+$. Note that $\lambda \in M$ (since $\lambda^+ \in M$ by elementarity), and $\exists \alpha \in M \cap \lambda^+$, $\alpha > \beta$, such that $|M \cap \alpha| = \lambda$, because $|M \cap \lambda^+| = \lambda^+$. M contains a bijection g from λ onto α, hence $|M \cap \lambda| = \lambda$, and so $\lambda \subseteq M$ (applying the lemma to the hypothesis that there is a Jonsson algebra on λ). Therefore $\alpha = \text{range}(g) \subseteq M$. In particular, $\beta \in M$. Since β was arbitrary, it follows that $\lambda^+ \subseteq M$. ■

In fact, for a little more effort and terminology, a stronger result is provable:

Theorem. (Tryba [T], Woodin) *If λ is regular and there is non-*

reflecting stationary subset of λ, then there is a Jonsson algebra on λ.

The short proof can be found in [BM]. In particular, there is a Jonsson algebra on λ^+ whenever λ is regular, since the set $\{\alpha < \lambda^+ : \mathrm{cf}(\alpha) = \lambda\}$ is non-reflecting and stationary in λ^+.

Example 1 and Theorem 3 yield a corollary:

Corollary 4. $(\forall n \in \mathbf{N})$*(There is a Jonsson algebra on* $\aleph_n$*).*

The simplest unanswered questions (at least to formulate) thus far are whether there are Jonsson algebras on $\aleph_\omega$, on $\aleph_{\omega+1}$, and more generally on the successors of singular cardinals. We shall discuss these questions in section 3.

For completeness, let me mention two other equivalent conditions for the existence of a Jonsson algebra. The first is based on results of Łos and Sierpiński (see [D]):

Theorem. *There is a Jonsson algebra of cardinality λ iff there is a Jonsson algebra of cardinality λ with exactly one commutative binary operation.*

The second characterization is related to a question of Mycielski about locally finite algebras:

Definition An algebra $A = (A, \{f_n : n \in \mathbf{N}\})$ is locally finite iff whenever X is a finite subset of A, then $A|X$ (the subalgebra of A generated by X) is finite.

What can one say about locally finite Jonsson algebras? Improving a theorem of Erdös and Hajnal, Devlin proved:

Theorem. [D] *There is a locally finite Jonsson algebra of cardinality λ iff there is a Jonsson algebra of cardinality λ.*

3. pcf theory

Possible cofinality (pcf) theory is the study of the cofinalities of ultraproducts of sets of cardinals. It was discovered (invented) by Shelah, and developed in its fullest form in his work on cardinal arithmetic, [Sh]. The theory has found applications in set theory, infinitary combinatorics (partition calculus), model theory, algebra (infinite abelian groups), set-theoretic topology, Boolean algebras (productivity of chain conditions) and Jonsson algebras. We select just the definitions and result that are necessary to prove that there is a Jonsson algebra on $\aleph_{\omega+1}$. A lucid introduction to pcf theory is available in the paper by Burke and Magidor, [BM], which serves also as an excellent entry-point to Shelah's treatise.

Suppose that a is a set of regular cardinals and $\min(a) > |a|$. Let D be an ultrafilter on a. The elements of Πa are functions f such that $\mathrm{dom}(f) = a$ and $(\forall \alpha \in a)(f(\alpha) < \alpha)$. We can define an equivalence relation $=_D$ on Πa by

$$f =_D g \text{ iff } \{\alpha \in a : f(\alpha) = g(\alpha)\} \in D,$$

and use the notation f/D $(\Pi a/D)$ for the equivalence class of f (the set $\{f/D : f \in \Pi a\}$). The ultraproduct $(\Pi a/D, \leq_D)$, where

$$f \leq_D g \text{ iff } \{\alpha \in a : f(\alpha) \leq g(\alpha)\} \in D,$$

is a linear order since D is an ultrafilter. Hence it has a **true cofinality**:

Definition Suppose that λ is a cardinal. We say that λ is the **true cofinality** of $\Pi a/D$, and write $\lambda = \mathrm{tcf}(\Pi a/D)$, iff:

(1) λ is regular;

(2) $\exists$ a strictly increasing cofinal sequence $\{f_\zeta \in \Pi a : \zeta < \lambda\}$ in $\Pi a/D$, i.e.

