# FUNCTION ALGEBRAS

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## 1. Introduction

The theory of function algebras forms an important branch of functional analysis. Its central problem is to determine whether a given complex-valued function can be uniformly approximated by the elements of a prescribed algebra of functions. One of the attractive features of the subject is that it involves a beautiful interplay of ideas and methods from a variety of sources, such as topology and algebra, and especially from functional analysis and the theory of analytic functions. Moreover, it has important applications, for instance, to classical analysis and to operator theory. Indeed, the concepts and techniques of function algebra theory often produce new insights into the classical theory of approximation by analytic functions, and raise new questions, which serve to enliven and reinvigorate that subject.

The theory of function algebras is so extensive that any short account must be selective, and this is the case for the present exposition. The intention here is to explain some of the core ideas and problems, and to give an account of an aspect of the theory of particular interest to the author. This aspect is the theory of generalized Hardy $H^p$ spaces, and it is of interest in operator theory because of its applications to the theory of Toeplitz operators. Part of its importance in the theory of function algebras relates to one of the major problems of the subject, namely, the determination of conditions under which it is possible to embed analytic structure into the spectrum of a function algebra.

The paper is organized as follows: In Section 2 we discuss the basic concepts and give an illustration of one of these concepts,





that of a representing measure, to prove a maximality theorem of Wermer. In Section 3, we discuss some important classes of function algebras. In Section 4 we give a brief account of generalized $H^p$ space theory and indicate how this theory is applied to the problem of embedding analytic structure. Finally, in Section 5, we discuss some connections with operator theory.

## 2. Representing measures

A *function algebra* on a compact Hausdorff space $\Omega$ is a closed subalgebra $A$ of the algebra $C(\Omega)$ of all continuous functions on $\Omega$, that contains the constants and separates the points of $\Omega$. (The operations on $C(\Omega)$ are the usual pointwise-defined ones and the norm is the supremum norm, given by $\|\varphi\| = \sup_{s\in\Omega} |\varphi(s)|$.)

Of course, $C(\Omega)$ is a function algebra, but it is not typical of the algebras that are of primary interest to function algebraists. Rather, the prototypical function algebra is the disc algebra. This is the set $\mathbf{A}$ of all continuous functions on the closed unit disc $\mathbf{D}$ that are analytic in the interior, and it is easily seen that $\mathbf{A}$ is indeed a function algebra. This algebra can also be realized as a function algebra on the unit circle $\mathbf{T}$, because the homomorphism from $C(\mathbf{D})$ to $C(\mathbf{T})$ got by sending a function on $\mathbf{D}$ to its restriction to $\mathbf{T}$ induces an isometric algebra isomorphism of $\mathbf{A}$ onto the closed subalgebra of $C(\mathbf{T})$ generated by the inclusion function $z: \mathbf{T} \to \mathbf{C}$ (always, $z$ will denote this function). It is usual to refer to $\mathbf{A}$ as the disc algebra on the disc and to its image on $\mathbf{T}$ as the disc algebra on the circle. It follows easily from Fejér's theorem that the disc algebra on $\mathbf{T}$ is the set of all elements of $C(\mathbf{T})$ whose Fourier transform is supported in the set $\mathbf{Z}^+$ of all non-negative integers.

We defer giving more examples of function algebras to the next section. Instead, we introduce the concept of a representing measure, one of the most important ideas of the theory.

Let $A$ be a function algebra on a compact Hausdorff space $\Omega$. If $\tau$ is a bounded linear functional on $A$, then, by the Hahn–Banach theorem and the Riesz–Kakutani representation theorem, there is a regular Borel complex measure $m$ on $\Omega$, called a *representing measure* for $\tau$, such that

$$\tau(\varphi) = \int \varphi \, dm \qquad (\varphi \in A)$$

and $\|\tau\| = \|m\|$, where $\|m\|$ is the total variation norm, $\|m\| = |m|(\Omega)$. (In general, the representing measure $m$ is not unique.) The case of most interest for the theory of function algebras is that in which $\tau$ is a *character* of $A$, that is, a non-zero multiplicative linear functional on $A$. Characters are automatically of norm one and map the unit to 1, from which it follows that any representing measure of a character is positive and of total mass one, that is, a probability measure.

A point concerning notation is needed before we proceed further: If $g \in C(\Omega)$ and $\mu$ is a regular Borel complex measure on $\Omega$, we denote by $g\mu$ the regular Borel complex measure on $\Omega$ corresponding to the bounded linear functional on $C(\Omega)$ given by $f \mapsto \int fg \, d\mu$.

To illustrate the idea of a representing measure, let us consider again the disc algebra $\mathbf{A}$ on the circle. Let $m$ be normalized Lebesgue measure on $\mathbf{T}$, so

$$\int \varphi \, dm = \frac{1}{2\pi} \int_0^{2\pi} \varphi(e^{it}) \, dt \qquad (\varphi \in C(\mathbf{T})).$$

Since $\int z^n \, dm = 0$ for all $n > 0$, it follows that $\int \varphi z \, dm = 0$ for all $\varphi \in A$. Now let $s$ be a point in the unit disc $\mathbf{D}$. Then $s$ defines a character $\tau_s$ on $A$ by setting $\tau_s(\varphi) = \varphi(s)$. (Recall that $\mathbf{A}$ is the isomorphic image of the disc algebra on $\mathbf{D}$. We are using this to identify elements of $\mathbf{A}$ with their corresponding extensions to $\mathbf{D}$.) Suppose now that $|s| < 1$. If $\varphi \in \mathbf{A}$, then $\varphi = \varphi(s) + (z - s)\psi$, for some $\psi \in \mathbf{A}$. Setting $\mu = (1 - \bar{z}s)^{-1} m$, we have $\int 1 \, d\mu = \int \sum_{n=0}^{\infty} (s\bar{z})^n \, dm = \sum_{n=0}^{\infty} s^n \int \bar{z}^n \, dm = 1$, so $\int \varphi \, d\mu = \varphi(s) + \int \psi z \, dm = \varphi(s)$. This is, of course, the Cauchy integral formula. The norm of the measure $\mu$ is not equal to 1 if $s$ is non-zero, so $\mu$ is not a representing measure for $\tau_s$, but with a little further manipulation we get a representing measure: Since $(1 - \bar{s}z)^{-1}$ belongs to $\mathbf{A}$, therefore

$$\frac{\varphi(s)}{1 - |s|^2} = \int \frac{\varphi}{1 - \bar{s}z} \, d\mu = \int \frac{\varphi}{|1 - s\bar{z}|^2} \, dm,$$

so $\varphi(s) = \int \varphi \, dm_s$, where $m_s$ is the probability measure

$$m_s = \frac{1 - |s|^2}{|1 - s\bar{z}|^2} m.$$

Thus, $m_s$ is a representing measure for $\tau_s$. The equation $\varphi(s) = \int \varphi \, dm_s$ is the Poisson integral formula.

It is easily seen that for any character $\tau$ on $\mathbf{A}$, there is only one representing measure for $\tau$ (see Section 3). We shall now give an application of this to prove the following theorem due to Wermer (the proof is due to Hoffman and Singer). First, recall an elementary fact from Banach algebra theory: If $B$ is a unital abelian Banach algebra, then an element $b \in B$ is invertible if and only if $\tau(b) \neq 0$ for all characters $\tau$ of $B$.