(2.1) $\zeta < \xi < \lambda$ implies $f_\zeta <_D f_\xi$

and (2.2) $(\forall h \in \Pi a)(\exists \zeta < \lambda)(h <_D f_\zeta)$.

To illustrate the idea, we compute some true cofinalities.

Example 4: If D is a principal ultrafilter on a (so D is generated by a singleton subset $\{\alpha\}$ of a say), then $\mathrm{tcf}(\Pi a/D) = \alpha$.

Example 5: Suppose that $a = \{\aleph_n : 1 < n < \omega\}$. If D is a non-principal ultrafilter on a, then $\mathrm{tcf}(\Pi a/D) > \aleph_\omega$. Why? Well, if $\{f_i \in \Pi a : i < \aleph_k\}$ is $<_D$-increasing and $k < \omega$, then $(\forall n > k)(\{f_i(\aleph_n) : i < \aleph_k\}$ is bounded in $\aleph_n$ by β_i say, and so the function

$$g(\aleph_m) = \sup\{\beta_i : i < \aleph_k\} + 1, \text{ if } m > k (= 0, \text{if } 1 < m < k)$$

is an element of Πa and $(\forall i < \aleph_k)(f_i <_D g)$, since D contains the co-finite filter on a. In other words, $\{f_i \in \Pi a : i < \aleph_k\}$ is not cofinal in $\Pi a/D$.

One of the fundamental tasks in pcf theory is to determine which cardinals are the true cofinalities of the ultraproducts $\Pi a/D$, or how many possible cofinalities the set a supports.

Definition We define pcf(a), the possible cofinalities of the set a, to be the collection

$$\{\lambda : \text{for some ultrafilter } D \text{ on } a, \mathrm{tcf}(\Pi a/D) = \lambda\}.$$

Example 4 tells us that $a \subseteq \mathrm{pcf}(a)$. We know too that there are $2^{2^{|a|}}$ ultrafilters on a (since every ultrafilter belongs to $P(P(a))$). Thus trivially

$$|a| \leq |\mathrm{pcf}(a)| \leq 2^{2^{|a|}}.$$

It can be shown that $|\mathrm{pcf}(a)| \leq 2^{|a|}$ and, for more money, $|\mathrm{pcf}(a)| \leq |a|^{+3}$. The major open question in pcf theory is whether $|\mathrm{pcf}(a)| = |a|$.

For our purposes, we shall need a special case of one of Shelah's theorems:

Theorem. [Sh] *Suppose that* $a = \{\aleph_n : 1 < n < \omega\}$. *Then* $\aleph_{\omega+1} \in \mathrm{pcf}(a)$.

So one can represent $\aleph_{\omega+1}$ as the true cofinality of $\Pi a/D$ for some ultrafilter D on a. Note that this does not tell us anything about $\aleph_\omega$ or other singular cardinals, since they can never appear in a set of possible cofinalities (which are by definition regular). Shelah demonstrated the power which this representation provides in his proof of the existence of a Jonsson algebra of cardinality $\aleph_{\omega+1}$:

Theorem. [Sh] *There is a Jonsson algebra on* $\aleph_{\omega+1}$.

Proof: Let $\mu = \aleph_\omega$, $\theta = (2^\mu)^+$, and fix $\mathbf{M} \leq H(\theta)$, $\mu^+ \in M$, $|M \cap \mu^+| = \mu^+$. We show that $\mu^+ \subseteq M$ (and appeal to Lemma 2).