**2.1. Theorem.** *If $B$ is a function algebra on the circle $\mathbf{T}$ containing the disc algebra $\mathbf{A}$, then either $B = \mathbf{A}$ or $B = C(\mathbf{T})$.*

***Proof.*** First suppose that $\tau(z) \neq 0$ for all characters $\tau$ of $B$. Then $z$ is invertible in $B$, and therefore $\bar{z} = 1/z \in B$, so by Fejér's theorem, $B = C(\mathbf{T})$. Suppose on the other hand that for some character $\tau$ of $B$, we have $\tau(z) = 0$. Then $\tau(\varphi) = \varphi(0)$ for all $\varphi \in \mathbf{A}$, so if $\mu$ is a representing measure for $\tau$ (as a character of $B$), then for all $\varphi \in \mathbf{A}$ we have $\int \varphi \, dm = \varphi(0) = \int \varphi \, d\mu$, and therefore $\mu = m$, by uniqueness of representing measures for $\mathbf{A}$. Hence, for all $\varphi \in B$ and for all $n > 0$, we have $\int \varphi z^n \, dm = \tau(\varphi z^n) = \tau(\varphi)\tau(z)^n = 0$. Thus, the Fourier transform of $\varphi$ is supported in $\mathbf{Z}^+$, and therefore $\varphi \in \mathbf{A}$. Hence, $B = \mathbf{A}$. □

This maximality result for $\mathbf{A}$ has some interesting consequences. One of them is that there are many continuous functions that are boundary values of analytic functions. Specifically, let $B$ be the set of all $\varphi \in C(\mathbf{T})$ for which there exists an analytic function $\psi$ on the open unit disc such that for almost all $t$ in $[0, 2\pi)$, $\psi(\lambda)$ converges to $\varphi(e^{it})$ as $\lambda$ converges to $e^{it}$ non-tangentially. Then $B$ is a subalgebra of $C(\mathbf{T})$ containing $\mathbf{A}$. It can be shown that the containment is proper, so by Wermer's theorem, $B$ is dense in $C(\mathbf{T})$. For details see [11].

Another application of the maximality theorem asserts that if $F$ is a proper closed subset of $\mathbf{T}$, then every continuous function on $F$ can be uniformly approximated by polynomials (*loc. cit.*).

The concept of a representing measure may perhaps seem, at first sight, not to be a very significant idea. The preceding examples and applications help to give some inkling of its power, more particularly in the case of a unique representing measure for a character. In fact, in the latter setting one can generalize a large amount of the classical theory of $H^p$ spaces. Moreover, this generalization has useful applications, one of which we shall see in Section 4.

## 3. Dirichlet and logmodular algebras

Let $K$ be a non-empty compact subset of $\mathbf{C}^n$. There are a number of important function algebras associated with $K$: The algebra $A(K)$ is defined to be the set of all continuous functions on $K$ that are holomorphic on the interior of $K$, and the algebra $R(K)$ is defined to be the set of all continuous functions on $K$ that are uniformly approximable by rational functions with no poles on $K$. By $P(K)$ we denote the algebra of all continuous functions on $K$ which are uniformly approximable by polynomials. Clearly, $P(K) \subseteq R(K) \subseteq A(K) \subseteq C(K)$, and a part of the theory of function algebras is given over to the problem of determining when one has equality at some point in this chain of inclusions.

Another significant class of function algebras is formed by the Arens–Singer algebras, which are generalizations of the disc algebra on the circle. Let $G$ be a non-trivial subgroup of the additive group $\mathbf{R}$, endowed with the discrete topology, so that the Pontryagin dual group $\hat{G}$ is therefore compact. Each element $x \in G$ defines a continuous character $\varepsilon_x$ on $\hat{G}$ by evaluation. Denote by $\mathbf{A(G)}$ the closed linear span in $C(\hat{G})$ of the characters $\varepsilon_x$ ($x \in G^+$), where $G^+ = \mathbf{R}^+ \cap G$. Then $\mathbf{A(G)}$ is a function algebra on $\hat{G}$. It can be shown to be isomorphic to an algebra of analytic almost-periodic functions in the upper half-plane, see [10], for example.



As the theory of function algebras developed, analyticity, at least in residual form, seemed to pervade the subject, and a natural question presented itself, namely, if $A$ is a function algebra on $\Omega$, and $A \neq C(\Omega)$, to what extent do the functions in $A$ behave like analytic functions? One observes that, for example, there is a shortage of real-valued functions among analytic functions, and a shortage of real-valued functions persists whenever $A \neq C(\Omega)$, as a consequence of the Stone–Weierstrass theorem ($A = C(\Omega)$ if and only if the real functions in $A$ separate the points of $\Omega$). Moreover, many analytic-type phenomena, such as Jensen's inequality and the maximum modulus principle, were observed to appear in great generality. Indeed, it was discovered that genuine analyticity existed in situations of a very general character.

We shall make this a little more precise: If $A$ is a function algebra on $\Omega$, then one can embed $\Omega$ homeomorphically into the spectrum $X$ of $A$. (As a set, $X$ consists of the characters of $A$ and it is made into a compact Hausdorff space by endowing it with the relative weak* topology induced from the dual space $A^*$.) It is easily seen that if $\tau_\omega$ is defined by $\tau_\omega(\varphi) = \varphi(\omega)$, then the map

$$\Omega \to X, \quad \omega \mapsto \tau_\omega,$$

is a homeomorphism of $\Omega$ onto a subspace of $X$. At one time it was conjectured that whenever $X$ is larger than $\Omega$, there had to be some analytic structure in $X$, in the sense that there should be an embedding $\theta$ of a disc into $X$ such that for all $\varphi \in A$, the functions $\hat{\varphi} \circ \theta$ are analytic. (By $\hat{\varphi}$ we denote the Gelfand transform of $\varphi$, that is, the continuous function on $\Omega$ defined by $\hat{\varphi}(\tau) = \tau(\varphi)$.) Support for this conjecture came from a remarkable theorem of Wermer, concerning embedding analytic structure in a certain very large class of function algebras. We shall discuss this result in more detail presently. However, in 1963, G. Stolzenberg gave an example which falsified the conjecture and Garnett later exhibited examples that showed that the spectrum could, in a sense, be quite arbitrary. We shall shortly make this more precise. First, we discuss some positive results.

Again suppose that $A$ is a function algebra on $\Omega$. Define the relation $\sim$ on the spectrum $X$ of $A$ by $\tau \sim \sigma$ if $\|\tau - \sigma\| < 2$. This relation is an equivalence relation on $X$. The search for analytic structure in the spectrum has been conducted in terms of the corresponding equivalence classes, called *Gleason parts*.

We now consider a large class of function algebras, first introduced by A. M. Gleason, called *Dirichlet* algebras. A function algebra $A$ on $\Omega$ is such an algebra if every real-valued continuous function on $\Omega$ can be uniformly approximated by real parts of functions in $A$.

Of course, $C(\Omega)$ is trivially a Dirichlet algebra, but these are not the interesting examples. The prototype is, as usual in this subject, the disc algebra $\mathbf{A}$. It follows easily from the density of the set of trigonometrical polynomials in $C(\mathbf{T})$ that $\mathbf{A} + \bar{\mathbf{A}}$ is dense in $C(\mathbf{T})$, and hence that $\mathbf{A}$ is a Dirichlet algebra ($\bar{\mathbf{A}}$ denotes the set of complex conjugates of elements of $\mathbf{A}$). It is an immediate consequence of the definition that a representing measure for a character on a Dirichlet algebra is unique, so in particular, this applies to $\mathbf{A}$, and proves the uniqueness claim we made in Section 2.

The Arens–Singer algebras $\mathbf{A}(\mathbf{G})$ are Dirichlet algebras, for the same reason as in the case of the disc algebra, namely density of the (generalized) trigonometric polynomials.

If $K$ is a compact subset of the plane whose complement is connected, and if $\partial K$ is the boundary of $K$, then $P(\partial K)$ is a Dirichlet algebra on $\partial K$ [3].