We know that $\mu^+ \in \mathrm{pcf}(a)$, where $a = \{\aleph_n : 1 < n < \omega\}$. By elementarity, it follows that:

(1) $a \in M$;

(2) $a \subseteq M$;

(3) there is an ultrafilter D on a, $D \in M$, and a sequence

$$\{f_i : i < \mu^+\} \in M \cap \Pi a$$

such that $\{f_i/D : i < \mu^+\}$ is increasing and cofinal in $\Pi a/D$.

Claim: $\{\alpha \in a : |M \cap \alpha| = \alpha\}$ is cofinal in a.

Proof of claim: Otherwise, let $g(\alpha) = \sup(M \cap \alpha)$ ($g(\alpha) = 0$ if $\sup(M \cap \alpha) = \alpha$). So $g \in \Pi a$, and hence by (3) there is $\kappa < \mu^+$ such that $g/D < f_\kappa/D$. So for some $\alpha \in a$, $0 < g(\alpha) < f_\kappa(\alpha)$ (D contains the co-finite filter). But $f_\kappa(\alpha) \in M \cap \alpha$, and $g(\alpha) = \sup(M \cap \alpha)$, contradiction. Hence $\{\alpha \in a : |M \cap \alpha| = \alpha\}$ is cofinal in a. By Corollary 4, there is a Jonsson algebra on α for each $\alpha \in a$, and so $\alpha \subseteq M$ (by (2) and Lemma 2). Thus: $\mu \subseteq M$. Finally, for $\xi \in M \cap [\mu, \mu^+)$, there is a bijection ϕ from μ onto ξ, and hence $\xi \subseteq M$. But $|M \cap \mu^+| = \mu^+$, hence $\mu^+ \subseteq M$. By Lemma 2, this establishes that there is a Jonsson algebra on $\mu^+ = \aleph_{\omega+1}$. ■

Shelah has extended this result to cover a wide class of successors of singular cardinals and also the class of inaccessible cardinals which are not in some degree Mahlo or have a stationary subset not reflecting in any inaccessible cardinals. These results are presented in [Sh]. Their broad import is to make it progressively more difficult for Jonsson cardinals to exist. And indeed, if one increases one's axiomatic commitments beyond ordinary set theory (ZFC), this difficulty becomes an impossibility:

Theorem [Erdös-Hajnal-Rado, Keisler-Rowbottom]

(1) *If* $2^\lambda = \lambda^+$, *then there is a Jonsson algebra on* λ^+.

(2) *If* $V = L$, *then for every infinite cardinal* λ, *there is a Jonsson algebra on* λ.

The relatively easy proofs of these can be found in [BM] (or [J] or [EHMR]). Thus it is consistent that there are no Jonsson cardinals at all. For Jonsson groups, additional set-theoretic hypotheses also have decisive implications:

Theorem. [Sh80] *Suppose that* λ *is an uncountable cardinal and* $2^\lambda = \lambda^+$.

(1) *There is a Jonsson group of cardinality* λ^+.

(2) *Moreover this group is a Jonsson semigroup, is simple, and there is a natural number* n *such that for any subset* X *of the group of cardinality* λ^+, *any element of the group is equal to the product of* n *elements of* X.

Whether there can be a Jonsson algebra of singular cardinality (e.g. $\aleph_\omega$) appears more difficult to resolve and different in character from the regular case. In 1988, Koepke [K], building on the work of Jensen on inner models of set theory, proved results which indicate that the non-existence of Jonsson algebras of singular cardinality is essentially connected with large cardinal axioms: if there is a Jonsson cardinal $\aleph_\xi$ such that $\xi < \aleph_\xi$, then for each α there is a model of ZFC whose set of uncountable measurable cardinals has order type α. He also showed that if there is a singular Jonsson cardinal of uncountable cofinality κ, then there is an inner model of ZFC with κ measurable cardinals. These results establish that the assumption of the non-existence of a Jonsson algebra of singular cardinality is much stronger than the assumption that ZFC is consistent.

To conclude this brief survey, let me mention perhaps the most attractive open question about Jonsson algebras: can one prove in ordinary set theory (ZFC) that every successor cardinal carries a Jonsson algebra?

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