If $A$ denotes the closure in $C(\mathbf{T}^2)$ of the trigonometric polynomials $\varphi = \sum_{n=0}^{N} \sum_{m=0}^{N} \lambda_{nm} z_1^n z_2^m$, then $A$ is a function algebra on $\mathbf{T}^2$ that is not a Dirichlet algebra.



Now suppose that $S$ is a subset of the spectrum $X$ of a function algebra $A$. If $S$ can be given the structure of a connected complex manifold in such a way that the functions in $A$, or rather their Gelfand transforms, are analytic on $S$, then it can be shown that $S$ must lie entirely in a single Gleason part. The embedding theorem of Wermer referred to earlier in this section asserts that if $A$ is a Dirichlet algebra, then for each Gleason part of $X$ that is not a singleton, there a homeomorphism $\theta$ of the open unit disc onto the part, where the latter is endowed with the metric (norm) topology, such that $\hat{\varphi} \circ \theta$ is analytic for all $\varphi \in A$. The proof of this theorem involves generalized Hardy space theory and a sketch of the method of proof is given in Section 4.

Wermer's result was extended by K. Hoffman to a more general class of function algebras, namely to logmodular algebras. A function algebra $A$ on $\Omega$ is *logmodular* if every real-valued continuous function $\varphi$ on $\Omega$ can be uniformly approximated by elements of the form $\log |\psi|$, where $\psi$ is an invertible element of $A$. The equation $\mathrm{Re}(\varphi) = \log |e^{\varphi}|$ shows that Dirichlet algebras are logmodular, but the converse is false, as we shall see in Section 4. Logmodular algebras share an important property with Dirichlet algebras, namely, uniqueness of representing measures for characters.

Despite these positive results, Stolzenberg's example shows that unless restrictive hypotheses are imposed, analytic structure may not be present in the Gleason parts of a function algebra. Indeed, not much can be said about the structure of Gleason parts in general, as can be seen from the following result of Garnett [8]: Given any completely regular $\sigma$-compact space $Y$ there exists a function algebra in whose spectrum $Y$ can be embedded as a single Gleason part.

## 4. Generalized $\mathbf{H}^{\mathbf{p}}$ space theory

We motivate our considerations in this section by briefly considering the classical Hardy space theory. If $p$ is a real number not less than 1, the Hardy space $H^p$ is defined to be the set of all analytic functions $f$ on the open unit disc for which $\sup_{0\le r<1}\int_0^{2\pi}|f(re^{it})|^p\,dt$ is finite. These spaces arise naturally in the theory of Fourier series and it was realized at an early stage of their development that many properties of $H^p$ functions belong to real-variable theory. For, each $H^p$ function can be written as the Poisson integral of an $L^p$ function on the boundary, or, for $p=1$, of a measure. This allows some results to be deduced without using the theory of analytic functions, and it turned out that the portion of the theory susceptible to this treatment is considerable.

In this approach to $H^p$ space theory the basic vehicle for the analysis of the functions in the Hardy spaces is the disc algebra $\mathbb{A}$ on the circle. The space $H^p$ is identified as the closure of $\mathbb{A}$ in the $L^p$ space of $\mathbb{T}$.

In a series of papers, the Hardy space theory on the circle was generalized by Helson and Lowdenslager to the context of certain abelian compact groups. The great generality of their arguments was soon recognized and the theory was successively generalized, in the context of function algebras, first to Dirichlet algebras and then to logmodular algebras. It was ultimately realized that the theory could be extended to the situation where one had a unique representing measure for a character of a function algebra.

Suppose that $A$ is a function algebra on a compact Hausdorff space $\Omega$ and that $m$ is the unique representing measure for some character of $A$ (if, for example, $A$ is a logmodular algebra, then, as observed above, every character admits a unique representing measure). To avoid trivialities, we shall also assume that $m$ is not a point mass. For $1\le p\le\infty$, we denote by $L^p$ the Lebesgue space $L^p(\Omega,m)$ and for $p$ finite we denote by $H^p$ the norm closure of $A$ in $L^p$. We signify by $H^\infty$ the weak* closure of $A$ in $L^\infty = L^{1*}$. The spaces $H^p$ are Banach spaces and, in particular, $H^2$ is a Hilbert space and $H^\infty$ is a Banach algebra.

Let $\tilde\Omega$ be the spectrum of the algebra $L^\infty$. Then the Gelfand representation induces an isometric isomorphism of $L^\infty$ onto $C(\tilde\Omega)$. Moreover, the image $\tilde A$ of the algebra $H^\infty$ under this representation is a function algebra on $\tilde\Omega$ and it turns out that $\tilde A$ is logmodular. In fact, the following is true: If $\varphi$ is a real-valued function in $L^\infty$, then there exists an invertible element $\psi$ of $H^\infty$ such that $\varphi=\log|\psi|$. Since a Dirichlet algebra necessarily contains all continuous idempotent functions, since $L^\infty$ is the closed linear span of its idempotent elements, and finally since $H^\infty\ne L^\infty$, it follows that $\tilde A$ is an example of a logmodular algebra which is not Dirichlet.



In the case of the disc algebra on the circle, normalized Lebesgue measure is the unique representing measure for a character of $A$. The corresponding algebra $H^\infty$ is therefore logmodular. This enables one to embed analytic structure into the Gleason parts of its spectrum $X$. Actually, $X$ is a very complicated space. One can naturally embed the open unit disc into $X$. A deep result concerning $X$ is the well-known corona theorem of Carleson, which asserts that the disc is dense in $X$. This can be reformulated as follows: If $f_1,\dots,f_n$ are bounded analytic functions on the open unit disc such that $\sum_{k=1}^n|f_k|$ is bounded away from zero, then there exist bounded analytic functions $g_1,\dots,g_n$ such that $f_1g_1+\cdots+f_ng_n=1$.

Let us return to the general situation. We give some examples of the analytic-type properties enjoyed by the elements of $H^1$. Firstly, the only real-valued elements of $H^1$ are the real constants. In the case of the circle, this is easily seen, using the fact that an integrable function whose Fourier transform vanishes must itself vanish almost everywhere. In the general situation the proof requires more work.

Analytic functions cannot vanish on "big" sets without vanishing identically. A similar result holds for $H^1$ space functions. This is our second example of analyticity, and it is a consequence of Jensen's inequality: If $f$ is an element of $H^1$ such that $\int f\,dm\ne 0$, then $\log|f|$ is integrable with respect to $m$ and

$$\log\left|\int f\,dm\right|\le\int\log|f|\,dm.$$

Hence, $f$ cannot vanish on a set of positive measure. In the case of the circle, one can strengthen this to assert that if $f$ is a non-zero element of $H^1$, then $f$ cannot vanish on a set of positive measure, a result known as the little F. and M. Riesz theorem, one of the nicest results of the theory.

The little F. and M. Riesz theorem is an example of a classical result that does not carry over to the general situation. There exists an example of a "system" $\Omega$, $A$ and $m$, a non-zero function $f \in H^1$ and a set $E$ of positive measure (with respect to $m$) such that $f$ vanishes on $E$. Moreover, one may even take $f$ to be an element of $A$ and $E$ to be an open set [19]. This counterexample has consequences in the theory of Toeplitz operators. Incidentally, "systems" $\Omega$, $A$ and $m$ where this kind of behaviour occurs do not have to be pathological in any way.

In passing, one should perhaps observe that the surprising thing is not that some of the classical $H^p$ space theory does not extend to the general situation, but that so much of it does.

Since we have referred to a little F. and M. Riesz theorem, by implication there should be a big F. and M. Riesz theorem, and, of course, there is. In the case of the circle, this well known result asserts that if a Borel complex measure on $\mathbf{T}$ has Fourier–Stieltjes transform supported in $\mathbf{Z}^+$, then it is absolutely continuous with respect to Lebesgue measure on $\mathbf{T}$. This theorem *does* extend to the general situation, although in substantially modified form [3].

In the general context there is also a version of the celebrated invariant subspace theorem of Beurling. A closed vector subspace $M$ of $H^2$ is *invariant* if $\varphi M \subseteq M$ for all $\varphi \in A$. The generalized Beurling theorem asserts that if $M$ is an invariant space and if there exists $f \in M$ such that $\int f\,dm \neq 0$, then there exists a function $\varphi \in H^\infty$ such that $|\varphi| = 1$ almost everywhere with respect to $m$ and $M = \varphi H^2$ (such functions $\varphi$ are called *inner* functions).

We now give a very brief sketch of how Wermer's embedding theorem is established. We shall leave out all the technical details of the proof, but nevertheless, the sketch should give a reasonable outline idea of the argument.

Let $A$ be a Dirichlet algebra on $\Omega$. If $P$ is a Gleason part of the spectrum of $A$ that is not a singleton, and $m$ is the unique representing measure for a character $\tau$ in $P$, then one can show that there is an inner function $Z$ in $H^\infty$ such that

$$ZH^2 = \{f \in H^2 \mid \int f\,dm = 0\}.$$

If $\Delta$ denotes the open unit disc and $s \in \Delta$, one can define $\tau_s$ in the spectrum of $A$ by $\tau_s(\varphi) = \int \varphi(1 - s\bar{Z})^{-1}\,dm$. Observe that

$$\tau_s(\varphi) = \sum_{n=0}^{\infty}\Big(\int \varphi\bar{Z}^n\,dm\Big)s^n,$$

so the function

$$\Delta \to \mathbf{C}, \quad s \mapsto \tau_s(\varphi),$$

is analytic. Using Schwarz's lemma one can show that

$$\|\tau_s - \tau\| \leq 2|s| < 2$$

for all $s \in \Delta$; therefore, $\tau_s \in P$. One can now show that the map

$$\theta\colon \Delta \to P, \quad s \mapsto \tau_s$$

is a homeomorphism of $\Delta$ onto $P$, where $P$ has the metric topology. For all $\varphi \in A$, the composition $\hat{\varphi} \circ \theta$ is analytic, as $\hat{\varphi}\theta(s) = \tau_s(\varphi)$. Thus we have embedded analytic structure into the spectrum of $A$. For full details of this construction, see [3].



## 5. Applications to operator theory

The class of Toeplitz operators is an exceptional class of operators, for it is one of the few large classes of operators about which we have detailed knowledge. Toeplitz operators are related to multiplication operators, but their structure and properties are much more difficult to analyse.

If $\varphi \in L^\infty(\mathbf{T})$, the multiplication operator corresponding to $\varphi$ is the operator $M_\varphi$ on $L^2(\mathbf{T})$ defined by $M_\varphi(f) = \varphi f$. The compression of this operator to the Hardy space $H^2$ is the corresponding Toeplitz operator. Explicitly, if $P$ is the projection of $L^2$ onto

$H^2$, then the Toeplitz operator $T_\varphi$ is defined by $T_\varphi(f) = P(\varphi f)$. The fact that the map, $\varphi \mapsto M_\varphi$, is an algebra homomorphism helps enormously in the analysis of the operators $M_\varphi$. For Toeplitz operators the corresponding map, $\varphi \mapsto T_\varphi$, can be easily seen to be linear and to preserve involutions (that is, $T_{\bar\varphi} = T_\varphi^*$), but it is not multiplicative, a fact that makes the analysis of Toeplitz operators very different from that of multiplication operators.

Before proceeding to discuss Toeplitz operators in more detail, let us say a few words about the significance of this class of operators. As indicated above, they are important in operator theory, because they provide a highly non-trivial class of operators accessible to detailed analysis. There are beautiful connections with function theory, specifically $H^p$ space theory. Amongst other applications, there are applications to the analysis of boundary-value problems, to information theory and to time-series analysis in statistics. For more information on applications, see [2].

Suppose now that $\Omega$ is a compact Hausdorff space, $A$ is a function algebra on $\Omega$, and $m$ is the unique representing measure for a character of $A$. As before, we denote the corresponding Lebesgue and Hardy spaces by $L^p$ and $H^p$, respectively. Given $\varphi \in L^\infty$, one can define the Toeplitz operator $T_\varphi$ in the same way as in the case of the circle. It is obvious that $\|T_\varphi\| \leq \|\varphi\|_\infty$, and, in fact, equality holds, but this is very non-obvious. One can derive this from a stronger result, a spectral inclusion result which says that the spectrum of $T_\varphi$ contains the spectrum of $\varphi$. (The spectrum is an important invariant. For an element $a$ of a unital Banach algebra, its *spectrum* is the set of all scalars $\lambda$ such that $a - \lambda$ is not invertible.) This spectral inclusion result is due to Hartman and Wintner [9] in the case of the circle and to the author [17] in the general case, where the proof is quite different to that of the classical case. The proof uses the fact that every real-valued function in $L^\infty$ is the logarithm of the modulus of an element of $H^\infty$.

One of the deepest results concerning Toeplitz operators on the circle is the theorem of Widom [6] which asserts that they have connected spectra. In the general situation the author has shown connectedness of the spectrum still persists for two important subclasses of Toeplitz operators namely, for Hermitian Toeplitz operators $T_\varphi$ (where $\bar\varphi = \varphi$) and for analytic Toeplitz operators (where $\varphi \in H^\infty$), see [17].



Let us illustrate the interplay of operator theory and function theory by considering a special simple result: A non-scalar Hermitian Toeplitz operator on the circle has no eigenvalues. The proof is easily reduced to showing that zero is not an eigenvalue. Suppose then $\varphi$ is a real-valued element of $L^\infty$ and that $f$ is an element of $H^2$ such that $T_\varphi(f) = 0$. Then $P(\varphi f) = 0$, so $\varphi\bar f$ belongs to $H^2$. Hence, $\varphi \bar f f$ is a real-valued element of $H^1$, and therefore it is almost everywhere equal to a constant, $c$ say. However, $c = \int c\,dm = \int \varphi \bar f f\,dm = \langle T_\varphi(f), f\rangle = 0$. Thus, $\varphi \bar f f = 0$ a.e. Now the assumption that $T_\varphi$ is non-scalar assures us that on some set of positive measure, $\varphi$ does not vanish. Hence, $f$ *does* vanish on a set of positive measure. Therefore, by the F. and M. Riesz theorem, $f = 0$ a.e.

We mentioned earlier that the little F. and M. Riesz theorem does not hold in the general situation. One way that this is reflected in the theory of Toeplitz operators is that non-scalar Hermitian Toeplitz operators on generalized Hardy spaces may have eigenvalues. Even here, however, a striking result is true. Non-zero eigenspaces must be infinite dimensional. For more details, see [17].

Limitations of space have allowed us to give no more than an inkling of the scope of the theory of Toeplitz operators and its profound connections with function theory. Some of the classical theory is covered in [6]. Both the classical and generalized theories are now so extensive that the bibliography that follows can indicate only a few of the many possibilities for further reading for the interested reader.

Toeplitz theory also has connections with the theory of C*-algebras and with $K$-theory. This aspect of the subject involves index theory, one of the most active areas of modern operator theory. Some references for this are: [4], [6], [12], [13], [16], [18] and [21].

As an introduction to function algebras, Browder's book [3] probably cannot be surpassed. For a more detailed treatment of the subject see the books of Gamelin [7] and Leibowitz [11], each of which contains extensive bibliographies. Two other books which may be consulted are [19] and [20]. Two journal articles that are particularly readable are [1] and [10].

## References

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